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# EXISTENCE OF SOLUTIONS TO THE LICHNEROWICZ EQUATION: A NEW PROOF

ROMAIN GICQUAUD

ABSTRACT. We provide a complete study of existence and uniqueness of solutions to the Lichnerowicz equation in general relativity with arbitrary mean curvature.

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## 1. INTRODUCTION

The Lichnerowicz equation is an elliptic equation that appears in the construction of initial data in general relativity. In the setting of this note, let  $(M, g)$  be a compact Riemannian manifold of dimension  $n > 2$ ,  $g \in W^{2,p}$ ,  $p > n/2$ , and assume given two functions  $\tau \in L^{2p}$  and  $A \in L^{2p}$ . The Lichnerowicz equation has a positive function  $\phi$  as unknown and reads

$$(1.1) \quad -\frac{4(n-1)}{n-2} \Delta \phi + \text{Scal } \phi = -\frac{n-1}{n} \tau^2 \phi^{N-1} + \frac{A^2}{\phi^{N+1}},$$

where  $\text{Scal}$  is the scalar curvature of  $g$  and  $N := \frac{2n}{n-2}$ .

We refer the reader to [3, 5] for an overview of the context in which this equation appears. It has attracted attention a couple of decades ago culminating at the classification of constant mean curvature initial data by J. Isenberg in [14]. Recently, important efforts have been put in constructing non-constant mean curvature initial data, see [12, 13, 16, 6] and [10].

The main aim of this note is to give a short proof of existence/non-existence of solutions to (1.1) in the generic case  $A \neq 0$ . This result is well-known to a large extent, see e.g. [16, Theorem 1]. The main novelty here is that there is no need to give separate proofs according to the sign of the Yamabe quotient of  $(M, g)$ . The particular case  $A \equiv 0$  is the prescribed scalar curvature equation which is similar to the problem addressed in [19, 17, 18, 20], see also [7, 8]. We will study it in Section 4.

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This paper is a byproduct of the techniques developed in [7, 8].

The outline of this paper is as follows. In Section 2, we introduce the main tool to discriminate which function  $\tau$  lead to existence of solutions to (1.1). In Section 3, we study the case  $A \neq 0$ . The main result of this section is Theorem 3.1 which is the main result of the paper. Section 4 is devoted to the case  $A \equiv 0$  which, as we indicated before, deserves a particular treatment.

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## 2. LOCAL YAMABE INVARIANT AND FIRST CONFORMAL EIGENVALUE

For any measurable subset  $V \subset M$ , we define the space

$$(2.1) \quad \mathcal{F}(V) := \{u \in W^{1,2}, u \equiv 0 \text{ a.e. on } M \setminus V\}$$

of Sobolev functions vanishing outside  $V$ . This set is obviously reduced to  $\{0\}$  if  $V$  has Lebesgue measure zero but there are larger  $V$  with  $\mathcal{F}(V) = \{0\}$ , see for example [1, Chapter 6]. Much of this section is adapted from [8].

For any  $u \in W^{1,2}$ , we set

$$(2.2) \quad G_g(u) := \int_M \left[ \frac{4(n-1)}{n-2} |du|^2 + \text{Scal } u^2 \right] d\mu^g$$

We also introduce, for any  $u \in W^{1,2}$ ,  $u \neq 0$ , the Rayleigh and the Yamabe quotients:

$$Q_g^R(u) := G(u) / \|u\|_{L^2}^2,$$

$$Q_g^Y(u) := G(u) / \|u\|_{L^N}^2.$$

With these definitions at hand, we introduce the local first conformal eigenvalue  $\lambda_g(V)$  and the local Yamabe invariant  $\mathcal{Y}_g(V)$  of any measurable subset  $V \subset M$  as follows:

$$\lambda_g(V) := \inf_{u \in \mathcal{F}(V) \setminus \{0\}} Q_g^R(u),$$

$$\mathcal{Y}_g(V) := \inf_{u \in \mathcal{F}(V) \setminus \{0\}} Q_g^Y(u).$$

From the definition of an infimum, we have  $\lambda_g(V) = \mathcal{Y}_g(V) = \infty$  if  $\mathcal{F}(V)$  is reduced to  $\{0\}$ .

**Proposition 2.1.** *The functional  $G$  defined in (2.2) is sequentially weakly lower semi-continuous on  $W^{1,2}$ : for every weakly converging sequence  $(u_k)_k$ ,  $u_k \rightharpoonup u_\infty$ , we have  $\liminf_{k \rightarrow \infty} G(u_k) \geq G(u_\infty)$ .*

*Proof.* Note that  $G_g$  can be decomposed as

$$(2.5) \quad G_g(u) = \frac{4(n-1)}{n-2} \int_M |du|^2 d\mu^g + \int_M \text{Scal } u^2 d\mu^g.$$

The first term is weakly lower semi-continuous with respect to  $u \in W^{1,2}$  as a continuous non-negative quadratic form. For the second one, we shall prove that, given a sequence  $(u_k)_k$  in  $W^{1,2}$ , converging weakly to  $u_\infty$ ,  $u_k \rightharpoonup_{k \rightarrow \infty} u_\infty$ , we have

$$\int_M \text{Scal } u_k^2 d\mu^g \rightarrow \int_M \text{Scal } u_\infty^2 d\mu^g.$$

To make the notation less cluttered, we denote the second term in (2.5) as  $S(u)$ :

$$S(u) := \int_M \text{Scal } u^2 d\mu^g.$$

Assume by contradiction that  $(S(u_k))_k$  does not converge to  $S(u_\infty)$ , there exists an  $\epsilon > 0$  such that, for an infinite number of integers  $k$ , we have

$$(2.6) \quad |S(u_k) - S(u_\infty)| > \epsilon.$$

Without loss of generality, we can assume that (2.6) holds for all integer  $k$  and also that  $(u_k)_k$  converges strongly in  $L^2$  to some  $\bar{u}_\infty \in L^2$  since the embedding  $W^{1,2} \hookrightarrow L^2$  is compact. Then we have  $u_\infty \equiv \bar{u}_\infty$  a.e. Indeed, the linear form

$$u \mapsto \int_M u(u_\infty - \bar{u}_\infty) d\mu^g$$

is (strongly) continuous for the  $L^2$ -topology and, hence, for the  $W^{1,2}$ -topology. As a consequence,

$$\int_M u_\infty(u_\infty - \bar{u}_\infty) d\mu^g = \lim_{k \rightarrow \infty} \int_M u_k(u_\infty - \bar{u}_\infty) d\mu^g = \int_M \bar{u}_\infty(u_\infty - \bar{u}_\infty) d\mu^g,$$

where the first equality holds by the  $W^{1,2}$ -weak convergence of  $(u_k)_k$  to  $u_\infty$  and the second one by the  $L^2$ -strong convergence of  $(u_k)_k$  to  $\bar{u}_\infty$ . Subtracting both equalities, we get

$$\int_M |u_\infty - \bar{u}_\infty|^2 d\mu^g = 0,$$

which proves that  $u_\infty \equiv \bar{u}_\infty$  a.e. Finally note that, since  $(u_k)_k$  is weakly convergent in  $W^{1,2}$ , it is bounded and thus (by interpolation) converges in all  $L^q$  spaces,  $q \in [2, N)$ . Since  $\text{Scal} \in L^p$ ,  $p > n/2$ , letting  $q$  be such that  $1 = \frac{1}{p} + \frac{2}{q}$ , we have  $q \in [2, N)$  and, by Hölder's inequality,  $S$  is a bounded quadratic form on  $L^q$ . In particular  $S$  is continuous on  $L^q$ :

$$S(u_k) \rightarrow S(u_\infty).$$

This contradicts (2.5):  $S$  is sequentially weakly continuous on  $W^{1,2}$ . This ends the proof of Proposition 2.1.  $\square$

In what follows, we let  $s > 0$  be the largest constant so that

$$(2.7) \quad \|u\|_{W^{1,2}}^2 \geq s \|u\|_{L^N}^2 \quad \forall u \in W^{1,2}.$$

**Proposition 2.2.** *Given any measurable set  $V \subset M$ ,  $\lambda_g(V)$  and  $\mathcal{Y}_g(V)$  have the same sign (i.e. they are either both positive, both negative or both zero).*

*Proof.* We can assume, without loss of generality, that  $\mathcal{F}(V) \neq \{0\}$  for otherwise  $\mathcal{Y}_g(V) = \lambda_g(V) = \infty$ . If  $\mathcal{Y}_g(V) < 0$ , there exists  $u \in \mathcal{F}(V)$  such that  $G_g(u) < 0$  so  $\lambda_g(V) < 0$ . Assume now that  $\mathcal{Y}_g(V) > 0$ , then, for all  $u \in \mathcal{F}(V) \setminus \{0\}$ , we have

$$Q_g^R(u) = \frac{G_g(u)}{\|u\|_{L^2}^2} \geq \frac{G_g(u)}{\|u\|_{L^N}^2 \text{Vol}_g(V)^{2/n}} \geq \frac{\mathcal{Y}_g(V)}{\text{Vol}_g(V)^{2/n}}.$$

We conclude that

$$\lambda_g(V) \geq \frac{\mathcal{Y}_g(V)}{\text{Vol}_g(V)^{2/n}} > 0.$$

All we have to show now is that, if  $\mathcal{Y}_g(V) = 0$ , we have  $\lambda_g(V) = 0$ . Assume for the rest of the proof that  $\mathcal{Y}_g(V) = 0$ . If  $\lambda_g(V)$  were negative, there would exist  $u \in \mathcal{F}(V)$  such that  $G_g(u) < 0$  so  $\mathcal{Y}_g(V) \leq Q_g^Y(u) < 0$ . This proves that  $\lambda_g(V) \geq 0$ . Since  $\mathcal{Y}_g(V) = 0$ , there exists a sequence

of functions  $u_k \in \mathcal{F}(V)$  such that  $Q_g^Y(u_k) \rightarrow 0$ . Without loss of generality, we can assume that  $\|u_k\|_{L^N} = 1$  so  $G_g(u_k) \rightarrow 0$ .

Let  $q$  be as in the proof of the previous proposition. Then we have that

$$\begin{aligned} G(u_k) &\geq \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2 - \frac{4(n-1)}{n-2} \|u_k\|_{L^2}^2 - \|\text{Scal}\|_{L^p} \|u_k\|_{L^q}^2 \\ &\geq \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2 - \frac{4(n-1)}{n-2} \text{Vol}_g(V)^{1-2/q} \|u_k\|_{L^q}^2 - \|\text{Scal}\|_{L^p} \|u_k\|_{L^q}^2. \end{aligned}$$

Hence, setting  $C = \frac{4(n-1)}{n-2} \text{Vol}_g(V)^{1-2/q} + \|\text{Scal}\|_{L^p}$ , we arrive at

$$(2.8) \quad G_g(u_k) + C \|u_k\|_{L^q}^2 \geq \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2.$$

Since  $q < N$ , we have that  $\|u_k\|_{W^{1,2}}$  is bounded independently of  $k$ . Arguing as in the proof of the previous proposition, we can assume that  $(u_k)_k$  converges weakly in  $W^{1,2}$  and strongly in  $L^2$  to some  $u_\infty \in \mathcal{F}(V)$ . Combining Equation (2.8) with the Sobolev estimate (2.7), we get

$$G_g(u_k) + C \|u_k\|_{L^q}^2 \geq \frac{4(n-1)}{n-2} s \|u_k\|_{L^N}^2 = \frac{4(n-1)}{n-2} s.$$

Passing to the limit as  $k$  goes to infinity, we conclude that  $\|u_\infty\|_{L^q} > 0$ , i.e.  $u_\infty \not\equiv 0$ . By the lower semicontinuity of  $G_g$ , we have  $G_g(u_\infty) \leq \liminf_{k \rightarrow \infty} G_g(u_k) = 0$ . Since  $G_g(u_\infty) \geq 0$ , we have  $G_g(u_\infty) = 0$ . We have proven that

$$0 \leq \lambda_g(V) \leq Q_g^R(u_\infty) = 0,$$

i.e.  $\lambda_g(V) = 0$ . This concludes the proof of the fact that  $\mathcal{Y}_g(V)$  and  $\lambda_g(V)$  have the same sign.  $\square$

The reason why it is more convenient to work with  $\mathcal{Y}_g(V)$  than with  $\lambda_g(V)$  is given by the following proposition.

**Proposition 2.3.** *Assume that  $g$  and  $h$  are two conformally related metrics,  $h = \phi^{N-2}g$ , for some positive function  $\phi \in W^{2,p}$ . Then for any measurable  $V$  we have*

$$\mathcal{Y}_g(V) = \mathcal{Y}_h(V).$$

*Proof.* The proof is a simple calculation. Given any  $u \in W^{1,2}$ , we have

$$\begin{aligned}
G_h(u) &= \int_M \left[ \frac{4(n-1)}{n-2} |du|_h^2 + \text{Scal}^h u^2 \right] d\mu^h \\
&= \int_M \left[ \frac{4(n-1)}{n-2} \phi^{2-N} |du|_g^2 + \left( -\frac{4(n-1)}{n-2} \Delta^g \phi + \text{Scal}^g \phi \right) \phi^{1-N} u^2 \right] \phi^N d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} \phi^2 |du|_g^2 + \left( -\frac{4(n-1)}{n-2} \Delta^g \phi + \text{Scal}^g \phi \right) \phi u^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} (\phi^2 |du|_g^2 - (\phi \Delta^g \phi) u^2) + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} (\phi^2 |du|_g^2 + \langle d\phi, d(\phi u^2) \rangle_g) + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} (\phi^2 |du|_g^2 + u^2 |d\phi|_g^2 + 2\langle \phi d\phi, u du \rangle_g) + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} |d(\phi u)|_g^2 + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= G_g(\phi u).
\end{aligned}$$

Similarly,

$$\|u\|_{L_h^N} = \left( \int_M u^N d\mu^h \right)^{1/N} = \left( \int_M u^N \phi^N d\mu^g \right)^{1/N} = \|\phi u\|_{L_g^N}.$$

So

$$Q_h^{\mathcal{Y}}(u) = Q_g^{\mathcal{Y}}(\phi u).$$

Since  $\phi$  is bounded away from zero, multiplication by  $\phi$  defines an automorphism of  $\mathcal{F}(V)$ . Hence,

$$\mathcal{Y}_g(V) = \inf_{u \in \mathcal{F}(V)} Q_g^{\mathcal{Y}}(u) = \inf_{u \in \mathcal{F}(V)} Q_g^{\mathcal{Y}}(\phi u) = \inf_{u \in \mathcal{F}(V)} Q_h^{\mathcal{Y}}(u) = \mathcal{Y}_h(V).$$

□

### 3. EXISTENCE OF SOLUTIONS TO THE LICHNEROWICZ EQUATION

**Theorem 3.1.** *Let  $(M, g)$  be a compact Riemannian manifold with  $g \in W^{2,p}$ ,  $p > n/2$ . Assume that  $\tau \in L^{2p}$  is given. Then the following statements are equivalent:*

- (1) *There exists a solution to (1.1) for all  $A \in L^{2p}$ ,  $A \neq 0$*
- (2) *There exists a solution to (1.1) for at least one  $A \in L^{2p}$ ,  $A \neq 0$ ,*
- (3) *The set  $Z = \tau^{-1}(0)$  satisfies  $\mathcal{Y}_g(Z) > 0$ .*

*Further, the solution to (1.1), when it exists, is unique unless  $\mathcal{Y}_g(M) = 0$  and  $\tau, A \equiv 0$  for which all solutions are proportional one to another.*

It should be noted that the theorem can be applied in particular when  $Z$  has zero Lebesgue measure. This is the case if  $\tau$  never vanishes or if 0 is a regular value for  $\tau$ .

This theorem reproduces results from [12, 13, 15, 16] and references therein (see also [9]) in which several proofs are given according to the sign of  $\mathcal{Y}_g(M)$  and the nullity of  $\tau$ . The main novelty is that the proof establishes a direct link between existence of solutions to the Lichnerowicz equation and the fact that  $\mathcal{Y}_g(Z) > 0$ . We first state a lemma:

**Lemma 3.2.** *Under the assumptions of the theorem, if  $\mathcal{Y}_g(Z) > 0$ , there exists a constant  $K > 0$  such that the operator*

$$u \mapsto -\frac{4(n-1)}{n-2}\Delta u + \text{Scal } u + K\frac{n-1}{n}\tau^2 u$$

*has positive first eigenvalue.*

*Proof.* Assume by contradiction that for all  $k \in \mathbb{N}$ , the first eigenvalue of

$$L_k : u \mapsto -\frac{4(n-1)}{n-2}\Delta u + \text{Scal } u + k\frac{n-1}{n}\tau^2 u$$

is non-positive. We denote it by  $\lambda_k$  and let  $u_k \in W^{2,p/2}$  be the first eigenfunction normalized so that  $u_k \geq 0$  and  $\|u_k\|_{L^2} = 1$ . The sequence  $(\lambda_k)_k$  is increasing since

$$\begin{aligned} \lambda_{k+1} &= \int_M u_{k+1} L_{k+1} u_{k+1} d\mu^g \\ &= \int_M u_{k+1} L_k u_{k+1} d\mu^g + \int_M \frac{n-1}{n} \tau^2 u_{k+1}^2 \\ &\geq \int_M u_{k+1} L_k u_{k+1} d\mu^g \\ &\geq \lambda_k. \end{aligned}$$

We claim that the sequence  $(u_k)_k$  is bounded in  $W^{1,2}$ . Indeed, we have, using the Hölder inequality:

$$\begin{aligned} 0 &\geq \int_M \left[ \frac{4(n-1)}{n-2} |du_k|^2 + \text{Scal } u_k^2 \right] d\mu^g \\ &\geq \frac{4(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - \|\text{Scal}\|_{L^p} \|u_k\|_{L^N}^{\frac{n}{p}} \|u_k\|_{L^2}^{2-\frac{n}{p}} \\ &\geq \frac{4(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - 2\|\text{Scal}\|_{L^p} \left[ \frac{n}{p} \epsilon \|u_k\|_{L^N}^2 + \frac{2p-n}{2p} \|u_k\|_{L^2}^2 \epsilon^{-n/(2p-n)} \right] \\ &\geq \frac{4(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - \|\text{Scal}\|_{L^p} \left[ \frac{2n}{sp} \epsilon \|u_k\|_{W^{1,2}}^2 + \frac{2p-n}{p} \|u_k\|_{L^2}^2 \epsilon^{-n/(2p-n)} \right], \end{aligned}$$

where we used the  $\epsilon$ -Young inequality and the Sobolev inequality (2.7). Assuming that  $\text{Scal} \neq 0$  (if  $\text{Scal} \equiv 0$  the argument is simpler), we choose  $\epsilon$  such that

$$\|\text{Scal}\|_{L^p} \frac{2n}{sp} \epsilon = \frac{2(n-1)}{n-2},$$

so

$$0 \geq \frac{2(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - C \|u_k\|_{L^2}^2,$$

for some explicit constant  $C = C(n, s, p, \|\text{Scal}\|_{L^p})$ . Since  $\|u_k\|_{L^2} = 1$ , this proves the claim that  $(u_k)_k$  is bounded in  $L^2$ .

From Rellich theorem, we now extract a subsequence  $(k_i)_i$  of  $k$  such that

$$u_{k_i} \rightarrow u_\infty \text{ in } L^2$$

for some  $u_\infty \in W^{1,2}$ . In particular,  $\|u_\infty\|_{L^2} = 1$ . We can also assume that

$$u_{k_i} \rightharpoonup u_\infty \text{ in } W^{1,2}.$$

We claim that  $u_\infty \equiv 0$  a.e. on  $M \setminus Z$ . Otherwise,

$$\int_M \tau^2 u_{k_i}^2 d\mu^g \rightarrow \int_M \tau^2 u_\infty^2 d\mu^g \neq 0,$$

so

$$\begin{aligned} \lambda_{k_i} &= \int_M u_{k_i} L_{k_i} u_{k_i} d\mu^g \\ &= \int_M u_{k_i} L_0 u_{k_i} d\mu^g + k_i \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 d\mu^g \\ &\geq \lambda_0 + k_i \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 d\mu^g \\ &\rightarrow_{i \rightarrow \infty} \infty, \end{aligned}$$

contradicting the fact that  $(\lambda_k)_k$  is bounded. Since  $\|u_\infty\|_{L^2} = 1$  and belongs to  $\mathcal{F}(Z)$ , we have a contradiction if  $\mathcal{F}(Z) = \{0\}$ . In the case where  $\mathcal{F}(Z) \neq \{0\}$ , we also get a contradiction since

$$\lambda_{k_i} = G_g(u_{k_i}) + k \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 \geq G_g(u_{k_i}),$$

so, since  $G_g$  is weakly lower semicontinuous,

$$\liminf_{i \rightarrow \infty} \lambda_{k_i} \geq \liminf_{i \rightarrow \infty} G_g(u_{k_i}) \geq G_g(u_\infty) \geq \lambda_g(Z) > 0.$$

This gives the final contradiction.  $\square$

*Proof of Theorem 3.1.* The statement  $1 \Rightarrow 2$  is obvious. We now prove that  $2 \Rightarrow 3$ . The proof is similar to that of Proposition 2.3. If  $\mathcal{F}(Z) = \{0\}$ , Statement 3 is satisfied since  $\mathcal{Y}_g(Z) = \infty$ . Otherwise, assume given  $A \in L^{2p}$  and  $\phi \in W^{2,p}$  satisfying (1.1). We set  $\hat{g} = \phi^{N-2}g$  and  $\hat{u} = u\phi^{-1}$ . For all  $u \in \mathcal{F}(Z)$ , we have

$$\begin{aligned} G_g(u) &= G_g(\phi\hat{u}) \\ &= \int_M \left[ \frac{4(n-1)}{n-2} \left( \phi^2 |d\hat{u}|_g^2 + \langle \phi d\phi, d(\hat{u}^2) \rangle_g + \hat{u}^2 |d\phi|_g^2 \right) + \text{Scal } \phi^2 \hat{u}^2 \right] d\mu^g \\ &= \int_M \left[ \frac{4(n-1)}{n-2} \left( \phi^2 |d\hat{u}|_g^2 - (\phi\Delta\phi)\hat{u}^2 \right) + \text{Scal } \phi^2 \hat{u}^2 \right] d\mu^g \\ &= \int_M \left[ \frac{4(n-1)}{n-2} \phi^2 |d\hat{u}|_g^2 + \left( \frac{A^2}{\phi^N} - \frac{n-1}{n} \tau^2 \phi^N \right) \hat{u}^2 \right] d\mu^g \\ &= \int_M \left[ \frac{4(n-1)}{n-2} |d\hat{u}|_{\hat{g}}^2 + \left( \frac{A^2}{\phi^{2N}} - \frac{n-1}{n} \tau^2 \right) \hat{u}^2 \right] d\mu^{\hat{g}} \\ &\geq \int_M \left[ \frac{4(n-1)}{n-2} |d\hat{u}|_{\hat{g}}^2 + \frac{A^2}{\phi^{2N}} \hat{u}^2 \right] d\mu^{\hat{g}} \quad (\text{since } \hat{u} \in \mathcal{F}(Z)). \end{aligned}$$

This immediately rules out the possibility that  $\mathcal{Y}_g(Z) < 0$  since  $G_g(u) \geq 0$  for all  $u \in \mathcal{F}(Z)$ . Assume next that  $M \setminus Z$  has positive Lebesgue measure. Then,  $\hat{u} \equiv 0$  on  $M \setminus Z$ . As a consequence, from the Poincaré inequality, there is a constant  $\mu = \mu(g, \tau)$  so that

$$G_g(u) \geq \frac{4(n-1)}{n-2} \int_M |d\hat{u}|_{\hat{g}}^2 d\mu^{\hat{g}} \geq \mu \|\hat{u}\|_{W^{1,2}}^2,$$

(see e.g. [11, Lemma 7.16]) and, hence, from the Sobolev embedding theorem,

$$G_g(u) \geq s\mu \|\hat{u}\|_{L^N}^2.$$



This proves that

$$\mathcal{Y}_g(Z) = s\mu > 0.$$

The only remaining possibility is that  $\tau \equiv 0$  a.e. that is to say  $Z = M$  and  $\mathcal{Y}_g(M) = 0$ . From the proof of Proposition 2.2, there exists a function  $u_\infty \geq 0$ ,  $u_\infty \not\equiv 0$  so that  $G_g(u_\infty) = 0$ . From the inequality

$$G_g(u_\infty) \geq \frac{4(n-1)}{n-2} \int_M |d\hat{u}_\infty|_g^2 d\mu^{\hat{g}},$$

we have  $d\hat{u}_\infty \equiv 0$ :  $\hat{u}_\infty$  is a constant function. This gives a contradiction since

$$0 = G_g(u_\infty) = \int_M \frac{A^2}{\phi^{2N}} \hat{u}_\infty^2 d\mu^{\hat{g}} > 0.$$

We finally prove that 3  $\Rightarrow$  1. The proof goes as usual by the sub- and super-solution method (see e.g. [21, Chapter 14]). Let  $K$  be as in the statement of Lemma 3.2. We let  $u$  denote the solution to

$$(3.1) \quad -\frac{4(n-1)}{n-2} \Delta u + \text{Scal } u + K \frac{n-1}{n} \tau^2 u = A^2.$$

Since the operator on the left hand side is positive, its Green function is positive, so  $u \in W^{2,p}$  is also positive (note that  $u$  is Hölder continuous). We set

$$\begin{cases} u_+ = \lambda_+ u, \\ u_- = \lambda_- u \end{cases}$$

for some positive constants  $\lambda_\pm$  to be chosen later. We want  $u_+$  to be a super-solution to the Lichnerowicz equation (1.1), i.e.  $u_+$  has to satisfy

$$-\frac{4(n-1)}{n-2} \Delta u_+ + \text{Scal } u_+ + \frac{n-1}{n} \tau^2 u_+^{N-1} \geq \frac{A^2}{u_+^{N+1}}.$$

From Equation (3.1), this is equivalent to

$$\frac{n-1}{n} \tau^2 (\lambda_+^{N-1} u^{N-1} - K \lambda_+ u) + \lambda_+ A^2 \geq \frac{A^2}{\lambda_+^{N+1} u^{N+1}}.$$

This inequality holds true if both the following inequalities are fulfilled:

$$\begin{cases} \lambda_+^{N-2} u^{N-2} \geq K, \\ \lambda_+^{N+2} \geq u^{-N-1}. \end{cases}$$

Since  $u$  is bounded from above and away from zero, they are true for large enough  $\lambda_+$ . Calculations for the sub-solution are similar: if  $\lambda_-$  is a small enough positive constant  $u_-$  is a sub-solution to the Lichnerowicz equation (1.1). By the sub- and super-solution argument, we get existence of  $u \in W^{2,p}$  solving (1.1). Uniqueness of  $u$  will be proven in the next proposition.  $\square$

**Proposition 3.3.** *Let  $(M, g)$  be a compact Riemannian manifold with  $g \in W^{2,p}$ ,  $p > n/2$ . Let  $\tau, A \in L^{2p}$  be two given functions. Assume given two positive functions  $\phi_1, \phi_2 \in W^{2,p}$  solving the Lichnerowicz equation (1.1).*

- If  $\tau \not\equiv 0$  or  $A \not\equiv 0$ , we have  $\phi_1 \equiv \phi_2$ ,
- If  $\tau, A \equiv 0$ ,  $\phi_1$  and  $\phi_2$  are proportional.

*Proof.* The proof of this fact is well known, we refer the reader e.g. to [16, Proposition 2] or to [6]. We present here the argument from [4]. Since  $\phi_1$  and  $\phi_2$  are bounded from below, we have that  $\phi_1^2/\phi_2$  and  $\phi_2^2/\phi_1$  both belong to  $W^{2,p}$ . By an integration by parts and some routine calculations, we have

$$(3.2) \quad \begin{aligned} & \int_M \left( -\frac{\Delta\phi_1}{\phi_1} + \frac{\Delta\phi_2}{\phi_2} \right) (\phi_1^2 - \phi_2^2) d\mu^g \\ &= \int_M \left| d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 \right|^2 d\mu^g + \int_M \left| d\phi_2 - \frac{\phi_2}{\phi_1} d\phi_1 \right|^2 d\mu^g. \end{aligned}$$

If we set

$$f(\phi) := \frac{n-2}{4(n-1)} \left[ \frac{A^2}{\phi^{N+2}} - \text{Scal} - \frac{n-1}{n} \tau^2 \phi^{N-2} \right],$$

we have

$$-\frac{\Delta\phi_1}{\phi_1} = f(\phi_1) \quad \text{and} \quad -\frac{\Delta\phi_2}{\phi_2} = f(\phi_2),$$

so the identity (3.2) gives

$$\begin{aligned} & \int_M [f(\phi_1) - f(\phi_2)] (\phi_1^2 - \phi_2^2) d\mu^g \\ &= \int_M \left| d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 \right|^2 d\mu^g + \int_M \left| d\phi_2 - \frac{\phi_2}{\phi_1} d\phi_1 \right|^2 d\mu^g. \end{aligned}$$

Since  $f$  is a decreasing function, we have

$$[f(\phi_1) - f(\phi_2)] (\phi_1^2 - \phi_2^2) \leq 0 \text{ a.e.}$$

This impose that

$$\int_M \left| d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 \right|^2 d\mu^g + \int_M \left| d\phi_2 - \frac{\phi_2}{\phi_1} d\phi_1 \right|^2 d\mu^g = 0.$$

In particular, we have

$$d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 = 0 \text{ a.e.} \Leftrightarrow d\left(\frac{\phi_1}{\phi_2}\right) = 0 \text{ a.e.}$$

meaning that  $\phi_1$  and  $\phi_2$  are proportional one another and they are equal unless  $f$  is a constant function at all points of  $M$ , i.e. unless  $\tau, A \equiv 0$ .  $\square$

#### 4. EXISTENCE OF SOLUTIONS TO THE PRESCRIBED SCALAR CURVATURE EQUATION

Our focus in this section is Equation (1.1) with  $A \equiv 0$ , namely

$$(4.1) \quad -\frac{4(n-1)}{n-2} \Delta\phi + \text{Scal} \phi = -f\phi^{N-1},$$

where  $f = \frac{n-1}{n} \tau^2 \geq 0$ . This equation is the well-known prescribed scalar curvature equation (see e.g. [2] for an introduction). The aim of this section is to give a full proof of Theorem 4.1 with an argument that is simpler than the one in [19, 7], following the lines of [8]. One difficulty in the study of Equation (4.1) is to show that  $\phi \not\equiv 0$  since  $\phi \equiv 0$  is a trivial solution to (4.1). This is overcome by studying the asymptotics of  $\phi$  in the non-compact case while here the argument has to be different. The theorem we prove is the following:

**Theorem 4.1.** *Let  $(M, g)$  be a compact Riemannian manifold with  $g \in W^{2,p/2}$ ,  $p > n$ . Assume that  $f \in L^p$ ,  $f \geq 0$ ,  $f \not\equiv 0$ , is given. Then the following statements are equivalent:*

- (1) *There exists a positive solution  $\phi \in W^{2,p}$  to (4.1),*

(2) We have  $\mathcal{Y}_g(M) < 0$  and the set  $Z = f^{-1}(0)$  satisfies  $\mathcal{Y}_g(Z) > 0$ .

Further, the solution to (4.1), when it exists, is unique.

The proof of  $1 \Rightarrow \mathcal{Y}_g(Z) > 0$  is entirely similar to the one given in the proof of Theorem 3.1 so we omit it. Note also that the metric  $h := \phi^{N-2}g$  has scalar curvature  $-f$  so

$$\mathcal{Y}_g(M) = \mathcal{Y}_h(M) \leq Q_h^Y(1) = \frac{G_h(1)}{\text{Vol}_h(M)^{2/N}} < 0.$$

The proof of the converse implication will occupy the remaining of this note. We first prove it assuming that  $f \in L^\infty$  and deduce the general case from this particular case.

We introduce the functional  $F$  defined for all  $\phi \in W^{1,2}$  by

$$(4.2) \quad F(\phi) := \int_M \left[ \frac{4(n-1)}{n-2} |d\phi|^2 + \text{Scal} \phi^2 + \frac{2}{N} f |\phi|^N \right] d\mu^g$$

Note that the assumption that  $f \in L^\infty$  is required in order to ensure that

$$I(\phi) = \int_M f |\phi|^N d\mu^g < \infty$$

for all  $\phi \in W^{1,2}$ . Note that  $\phi \mapsto I(\phi)$  is continuous for the strong topology and convex since  $f \geq 0$ . In particular, it is weakly lower semi-continuous. From Proposition 2.1, we conclude that  $F$  is sequentially weakly lower semi-continuous.

We now show that  $F$  is coercive. This will imply the existence of a minimizer for  $F$ . The proof is similar (yet simpler) than the one given in [8, Proposition 4.8].

**Lemma 4.2.** *Assume that 2 in Theorem 4.1 is satisfied, then the functional  $F$  is coercive.*

*Proof.* We assume, by contradiction, that there exists a constant  $B > 0$  and a sequence of elements  $u_k \in W^{1,2}$  such that, for all  $k$ ,  $F(u_k) \leq B$  while  $\|u_k\|_{W^{1,2}} \rightarrow \infty$ .

We first remark that  $F(|u_k|) = F(u_k)$  so, upon replacing  $u_k$  by  $|u_k|$ , we can suppose that  $u_k \geq 0$ . Let  $q$  be as in the proof of Proposition 2.1. We have

$$\begin{aligned} \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2 &\leq F(u_k) + \int_M \left( \frac{4(n-1)}{n-2} + \text{Scal} \right) u_k^2 d\mu^g \\ &\leq B + \left( \text{Vol}_g(M)^{1-2/q} + \|\text{Scal}\|_{L^p} \right) \|u_k\|_{L^q}^2. \end{aligned}$$

This proves that  $\|u_k\|_{L^q} \rightarrow \infty$  and that  $\|u_k\|_{W^{1,2}} \lesssim \|u_k\|_{L^q}$ . We set  $\gamma_k = \|u_k\|_{L^q}$  and  $v_k := \gamma_k^{-1} u_k$  so that the sequence  $(v_k)_k$  is bounded in  $W^{1,2}$  and satisfies  $\|v_k\|_{L^q} = 1$ . We can assume, without loss of generality, that  $v_k$  converges weakly in  $W^{1,2}$  and strongly in  $L^q$  to some  $v \in W^{1,2}$ . Since  $\|v\|_{L^q} = 1$ , we have  $v \neq 0$ .

We now claim that  $v \in \mathcal{F}(Z)$ . Indeed, we have

$$(4.3) \quad B \geq F(u_k) = \gamma_k^2 G(v_k) + \gamma_k^N \int_M f v_k^N d\mu^g = \gamma_k^N \left( \int_M f v_k^N d\mu^g + o(1) \right).$$

If we were able to prove that

$$(4.4) \quad \int_M f v_k^N d\mu^g \rightarrow \int_M f v^N d\mu^g,$$

we would immediately conclude that

$$\int_M f v^N d\mu^g = 0.$$

Yet, convergence of  $(v_k)_k$  to  $v$  is so weak that proving that (4.4) (if true) holds is delicate. We bypass this issue by the following argument. Assume, by contradiction, that  $v \notin \mathcal{F}(Z)$ , then there exist a set  $W \subset M \setminus Z$  with positive measure and an  $\epsilon > 0$  such that  $v \geq \epsilon \mathbb{1}_W$  a.e. (here  $\mathbb{1}_W$  is the indicator function of  $W$ ). Then,

$$\int_M f v^q d\mu^g \geq \epsilon^q \int_M f \mathbb{1}_W^q d\mu^g = \epsilon^q \int_W f d\mu^g > 0.$$

As a consequence, we have, for  $k$  large enough,

$$\int_M f v_k^q d\mu^g \geq \frac{\epsilon^q}{2} \int_W f d\mu^g.$$

From Hölder's inequality, we have

$$\left( \int_M f v_k^N d\mu^g \right)^{q/N} \left( \int_M f d\mu^g \right)^{1-q/N} \geq \int_M f v_k^q d\mu^g \geq \frac{\epsilon^q}{2} \int_W f d\mu^g.$$

This shows that  $\int_M f v_k^N d\mu^g$  is bounded from below by a positive constant. This yields a contradiction with (4.3). As a consequence, we have  $v \in \mathcal{F}(Z)$ .

Due to our assumption on  $Z$ , we have  $G_g(v) \geq \mathcal{Y}_g(V) \|v\|_{L^N}^{2/N} > 0$ . So

$$\liminf_{k \rightarrow \infty} G_g(v_k) \geq G_g(v) > 0.$$

In particular, we have

$$\liminf_{k \rightarrow \infty} F(u_k) \geq \liminf_{k \rightarrow \infty} G_g(u_k) = \liminf_{k \rightarrow \infty} \gamma_k^2 G_g(v_k) = \infty.$$

This contradicts the assumption  $F(u_k) \leq B$ .  $\square$

We have now all the ingredients to conclude that  $F$  admits a minimizer  $\phi$ . Since  $F(|\phi|) = F(\phi)$ , we can assume, without loss of generality, that  $\phi \geq 0$ .  $\phi$  is then a solution in a weak sense to (4.1). By elliptic regularity, we conclude that  $\phi \in W^{2,p}$  and by Harnack's inequality that  $\phi > 0$  provided  $\phi \not\equiv 0$ .

We rule out the possibility that  $\phi \equiv 0$  as follows. Since  $\mathcal{Y}_g(M) < 0$ , there exists  $w \in W^{1,2}$  such that  $G_g(w) < 0$ . For any  $\lambda > 0$  we have

$$F(\lambda w) = \lambda^2 G_g(w) + \lambda^N I(w).$$

In particular, if  $\lambda$  is small enough we have  $F(\lambda w) < 0$ . This shows that the zero function is not a global minimum of  $F$ . This forces  $\phi \not\equiv 0$ .

Uniqueness of  $\phi$  is obtained by applying Proposition 3.3.

We now need to get rid of the assumption  $f \in L^\infty$ . For all  $k > 0$ , we set  $f_k := \min\{f, k\} \in L^\infty$ . Let  $\phi_k$  denote the solution to (4.1) with  $f$  replaced by  $f_k$ . Note that the zero set of  $f_k$  is the same as that of  $f$  so the preceding construction applies. It follows from the maximum principle that  $\phi_{k+1} \leq \phi_k$  for all  $k > 0$  (the argument is similar to the one in the proof of Proposition 3.3). Since  $\phi_1 \in W^{2,p} \subset L^\infty$ , the sequence  $f_k \phi_k^{N-1}$  is uniformly bounded in  $L^p$ . Hence, from elliptic regularity, the sequence  $(\phi_k)_k$  is bounded in  $W^{2,p}$ . By the compactness of the embedding  $W^{2,p} \hookrightarrow L^\infty$  together with elliptic regularity, there exists a subsequence  $(\phi_{\theta(k)})_k$  of  $(\phi_k)_k$  that converges to some  $\phi \in W^{2,p}$ ,  $\phi \geq 0$  solving (4.1). Note that, from Dini's theorem,  $(\phi_k)_k$  converges in  $L^\infty$  to  $\phi$ . All we need to do is to exclude that  $\phi \equiv 0$ .

This can be done as follows. Let  $w \in W^{1,2}$  be, as before, such that  $G_g(w) < 0$ . Since  $W^{2,p}$  is dense in  $W^{1,2}$ , we can assume that  $w \in W^{2,p} \subset L^\infty$ . As before, considering  $u = \lambda w$  in the

functional (4.2), we get existence of  $v$  such that  $F(v) < 0$ . Set

$$F_k(u) := \int_M \left[ \frac{4(n-1)}{n-2} |d\phi|^2 + \text{Scal } \phi^2 + \frac{2}{N} f_k |\phi|^N \right] d\mu^g$$

So we have  $F_k(\phi_k) \leq F_k(v) \leq F(v) < 0$ . Now remark that  $F_k(\phi_k) \rightarrow_{k \rightarrow \infty} F(\phi)$ . This forces  $F(\phi) \leq F(v) < 0$  which shows that  $\phi \neq 0$ . By construction  $\phi \geq 0$  and from Harnack's inequality, we have  $\phi > 0$ . This ends the proof of Theorem 4.1. Uniqueness is obtained from Proposition 3.3.

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