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EXISTENCE OF SOLUTIONS TO THE LICHNEROWICZ EQUATION: A NEW PROOF

ROMAIN GICQUAUD

ABSTRACT. We provide a complete study of existence and uniqueness of solutions to the Lichnerowicz equation in general relativity with arbitrary mean curvature.

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1. INTRODUCTION

The Lichnerowicz equation is an elliptic equation that appears in the construction of initial data in general relativity. In the setting of this note, let (M, g) be a compact Riemannian manifold of dimension $n > 2$, $g \in W^{2,p}$, $p > n/2$, and assume given two functions $\tau \in L^{2p}$ and $A \in L^{2p}$. The Lichnerowicz equation has a positive function ϕ as unknown and reads

$$-\frac{4(n-1)}{n-2}\Delta\phi + \mathrm{Scal}\,\phi = -\frac{n-1}{n}\tau^2\phi^{N-1} + \frac{A^2}{\phi^{N+1}}, \tag{1.1}$$

where Scal is the scalar curvature of g and $N := \dfrac{2n}{n-2}$.

We refer the reader to [3, 5] for an overview of the context in which this equation appears. It has attracted attention a couple of decades ago culminating at the classification of constant mean curvature initial data by J. Isenberg in [14]. Recently, important efforts have been put in constructing non-constant mean curvature initial data, see [12, 13, 16, 6] and [10].

The main aim of this note is to give a short proof of existence/non-existence of solutions to (1.1) in the generic case $A \not\equiv 0$. This result is well-known to a large extent, see e.g. [16, Theorem 1]. The main novelty here is that there is no need to give separate proofs according to the sign of the Yamabe quotient of (M, g). The particular case $A \equiv 0$ is the prescribed scalar curvature equation which is similar to the problem addressed in [19, 17, 18, 20], see also [7, 8]. We will study it in Section 4.

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This paper is a byproduct of the techniques developed in [7, 8].

The outline of this paper is as follows. In Section 2, we introduce the main tool to discrimitate which function τ lead to existence of solutions to (1.1). In Section 3, we study the case $A \not\equiv 0$. The main result of this section is Theorem 3.1 which is the main result of the paper. Section 4 is devoted to the case $A \equiv 0$ which, as we indicated before, deserves a particular treatment.

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2. Local Yamabe invariant and first conformal eigenvalue

For any measurable subset $V \subset M$, we define the space

$$\mathcal{F}(V) := \{u \in W^{1,2}, u \equiv 0 \text{ a.e. on } M \backslash V\} \tag{2.1}$$

of Sobolev functions vanishing outside V. This set is obviously reduced to $\{0\}$ if V has Lebesgue measure zero but there are larger V with $\mathcal{F}(V) = \{0\}$, see for example [1, Chapter 6]. Much of this section is adapted from [8].

For any $u \in W^{1,2}$, we set

$$G_g(u) := \int_M \left[\frac{4(n-1)}{n-2} |du|^2 + \text{Scal } u^2 \right] d\mu^g \tag{2.2}$$

We also introduce, for any $u \in W^{1,2}$, $u \not\equiv 0$, the Rayleigh and the Yamabe quotients:

$$Q_g^R(u) := G(u)/\|u\|_{L^2}^2,$$
$$Q_g^Y(u) := G(u)/\|u\|_{L^N}^2.$$

With these definitions at hand, we introduce the local first conformal eigenvalue $\lambda_g(V)$ and the local Yamabe invariant $\mathcal{Y}_g(V)$ of any measurable subset $V \subset M$ as follows:

$$\lambda_g(V) := \inf_{u \in \mathcal{F}(V) \backslash \{0\}} Q_g^R(u),$$
$$\mathcal{Y}_g(V) := \inf_{u \in \mathcal{F}(V) \backslash \{0\}} Q_g^Y(u).$$

From the definition of an infimum, we have $\lambda_g(V) = \mathcal{Y}_g(V) = \infty$ if $\mathcal{F}(V)$ is reduced to $\{0\}$.

Proposition 2.1. *The functional G defined in (2.2) is sequentially weakly lower semi-continuous on $W^{1,2}$: for every weakly converging sequence $(u_k)_k$, $u_k \rightharpoonup u_\infty$, we have $\liminf_{k\to\infty} G(u_k) \geqslant G(u_\infty)$.*

Proof. Note that G_g can be decomposed as

$$G_g(u) = \frac{4(n-1)}{n-2} \int_M |du|^2 d\mu^g + \int_M \text{Scal } u^2 d\mu^g. \tag{2.5}$$

The first term is weakly lower semi-continuous with respect to $u \in W^{1,2}$ as a continuous non-negative quadratic form. For the second one, we shall prove that, given a sequence $(u_k)_k$ in $W^{1,2}$, converging weakly to u_∞, $u_k \rightharpoonup_{k\to\infty} u_\infty$, we have

$$\int_M \text{Scal } u_k^2 d\mu^g \to \int_M \text{Scal } u_\infty^2 d\mu^g.$$

To make the notation less cluttered, we denote the second term in (2.5) as $S(u)$:

$$S(u) := \int_M \mathrm{Scal}\, u^2 d\mu^g.$$

Assume by contradiction that $(S(u_k))_k$ does not converge to $S(u_\infty)$, there exists an $\epsilon > 0$ such that, for an infinite number of integers k, we have

$$|S(u_k) - S(u_\infty)| > \epsilon. \tag{2.6}$$

Without loss of generality, we can assume that (2.6) holds for all integer k and also that $(u_k)_k$ converges strongly in L^2 to some $\overline{u}_\infty \in L^2$ since the embedding $W^{1,2} \hookrightarrow L^2$ is compact. Then we have $u_\infty \equiv \overline{u}_\infty$ a.e. Indeed, the linear form

$$u \mapsto \int_M u(u_\infty - \overline{u}_\infty) d\mu^g$$

is (strongly) continuous for the L^2-topology and, hence, for the $W^{1,2}$-topology. As a consequence,

$$\int_M u_\infty(u_\infty - \overline{u}_\infty) d\mu^g = \lim_{k\to\infty} \int_M u_k(u_\infty - \overline{u}_\infty) d\mu^g = \int_M \overline{u}_\infty(u_\infty - \overline{u}_\infty) d\mu^g,$$

where the first equality holds by the $W^{1,2}$-weak convergence of $(u_k)_k$ to u_∞ and the second one by the L^2-strong convergence of $(u_k)_k$ to $\overline{u}_\infty$. Subtracting both equalities, we get

$$\int_M |u_\infty - \overline{u}_\infty|^2 d\mu^g = 0,$$

which proves that $u_\infty \equiv \overline{u}_\infty$ a.e. Finally note that, since $(u_k)_k$ is weakly convergent in $W^{1,2}$, it is bounded and thus (by interpolation) converges in all L^q spaces, $q \in [2, N)$. Since $\mathrm{Scal} \in L^p$, $p > n/2$, letting q be such that $1 = \dfrac{1}{p} + \dfrac{2}{q}$, we have $q \in [2, N)$ and, by Hölder's inequality, S is a bounded quadratic form on L^q. In particular S is continuous on L^q:

$$S(u_k) \to S(u_\infty).$$

This contradicts (2.5): S is sequentially weakly continuous on $W^{1,2}$. This ends the proof of Proposition 2.1. □

In what follows, we let $s > 0$ be the largest constant so that

$$\|u\|^2_{W^{1,2}} \geqslant s\|u\|^2_{L^N} \quad \forall u \in W^{1,2}. \tag{2.7}$$

Proposition 2.2. *Given any measurable set $V \subset M$, $\lambda_g(V)$ and $\mathcal{Y}_g(V)$ have the same sign (i.e. they are either both positive, both negative or both zero).*

Proof. We can assume, without loss of generality, that $\mathcal{F}(V) \neq \{0\}$ for otherwise $\mathcal{Y}_g(V) = \lambda_g(V) = \infty$. If $\mathcal{Y}_g(V) < 0$, there exists $u \in \mathcal{F}(V)$ such that $G_g(u) < 0$ so $\lambda_g(V) < 0$. Assume now that $\mathcal{Y}_g(V) > 0$, then, for all $u \in \mathcal{F}(V)\backslash\{0\}$, we have

$$Q^R_g(u) = \frac{G_g(u)}{\|u\|^2_{L^2}} \geqslant \frac{G_g(u)}{\|u\|^2_{L^N}\,\mathrm{Vol}_g(V)^{2/n}} \geqslant \frac{\mathcal{Y}_g(V)}{\mathrm{Vol}_g(V)^{2/n}}.$$

We conclude that

$$\lambda_g(V) \geqslant \frac{\mathcal{Y}_g(V)}{\mathrm{Vol}_g(V)^{2/n}} > 0.$$

All we have to show now is that, if $\mathcal{Y}_g(V) = 0$, we have $\lambda_g(V) = 0$. Assume for the rest of the proof that $\mathcal{Y}_g(V) = 0$. If $\lambda_g(V)$ were negative, there would exits $u \in \mathcal{F}(V)$ such that $G_g(u) < 0$ so $\mathcal{Y}_g(V) \leqslant Q^Y_g(u) < 0$. This proves that $\lambda_g(V) \geqslant 0$. Since $\mathcal{Y}_g(V) = 0$, there exists a sequence

of functions $u_k \in \mathcal{F}(V)$ such that $Q_g^Y(u_k) \to 0$. Without loss of generality, we can assume that $\|u_k\|_{L^N} = 1$ so $G_g(u_k) \to 0$.

Let q be as in the proof of the previous proposition. Then we have that

$$\begin{aligned} G(u_k) &\geqslant \frac{4(n-1)}{n-2}\|u_k\|^2_{W^{1,2}} - \frac{4(n-1)}{n-2}\|u_k\|^2_{L^2} - \|\mathrm{Scal}\|_{L^p}\|u_k\|^2_{L^q} \\ &\geqslant \frac{4(n-1)}{n-2}\|u_k\|^2_{W^{1,2}} - \frac{4(n-1)}{n-2}\mathrm{Vol}_g(V)^{1-2/q}\|u_k\|^2_{L^q} - \|\mathrm{Scal}\|_{L^p}\|u_k\|^2_{L^q}. \end{aligned}$$

Hence, setting $C = \frac{4(n-1)}{n-2}\mathrm{Vol}_g(V)^{1-2/q} + \|\mathrm{Scal}\|_{L^p}$, we arrive at

$$G_g(u_k) + C\|u_k\|^2_{L^q} \geqslant \frac{4(n-1)}{n-2}\|u_k\|^2_{W^{1,2}}. \tag{2.8}$$

Since $q < N$, we have that $\|u_k\|_{W^{1,2}}$ is bounded independently of k. Arguing as in the proof of the previous proposition, we can assume that $(u_k)_k$ converges weakly in $W^{1,2}$ and strongly in L^2 to some $u_\infty \in \mathcal{F}(V)$. Combining Equation (2.8) with the Sobolev estimate (2.7), we get

$$G_g(u_k) + C\|u_k\|^2_{L^q} \geqslant \frac{4(n-1)}{n-2}s\|u_k\|^2_{L^N} = \frac{4(n-1)}{n-2}s.$$

Passing to the limit as k goes to infinity, we conclude that $\|u_\infty\|_{L^q} > 0$, i.e. $u_\infty \not\equiv 0$. By the lower semicontinuity of G_g, we have $G_g(u_\infty) \leqslant \liminf_{k\to\infty} G_g(u_k) = 0$. Since $G_g(u_\infty) \geqslant 0$, we have $G_g(u_\infty) = 0$. We have proven that

$$0 \leqslant \lambda_g(V) \leqslant Q_g^R(u_\infty) = 0,$$

i.e. $\lambda_g(V) = 0$. This concludes the proof of the fact that $\mathcal{Y}_g(V)$ and $\lambda_g(V)$ have the same sign. □

The reason why it is more convenient to work with $\mathcal{Y}_g(V)$ than with $\lambda_g(V)$ is given by the following proposition.

Proposition 2.3. *Assume that g and h are two conformally related metrics, $h = \phi^{N-2}g$, for some positive function $\phi \in W^{2,p}$. Then for any measurable V we have*

$$\mathcal{Y}_g(V) = \mathcal{Y}_h(V).$$

Proof. The proof is a simple calculation. Given any $u \in W^{1,2}$, we have

$$\begin{aligned}
G_h(u) &= \int_M \left[\frac{4(n-1)}{n-2}|du|_h^2 + \mathrm{Scal}^h\, u^2\right] d\mu^h \\
&= \int_M \left[\frac{4(n-1)}{n-2}\phi^{2-N}|du|_g^2 + \left(-\frac{4(n-1)}{n-2}\Delta^g\phi + \mathrm{Scal}^g\,\phi\right)\phi^{1-N}u^2\right]\phi^N d\mu^g \\
&= \int_M \left[\frac{4(n-1)}{n-2}\phi^{2}|du|_g^2 + \left(-\frac{4(n-1)}{n-2}\Delta^g\phi + \mathrm{Scal}^g\,\phi\right)\phi u^2\right] d\mu^g \\
&= \int_M \left[\frac{4(n-1)}{n-2}\left(\phi^2|du|_g^2 - (\phi\Delta^g\phi)u^2\right) + \mathrm{Scal}^g\,(\phi u)^2\right] d\mu^g \\
&= \int_M \left[\frac{4(n-1)}{n-2}\left(\phi^2|du|_g^2 + \langle d\phi, d(\phi u^2)\rangle_g\right) + \mathrm{Scal}^g\,(\phi u)^2\right] d\mu^g \\
&= \int_M \left[\frac{4(n-1)}{n-2}\left(\phi^2|du|_g^2 + u^2|d\phi|_g^2 + 2\langle \phi d\phi, u du\rangle_g\right) + \mathrm{Scal}^g\,(\phi u)^2\right] d\mu^g \\
&= \int_M \left[\frac{4(n-1)}{n-2}|d(\phi u)|_g^2 + \mathrm{Scal}^g\,(\phi u)^2\right] d\mu^g \\
&= G_g(\phi u).
\end{aligned}$$

Similarly,

$$\|u\|_{L_h^N} = \left(\int_M u^N d\mu^h\right)^{1/N} = \left(\int_M u^N\phi^N d\mu^g\right)^{1/N} = \|\phi u\|_{L_g^N}.$$

So

$$Q_h^{\mathcal{Y}}(u) = Q_g^{\mathcal{Y}}(\phi u).$$

Since ϕ is bounded away from zero, multiplication by ϕ defines an automorphism of $\mathcal{F}(V)$. Hence,

$$\mathcal{Y}_g(V) = \inf_{u\in\mathcal{F}(V)} Q_g^{\mathcal{Y}}(u) = \inf_{u\in\mathcal{F}(V)} Q_g^{\mathcal{Y}}(\phi u) = \inf_{u\in\mathcal{F}(V)} Q_h^{\mathcal{Y}}(u) = \mathcal{Y}_h(V).$$

□

3. Existence of solutions to the Lichnerowicz equation

Theorem 3.1. *Let (M, g) be a compact Riemannian manifold with $g \in W^{2,p}$, $p > n/2$. Assume that $\tau \in L^{2p}$ is given. Then the following statements are equivalent:*

(1) There exists a solution to (1.1) for all $A \in L^{2p}$, $A \not\equiv 0$
(2) There exists a solution to (1.1) for at least one $A \in L^{2p}$, $A \not\equiv 0$,
(3) The set $Z = \tau^{-1}(0)$ satisfies $\mathcal{Y}_g(Z) > 0$.

Further, the solution to (1.1), when it exists, is unique unless $\mathcal{Y}_g(M) = 0$ and $\tau, A \equiv 0$ for which all solutions are proportional one to another.

It should be noted that the theorem can be applied in particular when $\mathcal{Z}$ has zero Lebesgue measure. This is the case if τ never vanishes or if 0 is a regular value for τ.

This theorem reproduces results from [12, 13, 15, 16] and references therein (see also [9]) in which several proofs are given according to the sign of $\mathcal{Y}_g(M)$ and the nullity of τ. The main novelty is that the proof establishes a direct link between existence of solutions to the Lichnerowicz equation and the fact that $\mathcal{Y}_g(Z) > 0$. We first state a lemma:

Lemma 3.2. *Under the assumptions of the theorem, if $\mathcal{Y}_g(Z) > 0$, there exists a constant $K > 0$ such that the operator*

$$u \mapsto -\frac{4(n-1)}{n-2}\Delta u + \mathrm{Scal}\, u + K\frac{n-1}{n}\tau^2 u$$

has positive first eigenvalue.

Proof. Assume by contradiction that for all $k \in \mathbb{N}$, the first eigenvalue of

$$L_k : u \mapsto -\frac{4(n-1)}{n-2}\Delta u + \mathrm{Scal}\, u + k\frac{n-1}{n}\tau^2 u$$

is non-positive. We denote it by λ_k and let $u_k \in W^{2,p/2}$ be the first eigenfunction normalized so that $u_k \geqslant 0$ and $\|u_k\|_{L^2} = 1$. The sequence $(\lambda_k)_k$ is increasing since

$$\begin{aligned}
\lambda_{k+1} &= \int_M u_{k+1} L_{k+1} u_{k+1} d\mu^g \\
&= \int_M u_{k+1} L_k u_{k+1} d\mu^g + \int_M \frac{n-1}{n}\tau^2 u_{k+1}^2 \\
&\geqslant \int_M u_{k+1} L_k u_{k+1} d\mu^g \\
&\geqslant \lambda_k.
\end{aligned}$$

We claim that the sequence $(u_k)_k$ is bounded in $W^{1,2}$. Indeed, we have, using the Hölder inequality:

$$\begin{aligned}
0 &\geqslant \int_M \left[\frac{4(n-1)}{n-2}|du_k|^2 + \mathrm{Scal}\, u_k^2\right] d\mu^g \\
&\geqslant \frac{4(n-1)}{n-2}\int_M |du_k|^2 d\mu^g - \|\mathrm{Scal}\|_{L^p}\|u_k\|_{L^N}^{\frac{n}{p}}\|u_k\|_{L^2}^{2-\frac{n}{p}} \\
&\geqslant \frac{4(n-1)}{n-2}\int_M |du_k|^2 d\mu^g - 2\|\mathrm{Scal}\|_{L^p}\left[\frac{n}{p}\epsilon\|u_k\|_{L^N}^2 + \frac{2p-n}{2p}\|u_k\|_{L^2}^2\epsilon^{-n/(2p-n)}\right] \\
&\geqslant \frac{4(n-1)}{n-2}\int_M |du_k|^2 d\mu^g - \|\mathrm{Scal}\|_{L^p}\left[\frac{2n}{sp}\epsilon\|u_k\|_{W^{1,2}}^2 + \frac{2p-n}{p}\|u_k\|_{L^2}^2\epsilon^{-n/(2p-n)}\right],
\end{aligned}$$

where we used the ϵ-Young inequality and the Sobolev inequality (2.7). Assuming that $\mathrm{Scal} \not\equiv 0$ (if $\mathrm{Scal} \equiv 0$ the argument is simpler), we choose ϵ such that

$$\|\mathrm{Scal}\|_{L^p}\frac{2n}{sp}\epsilon = \frac{2(n-1)}{n-2},$$

so

$$0 \geqslant \frac{2(n-1)}{n-2}\int_M |du_k|^2 d\mu^g - C\|u_k\|_{L^2}^2,$$

for some explicit constant $C = C(n, s, p, \|\mathrm{Scal}\|_{L^p})$. Since $\|u_k\|_{L^2} = 1$, this proves the claim that $(u_k)_k$ is bounded in L^2.

From Rellich theorem, we now extract a subsequence $(k_i)_i$ of k such that

$$u_{k_i} \to u_\infty \text{ in } L^2$$

for some $u_\infty \in W^{1,2}$. In particular, $\|u_\infty\|_{L^2} = 1$. We can also assume that

$$u_{k_i} \rightharpoonup u_\infty \text{ in } W^{1,2}.$$

We claim that $u_\infty \equiv 0$ a.e. on $M\backslash Z$. Otherwise,

$$\int_M \tau^2 u_{k_i}^2 d\mu^g \to \int_M \tau^2 u_\infty^2 d\mu^g \neq 0,$$

so

$$\begin{aligned}\lambda_{k_i} &= \int_M u_{k_i} L_{k_i} u_{k_i} d\mu^g \\ &= \int_M u_{k_i} L_0 u_{k_i} d\mu^g + k_i \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 d\mu^g \\ &\geqslant \lambda_0 + k_i \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 d\mu^g \\ &\to_{i\to\infty} \infty,\end{aligned}$$

contradicting the fact that $(\lambda_k)_k$ is bounded. Since $\|u_\infty\|_{L^2} = 1$ and belongs to $\mathcal{F}(Z)$, we have a contradiction if $\mathcal{F}(Z) = \{0\}$. In the case where $\mathcal{F}(Z) \neq \{0\}$, we also get a contradiction since

$$\lambda_{k_i} = G_g(u_{k_i}) + k \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 \geqslant G_g(u_{k_i}),$$

so, since G_g is weakly lower semicontinuous,

$$\liminf_{i\to\infty} \lambda_{k_i} \geqslant \liminf_{i\to\infty} G_g(u_{k_i}) \geqslant G_g(u_\infty) \geqslant \lambda_g(Z) > 0.$$

This gives the final contradiction. □

Proof of Theorem 3.1. The statement 1 ⇒ 2 is obvious. We now prove that 2 ⇒ 3. The proof is similar to that of Proposition 2.3. If $\mathcal{F}(Z) = \{0\}$, Statement 3 is satisfied since $\mathcal{Y}_g(Z) = \infty$. Otherwise, assume given $A \in L^{2p}$ and $\phi \in W^{2,p}$ satisfying (1.1). We set $\widehat{g} = \phi^{N-2} g$ and $\widehat{u} = u\phi^{-1}$. For all $u \in \mathcal{F}(Z)$, we have

$$\begin{aligned}G_g(u) &= G_g(\phi\widehat{u}) \\ &= \int_M \left[\frac{4(n-1)}{n-2}\left(\phi^2 |d\widehat{u}|_g^2 + \left\langle \phi d\phi, d(\widehat{u}^2)\right\rangle_g + \widehat{u}^2 |d\phi|_g^2\right) + \mathrm{Scal}\, \phi^2 \widehat{u}^2\right] d\mu^g \\ &= \int_M \left[\frac{4(n-1)}{n-2}\left(\phi^2 |d\widehat{u}|_g^2 - (\phi\Delta\phi)\widehat{u}^2\right) + \mathrm{Scal}\, \phi^2 \widehat{u}^2\right] d\mu^g \\ &= \int_M \left[\frac{4(n-1)}{n-2}\phi^2 |d\widehat{u}|_g^2 + \left(\frac{A^2}{\phi^N} - \frac{n-1}{n}\tau^2\phi^N\right)\widehat{u}^2\right] d\mu^g \\ &= \int_M \left[\frac{4(n-1)}{n-2} |d\widehat{u}|_{\widehat{g}}^2 + \left(\frac{A^2}{\phi^{2N}} - \frac{n-1}{n}\tau^2\right)\widehat{u}^2\right] d\mu^{\widehat{g}} \\ &\geqslant \int_M \left[\frac{4(n-1)}{n-2} |d\widehat{u}|_{\widehat{g}}^2 + \frac{A^2}{\phi^{2N}}\widehat{u}^2\right] d\mu^{\widehat{g}} \qquad (\text{since } \widehat{u} \in \mathcal{F}(Z)).\end{aligned}$$

This immediately rules out the possibility that $\mathcal{Y}_g(Z) < 0$ since $G_g(u) \geqslant 0$ for all $u \in \mathcal{F}(Z)$. Assume next that $M\backslash Z$ has positive Lebesgue measure. Then, $\widehat{u} \equiv 0$ on $M\backslash Z$. As a consequence, from the Poincaré inequality, there is a constant $\mu = \mu(g, \tau)$ so that

$$G_g(u) \geqslant \frac{4(n-1)}{n-2} \int_M |d\widehat{u}|_{\widehat{g}}^2 d\mu^{\widehat{g}} \geqslant \mu \|\widehat{u}\|_{W^{1,2}}^2,$$

(see e.g. [11, Lemma 7.16]) and, hence, from the Sobolev embedding theorem,

$$G_g(u) \geqslant s\mu \|\widehat{u}\|_{L^N}^2.$$

This proves that

$$\mathcal{Y}_g(Z) = s\mu > 0.$$

The only remaining possibility is that $\tau \equiv 0$ a.e. that is to say $Z = M$ and $\mathcal{Y}_g(M) = 0$. From the proof of Proposition 2.2, there exists a function $u_\infty \geqslant 0$, $u_\infty \not\equiv 0$ so that $G_g(u_\infty) = 0$. From the inequality

$$G_g(u_\infty) \geqslant \frac{4(n-1)}{n-2} \int_M |d\widehat{u}_\infty|^2_{\hat{g}}\, d\mu^{\hat{g}},$$

we have $d\widehat{u}_\infty \equiv 0$: $\widehat{u}_\infty$ is a constant function. This gives a contradiction since

$$0 = G_g(u_\infty) = \int_M \frac{A^2}{\phi^{2N}} \widehat{u}_\infty^2 d\mu^{\hat{g}} > 0.$$

We finally prove that 3 $\Rightarrow$ 1. The proof goes as usual by the sub- and super-solution method (see e.g. [21, Chapter 14]). Let K be as in the statement of Lemma 3.2. We let u denote the solution to

$$-\frac{4(n-1)}{n-2}\Delta u + \mathrm{Scal}\, u + K\frac{n-1}{n}\tau^2 u = A^2. \tag{3.1}$$

Since the operator on the left hand side is positive, its Green function is positive, so $u \in W^{2,p}$ is also positive (note that u is Hölder continuous). We set

$$\begin{cases} u_+ = \lambda_+ u, \\ u_- = \lambda_- u \end{cases}$$

for some positive constants $\lambda_\pm$ to be chosen later. We want u_+ to be a super-solution to the Lichnerowicz equation (1.1), i.e. u_+ has to satisfy

$$-\frac{4(n-1)}{n-2}\Delta u_+ + \mathrm{Scal}\, u_+ + \frac{n-1}{n}\tau^2 u_+^{N-1} \geqslant \frac{A^2}{u_+^{N+1}}.$$

From Equation (3.1), this is equivalent to

$$\frac{n-1}{n}\tau^2\left(\lambda_+^{N-1}u^{N-1} - K\lambda_+ u\right) + \lambda_+ A^2 \geqslant \frac{A^2}{\lambda_+^{N+1}u^{N+1}}.$$

This inequality holds true if both the following inequalities are fulfilled:

$$\begin{cases} \lambda_+^{N-2}u^{N-2} \geqslant K, \\ \lambda_+^{N+2} \geqslant u^{-N-1}. \end{cases}$$

Since u is bounded from above and away from zero, they are true for large enough λ_+. Calculations for the sub-solution are similar: if λ_- is a small enough positive constant u_- is a sub-solution to the Lichnerowicz equation (1.1). By the sub- and super-solution argument, we get existence of $u \in W^{2,p}$ solving (1.1). Uniqueness of u will be proven in the next proposition. □

Proposition 3.3. *Let (M, g) be a compact Riemannian manifold with $g \in W^{2,p}$, $p > n/2$. Let $\tau, A \in L^{2p}$ be two given functions. Assume given two positive functions $\phi_1, \phi_2 \in W^{2,p}$ solving the Lichnerowicz equation (1.1).*

- *If $\tau \not\equiv 0$ or $A \not\equiv 0$, we have $\phi_1 \equiv \phi_2$,*
- *If $\tau, A \equiv 0$, ϕ_1 and ϕ_2 are proportional.*

Proof. The proof of this fact is well known, we refer the reader e.g. to [16, Proposition 2] or to [6]. We present here the argument from [4]. Since ϕ_1 and ϕ_2 are bounded from below, we have that ϕ_1^2/ϕ_2 and ϕ_2^2/ϕ_1 both belong to $W^{2,p}$. By an integration by parts and some routine calculations, we have

$$\begin{aligned}\int_M \left(-\frac{\Delta\phi_1}{\phi_1}+\frac{\Delta\phi_2}{\phi_2}\right)(\phi_1^2-\phi_2^2)d\mu^g \\ = \int_M \left|d\phi_1-\frac{\phi_1}{\phi_2}d\phi_2\right|^2 d\mu^g + \int_M \left|d\phi_2-\frac{\phi_2}{\phi_1}d\phi_1\right|^2 d\mu^g.\end{aligned} \tag{3.2}$$

If we set

$$f(\phi) := \frac{n-2}{4(n-1)}\left[\frac{A^2}{\phi^{N+2}} - \mathrm{Scal} - \frac{n-1}{n}\tau^2\phi^{N-2}\right],$$

we have

$$-\frac{\Delta\phi_1}{\phi_1} = f(\phi_1) \quad \text{and} \quad -\frac{\Delta\phi_2}{\phi_2} = f(\phi_2),$$

so the identity (3.2) gives

$$\begin{aligned}\int_M [f(\phi_1)-f(\phi_2)](\phi_1^2-\phi_2^2)d\mu^g \\ = \int_M \left|d\phi_1-\frac{\phi_1}{\phi_2}d\phi_2\right|^2 d\mu^g + \int_M \left|d\phi_2-\frac{\phi_2}{\phi_1}d\phi_1\right|^2 d\mu^g.\end{aligned}$$

Since f is a decreasing function, we have

$$[f(\phi_1)-f(\phi_2)](\phi_1^2-\phi_2^2) \leqslant 0 \text{ a.e.}$$

This impose that

$$\int_M \left|d\phi_1-\frac{\phi_1}{\phi_2}d\phi_2\right|^2 d\mu^g + \int_M \left|d\phi_2-\frac{\phi_2}{\phi_1}d\phi_1\right|^2 d\mu^g = 0.$$

In particular, we have

$$d\phi_1 - \frac{\phi_1}{\phi_2}d\phi_2 = 0 \text{ a.e.} \Leftrightarrow d\left(\frac{\phi_1}{\phi_2}\right) = 0 \text{ a.e.}$$

meaning that ϕ_1 and ϕ_2 are proportional one another and they are equal unless f is a constant function at all points of M, i.e. unless $\tau, A \equiv 0$. □

4. Existence of solutions to the prescribed scalar curvature equation

Our focus in this section is Equation (1.1) with $A \equiv 0$, namely

$$-\frac{4(n-1)}{n-2}\Delta\phi + \mathrm{Scal}\,\phi = -f\phi^{N-1}, \tag{4.1}$$

where $f = \frac{n-1}{n}\tau^2 \geqslant 0$. This equation is the well-known prescribed scalar curvature equation (see e.g. [2] for an introduction). The aim of this section is to give a full proof of Theorem 4.1 with an argument that is simpler than the one in [19, 7], following the lines of [8]. One difficulty in the study of Equation (4.1) is to show that $\phi \not\equiv 0$ since $\phi \equiv 0$ is a trivial solution to (4.1). This is overcome by studying the asymptotics of ϕ in the non-compact case while here the argument has to be different. The theorm we prove is the following:

Theorem 4.1. *Let (M,g) be a compact Riemannian manifold with $g \in W^{2,p/2}$, $p > n$. Assume that $f \in L^p$, $f \geqslant 0$, $f \not\equiv 0$, is given. Then the following statements are equivalent:*

(1) There exists a positive solution $\phi \in W^{2,p}$ to (4.1),

(2) We have $\mathcal{Y}_g(M) < 0$ and the set $Z = f^{-1}(0)$ satisfies $\mathcal{Y}_g(Z) > 0$.

Further, the solution to (4.1), when it exists, is unique.

The proof of 1$\Rightarrow \mathcal{Y}_g(Z) > 0$ is entirely similar to the one given in the proof of Theorem 3.1 so we omit it. Note also that the metric $h := \phi^{N-2} g$ has scalar curvature $-f$ so

$$\mathcal{Y}_g(M) = \mathcal{Y}_h(M) \leqslant Q_h^Y(1) = \frac{G_h(1)}{\mathrm{Vol}_h(M)^{2/N}} < 0.$$

The proof of the converse implication will occupy the remaining of this note. We first prove it assuming that $f \in L^\infty$ and deduce the general case from this particular case.

We introduce the functional F defined for all $\phi \in W^{1,2}$ by

$$F(\phi) := \int_M \left[\frac{4(n-1)}{n-2} |d\phi|^2 + \mathrm{Scal}\, \phi^2 + \frac{2}{N} f |\phi|^N \right] d\mu^g \tag{4.2}$$

Note that the assumption that $f \in L^\infty$ is required in order to ensure that

$$I(\phi) = \int_M f |\phi|^N d\mu^g < \infty$$

for all $\phi \in W^{1,2}$. Note that $\phi \mapsto I(\phi)$ is continuous for the strong topology and convex since $f \geqslant 0$. In particular, it is weakly lower semi-continuous. From Proposition 2.1, we conclude that F is sequentially weakly lower semi-continuous.

We now show that F is coercive. This will imply the existence of a minimizer for F. The proof is similar (yet simpler) than the one given in [8, Proposition 4.8].

Lemma 4.2. *Assume that 2 in Theorem 4.1 is satisfied, then the functional F is coercive.*

Proof. We assume, by contradiction, that there exists a constant $B > 0$ and a sequence of elements $u_k \in W^{1,2}$ such that, for all k, $F(u_k) \leqslant B$ while $\|u_k\|_{W^{1,2}} \to \infty$.

We first remark that $F(|u_k|) = F(u_k)$ so, upon replacing u_k by $|u_k|$, we can suppose that $u_k \geqslant 0$. Let q be as in the proof of Proposition 2.1. We have

$$\begin{aligned} \frac{4(n-1)}{n-2} \|u_k\|^2_{W^{1,2}} &\leqslant F(u_k) + \int_M \left(\frac{4(n-1)}{n-2} + \mathrm{Scal} \right) u_k^2 d\mu^g \\ &\leqslant B + \left(\mathrm{Vol}_g(M)^{1-2/q} + \|\mathrm{Scal}\|_{L^p} \right) \|u_k\|^2_{L^q}. \end{aligned}$$

This proves that $\|u_k\|_{L^q} \to \infty$ and that $\|u_k\|_{W^{1,2}} \lesssim \|u_k\|_{L^q}$. We set $\gamma_k = \|u_k\|_{L^q}$ and $v_k := \gamma_k^{-1} u_k$ so that the sequence $(v_k)_k$ is bounded in $W^{1,2}$ and satisfies $\|v_k\|_{L^q} = 1$. We can assume, without loss of generality, that v_k converges weakly in $W^{1,2}$ and strongly in L^q to some $v \in W^{1,2}$. Since $\|v\|_{L^q} = 1$, we have $v \not\equiv 0$.

We now claim that $v \in \mathcal{F}(Z)$. Indeed, we have

$$B \geqslant F(u_k) = \gamma_k^2 G(v_k) + \gamma_k^N \int_M f v_k^N d\mu^g = \gamma_k^N \left(\int_M f v_k^N d\mu^g + o(1) \right). \tag{4.3}$$

If we were able to prove that

$$\int_M f v_k^N d\mu^g \to \int_M f v^N d\mu^g, \tag{4.4}$$

we would immediately conclude that

$$\int_M f v^N d\mu^g = 0.$$

Yet, convergence of $(v_k)_k$ to v is so weak that proving that (4.4) (if true) holds is delicate. We bypass this issue by the following argument. Assume, by contradiction, that $v \notin \mathcal{F}(Z)$, then there exist a set $W \subset M \backslash Z$ with positive measure and an $\epsilon > 0$ such that $v \geqslant \epsilon \mathbb{1}_W$ a.e. (here $\mathbb{1}_W$ is the indicator function of W). Then,

$$\int_M f v^q d\mu^g \geqslant \epsilon^q \int_M f \mathbb{1}_W^q d\mu^g = \epsilon^q \int_W f d\mu^g > 0.$$

As a consequence, we have, for k large enough,

$$\int_M f v_k^q d\mu^g \geqslant \frac{\epsilon^q}{2} \int_W f d\mu^g.$$

From Hölder's inequality, we have

$$\left(\int_M f v_k^N d\mu^g\right)^{q/N} \left(\int_M f d\mu^g\right)^{1-q/N} \geqslant \int_M f v_k^q d\mu^g \geqslant \frac{\epsilon^q}{2} \int_W f d\mu^g.$$

This shows that $\int_M f v_k^N d\mu^g$ is bounded from below by a positive constant. This yields a contradiction with (4.3). As a consequence, we have $v \in \mathcal{F}(Z)$.

Due to our assumption on Z, we have $G_g(v) \geqslant \mathcal{Y}_g(V) \|v\|_{L^N}^{2/N} > 0$. So

$$\liminf_{k \to \infty} G_g(v_k) \geqslant G_g(v) > 0.$$

In particular, we have

$$\liminf_{k \to \infty} F(u_k) \geqslant \liminf_{k \to \infty} G_g(u_k) = \liminf_{k \to \infty} \gamma_k^2 G_g(v_k) = \infty.$$

This contradicts the assumption $F(u_k) \leqslant B$. □

We have now all the ingredients to conclude that F admits a minimizer ϕ. Since $F(|\phi|) = F(\phi)$, we can assume, without loss of generality, that $\phi \geqslant 0$. ϕ is then a solution in a weak sense to (4.1). By elliptic regularity, we conclude that $\phi \in W^{2,p}$ and by Harnack's inequality that $\phi > 0$ provided $\phi \not\equiv 0$.

We rule out the possibility that $\phi \equiv 0$ as follows. Since $\mathcal{Y}_g(M) < 0$, there exists $w \in W^{1,2}$ such that $G_g(w) < 0$. For any $\lambda > 0$ we have

$$F(\lambda w) = \lambda^2 G_g(w) + \lambda^N I(w).$$

In particular, if λ is small enough we have $F(\lambda w) < 0$. This shows that the zero function is not a global mimimum of F. This forces $\phi \not\equiv 0$.

Uniqueness of ϕ is obtained by applying Proposition 3.3.

We now need to get rid of the assumption $f \in L^\infty$. For all $k > 0$, we set $f_k := \min\{f, k\} \in L^\infty$. Let ϕ_k denote the solution to (4.1) with f replaced by f_k. Note that the zero set of f_k is the same as that of f so the preceding construction applies. It follows from the maximum principle that $\phi_{k+1} \leqslant \phi_k$ for all $k > 0$ (the argument is similar to the one in the proof of Proposition 3.3). Since $\phi_1 \in W^{2,p} \subset L^\infty$, the sequence $f_k \phi_k^{N-1}$ is uniformly bounded in L^p. Hence, from elliptic regularity, the sequence $(\phi_k)_k$ is bounded in $W^{2,p}$. By the compactness of the embedding $W^{2,p} \hookrightarrow L^\infty$ together with elliptic regularity, there exists a subsequence $(\phi_{\theta(k)})_k$ of $(\phi_k)_k$ that converges to some $\phi \in W^{2,p}$, $\phi \geqslant 0$ solving (4.1). Note that, from Dini's theorem, $(\phi_k)_k$ converges in L^∞ to ϕ. All we need to do is to exclude that $\phi \equiv 0$.

This can be done as follows. Let $w \in W^{1,2}$ be, as before, such that $G_g(w) < 0$. Since $W^{2,p}$ is dense in $W^{1,2}$, we can assume that $w \in W^{2,p} \subset L^\infty$. As before, considering $u = \lambda w$ in the

functional (4.2), we get existence of v such that $F(v) < 0$. Set

$$F_k(u) := \int_M \left[\frac{4(n-1)}{n-2} |d\phi|^2 + \mathrm{Scal}\, \phi^2 + \frac{2}{N} f_k |\phi|^N \right] d\mu^g$$

So we have $F_k(\phi_k) \leqslant F_k(v) \leqslant F(v) < 0$. Now remark that $F_k(\phi_k) \to_{k\to\infty} F(\phi)$. This forces $F(\phi) \leqslant F(v) < 0$ which shows that $\phi \not\equiv 0$. By construction $\phi \geqslant 0$ and from Harnack's inequality, we have $\phi > 0$. This ends the proof of Theorem 4.1. Uniqueness is obtained from Proposition 3.3.

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