



**HAL**  
open science

# Existence of solutions to the Lichnerowicz equation: A new proof

Romain Gicquaud

► **To cite this version:**

Romain Gicquaud. Existence of solutions to the Lichnerowicz equation: A new proof. Journal of Mathematical Physics, 2022, 63 (2), pp.022501. 10.1063/5.0050920 . hal-02423741

**HAL Id: hal-02423741**

**<https://hal.science/hal-02423741v1>**

Submitted on 27 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# EXISTENCE OF SOLUTIONS TO THE LICHNEROWICZ EQUATION: A NEW PROOF

ROMAIN GICQUAUD

ABSTRACT. We provide a complete study of existence and uniqueness of solutions to the Lichnerowicz equation in general relativity with arbitrary mean curvature.

## CONTENTS

1. Introduction	1
2. Local Yamabe invariant and first conformal eigenvalue	2
3. Existence of solutions to the Lichnerowicz equation	5
4. Existence of solutions to the prescribed scalar curvature equation	9
References	12

## 1. INTRODUCTION

The Lichnerowicz equation is an elliptic equation that appears in the construction of initial data in general relativity. In the setting of this note, let  $(M, g)$  be a compact Riemannian manifold of dimension  $n > 2$ ,  $g \in W^{2,p}$ ,  $p > n/2$ , and assume given two functions  $\tau \in L^{2p}$  and  $A \in L^{2p}$ . The Lichnerowicz equation has a positive function  $\phi$  as unknown and reads

$$(1.1) \quad -\frac{4(n-1)}{n-2} \Delta \phi + \text{Scal } \phi = -\frac{n-1}{n} \tau^2 \phi^{N-1} + \frac{A^2}{\phi^{N+1}},$$

where  $\text{Scal}$  is the scalar curvature of  $g$  and  $N := \frac{2n}{n-2}$ .

We refer the reader to [3, 5] for an overview of the context in which this equation appears. It has attracted attention a couple of decades ago culminating at the classification of constant mean curvature initial data by J. Isenberg in [14]. Recently, important efforts have been put in constructing non-constant mean curvature initial data, see [12, 13, 16, 6] and [10].

The main aim of this note is to give a short proof of existence/non-existence of solutions to (1.1) in the generic case  $A \neq 0$ . This result is well-known to a large extent, see e.g. [16, Theorem 1]. The main novelty here is that there is no need to give separate proofs according to the sign of the Yamabe quotient of  $(M, g)$ . The particular case  $A \equiv 0$  is the prescribed scalar curvature equation which is similar to the problem addressed in [19, 17, 18, 20], see also [7, 8]. We will study it in Section 4.

---

*Date:* 18<sup>th</sup> November, 2019.

*2000 Mathematics Subject Classification.* 53C21 (Primary), 35Q75, 53C80, 83C05 (Secondary).

*Key words and phrases.* Einstein constraint equations, non-constant mean curvature, conformal method, Lichnerowicz equation, compact manifold, prescribed scalar curvature.

This paper is a byproduct of the techniques developed in [7, 8].

The outline of this paper is as follows. In Section 2, we introduce the main tool to discriminate which function  $\tau$  lead to existence of solutions to (1.1). In Section 3, we study the case  $A \neq 0$ . The main result of this section is Theorem 3.1 which is the main result of the paper. Section 4 is devoted to the case  $A \equiv 0$  which, as we indicated before, deserves a particular treatment.

**Acknowledgements:** The author is grateful to Marie-Françoise Bidaut-Véron for useful comments on a preliminary version of this article.

## 2. LOCAL YAMABE INVARIANT AND FIRST CONFORMAL EIGENVALUE

For any measurable subset  $V \subset M$ , we define the space

$$(2.1) \quad \mathcal{F}(V) := \{u \in W^{1,2}, u \equiv 0 \text{ a.e. on } M \setminus V\}$$

of Sobolev functions vanishing outside  $V$ . This set is obviously reduced to  $\{0\}$  if  $V$  has Lebesgue measure zero but there are larger  $V$  with  $\mathcal{F}(V) = \{0\}$ , see for example [1, Chapter 6]. Much of this section is adapted from [8].

For any  $u \in W^{1,2}$ , we set

$$(2.2) \quad G_g(u) := \int_M \left[ \frac{4(n-1)}{n-2} |du|^2 + \text{Scal } u^2 \right] d\mu^g$$

We also introduce, for any  $u \in W^{1,2}$ ,  $u \neq 0$ , the Rayleigh and the Yamabe quotients:

$$\begin{aligned} Q_g^R(u) &:= G(u) / \|u\|_{L^2}^2, \\ Q_g^Y(u) &:= G(u) / \|u\|_{L^N}^2. \end{aligned}$$

With these definitions at hand, we introduce the local first conformal eigenvalue  $\lambda_g(V)$  and the local Yamabe invariant  $\mathcal{Y}_g(V)$  of any measurable subset  $V \subset M$  as follows:

$$\begin{aligned} \lambda_g(V) &:= \inf_{u \in \mathcal{F}(V) \setminus \{0\}} Q_g^R(u), \\ \mathcal{Y}_g(V) &:= \inf_{u \in \mathcal{F}(V) \setminus \{0\}} Q_g^Y(u). \end{aligned}$$

From the definition of an infimum, we have  $\lambda_g(V) = \mathcal{Y}_g(V) = \infty$  if  $\mathcal{F}(V)$  is reduced to  $\{0\}$ .

**Proposition 2.1.** *The functional  $G$  defined in (2.2) is sequentially weakly lower semi-continuous on  $W^{1,2}$ : for every weakly converging sequence  $(u_k)_k$ ,  $u_k \rightharpoonup u_\infty$ , we have  $\liminf_{k \rightarrow \infty} G(u_k) \geq G(u_\infty)$ .*

*Proof.* Note that  $G_g$  can be decomposed as

$$(2.5) \quad G_g(u) = \frac{4(n-1)}{n-2} \int_M |du|^2 d\mu^g + \int_M \text{Scal } u^2 d\mu^g.$$

The first term is weakly lower semi-continuous with respect to  $u \in W^{1,2}$  as a continuous non-negative quadratic form. For the second one, we shall prove that, given a sequence  $(u_k)_k$  in  $W^{1,2}$ , converging weakly to  $u_\infty$ ,  $u_k \rightharpoonup_{k \rightarrow \infty} u_\infty$ , we have

$$\int_M \text{Scal } u_k^2 d\mu^g \rightarrow \int_M \text{Scal } u_\infty^2 d\mu^g.$$

To make the notation less cluttered, we denote the second term in (2.5) as  $S(u)$ :

$$S(u) := \int_M \text{Scal } u^2 d\mu^g.$$

Assume by contradiction that  $(S(u_k))_k$  does not converge to  $S(u_\infty)$ , there exists an  $\epsilon > 0$  such that, for an infinite number of integers  $k$ , we have

$$(2.6) \quad |S(u_k) - S(u_\infty)| > \epsilon.$$

Without loss of generality, we can assume that (2.6) holds for all integer  $k$  and also that  $(u_k)_k$  converges strongly in  $L^2$  to some  $\bar{u}_\infty \in L^2$  since the embedding  $W^{1,2} \hookrightarrow L^2$  is compact. Then we have  $u_\infty \equiv \bar{u}_\infty$  a.e. Indeed, the linear form

$$u \mapsto \int_M u(u_\infty - \bar{u}_\infty) d\mu^g$$

is (strongly) continuous for the  $L^2$ -topology and, hence, for the  $W^{1,2}$ -topology. As a consequence,

$$\int_M u_\infty(u_\infty - \bar{u}_\infty) d\mu^g = \lim_{k \rightarrow \infty} \int_M u_k(u_\infty - \bar{u}_\infty) d\mu^g = \int_M \bar{u}_\infty(u_\infty - \bar{u}_\infty) d\mu^g,$$

where the first equality holds by the  $W^{1,2}$ -weak convergence of  $(u_k)_k$  to  $u_\infty$  and the second one by the  $L^2$ -strong convergence of  $(u_k)_k$  to  $\bar{u}_\infty$ . Subtracting both equalities, we get

$$\int_M |u_\infty - \bar{u}_\infty|^2 d\mu^g = 0,$$

which proves that  $u_\infty \equiv \bar{u}_\infty$  a.e. Finally note that, since  $(u_k)_k$  is weakly convergent in  $W^{1,2}$ , it is bounded and thus (by interpolation) converges in all  $L^q$  spaces,  $q \in [2, N)$ . Since  $\text{Scal} \in L^p$ ,  $p > n/2$ , letting  $q$  be such that  $1 = \frac{1}{p} + \frac{2}{q}$ , we have  $q \in [2, N)$  and, by Hölder's inequality,  $S$  is a bounded quadratic form on  $L^q$ . In particular  $S$  is continuous on  $L^q$ :

$$S(u_k) \rightarrow S(u_\infty).$$

This contradicts (2.5):  $S$  is sequentially weakly continuous on  $W^{1,2}$ . This ends the proof of Proposition 2.1.  $\square$

In what follows, we let  $s > 0$  be the largest constant so that

$$(2.7) \quad \|u\|_{W^{1,2}}^2 \geq s \|u\|_{L^N}^2 \quad \forall u \in W^{1,2}.$$

**Proposition 2.2.** *Given any measurable set  $V \subset M$ ,  $\lambda_g(V)$  and  $\mathcal{Y}_g(V)$  have the same sign (i.e. they are either both positive, both negative or both zero).*

*Proof.* We can assume, without loss of generality, that  $\mathcal{F}(V) \neq \{0\}$  for otherwise  $\mathcal{Y}_g(V) = \lambda_g(V) = \infty$ . If  $\mathcal{Y}_g(V) < 0$ , there exists  $u \in \mathcal{F}(V)$  such that  $G_g(u) < 0$  so  $\lambda_g(V) < 0$ . Assume now that  $\mathcal{Y}_g(V) > 0$ , then, for all  $u \in \mathcal{F}(V) \setminus \{0\}$ , we have

$$Q_g^R(u) = \frac{G_g(u)}{\|u\|_{L^2}^2} \geq \frac{G_g(u)}{\|u\|_{L^N}^2 \text{Vol}_g(V)^{2/n}} \geq \frac{\mathcal{Y}_g(V)}{\text{Vol}_g(V)^{2/n}}.$$

We conclude that

$$\lambda_g(V) \geq \frac{\mathcal{Y}_g(V)}{\text{Vol}_g(V)^{2/n}} > 0.$$

All we have to show now is that, if  $\mathcal{Y}_g(V) = 0$ , we have  $\lambda_g(V) = 0$ . Assume for the rest of the proof that  $\mathcal{Y}_g(V) = 0$ . If  $\lambda_g(V)$  were negative, there would exist  $u \in \mathcal{F}(V)$  such that  $G_g(u) < 0$  so  $\mathcal{Y}_g(V) \leq Q_g^Y(u) < 0$ . This proves that  $\lambda_g(V) \geq 0$ . Since  $\mathcal{Y}_g(V) = 0$ , there exists a sequence

of functions  $u_k \in \mathcal{F}(V)$  such that  $Q_g^Y(u_k) \rightarrow 0$ . Without loss of generality, we can assume that  $\|u_k\|_{L^N} = 1$  so  $G_g(u_k) \rightarrow 0$ .

Let  $q$  be as in the proof of the previous proposition. Then we have that

$$\begin{aligned} G(u_k) &\geq \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2 - \frac{4(n-1)}{n-2} \|u_k\|_{L^2}^2 - \|\text{Scal}\|_{L^p} \|u_k\|_{L^q}^2 \\ &\geq \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2 - \frac{4(n-1)}{n-2} \text{Vol}_g(V)^{1-2/q} \|u_k\|_{L^q}^2 - \|\text{Scal}\|_{L^p} \|u_k\|_{L^q}^2. \end{aligned}$$

Hence, setting  $C = \frac{4(n-1)}{n-2} \text{Vol}_g(V)^{1-2/q} + \|\text{Scal}\|_{L^p}$ , we arrive at

$$(2.8) \quad G_g(u_k) + C \|u_k\|_{L^q}^2 \geq \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2.$$

Since  $q < N$ , we have that  $\|u_k\|_{W^{1,2}}$  is bounded independently of  $k$ . Arguing as in the proof of the previous proposition, we can assume that  $(u_k)_k$  converges weakly in  $W^{1,2}$  and strongly in  $L^2$  to some  $u_\infty \in \mathcal{F}(V)$ . Combining Equation (2.8) with the Sobolev estimate (2.7), we get

$$G_g(u_k) + C \|u_k\|_{L^q}^2 \geq \frac{4(n-1)}{n-2} s \|u_k\|_{L^N}^2 = \frac{4(n-1)}{n-2} s.$$

Passing to the limit as  $k$  goes to infinity, we conclude that  $\|u_\infty\|_{L^q} > 0$ , i.e.  $u_\infty \not\equiv 0$ . By the lower semicontinuity of  $G_g$ , we have  $G_g(u_\infty) \leq \liminf_{k \rightarrow \infty} G_g(u_k) = 0$ . Since  $G_g(u_\infty) \geq 0$ , we have  $G_g(u_\infty) = 0$ . We have proven that

$$0 \leq \lambda_g(V) \leq Q_g^R(u_\infty) = 0,$$

i.e.  $\lambda_g(V) = 0$ . This concludes the proof of the fact that  $\mathcal{Y}_g(V)$  and  $\lambda_g(V)$  have the same sign.  $\square$

The reason why it is more convenient to work with  $\mathcal{Y}_g(V)$  than with  $\lambda_g(V)$  is given by the following proposition.

**Proposition 2.3.** *Assume that  $g$  and  $h$  are two conformally related metrics,  $h = \phi^{N-2}g$ , for some positive function  $\phi \in W^{2,p}$ . Then for any measurable  $V$  we have*

$$\mathcal{Y}_g(V) = \mathcal{Y}_h(V).$$

*Proof.* The proof is a simple calculation. Given any  $u \in W^{1,2}$ , we have

$$\begin{aligned}
G_h(u) &= \int_M \left[ \frac{4(n-1)}{n-2} |du|_h^2 + \text{Scal}^h u^2 \right] d\mu^h \\
&= \int_M \left[ \frac{4(n-1)}{n-2} \phi^{2-N} |du|_g^2 + \left( -\frac{4(n-1)}{n-2} \Delta^g \phi + \text{Scal}^g \phi \right) \phi^{1-N} u^2 \right] \phi^N d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} \phi^2 |du|_g^2 + \left( -\frac{4(n-1)}{n-2} \Delta^g \phi + \text{Scal}^g \phi \right) \phi u^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} (\phi^2 |du|_g^2 - (\phi \Delta^g \phi) u^2) + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} (\phi^2 |du|_g^2 + \langle d\phi, d(\phi u^2) \rangle_g) + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} (\phi^2 |du|_g^2 + u^2 |d\phi|_g^2 + 2\langle \phi d\phi, u du \rangle_g) + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= \int_M \left[ \frac{4(n-1)}{n-2} |d(\phi u)|_g^2 + \text{Scal}^g (\phi u)^2 \right] d\mu^g \\
&= G_g(\phi u).
\end{aligned}$$

Similarly,

$$\|u\|_{L_h^N} = \left( \int_M u^N d\mu^h \right)^{1/N} = \left( \int_M u^N \phi^N d\mu^g \right)^{1/N} = \|\phi u\|_{L_g^N}.$$

So

$$Q_h^{\mathcal{Y}}(u) = Q_g^{\mathcal{Y}}(\phi u).$$

Since  $\phi$  is bounded away from zero, multiplication by  $\phi$  defines an automorphism of  $\mathcal{F}(V)$ . Hence,

$$\mathcal{Y}_g(V) = \inf_{u \in \mathcal{F}(V)} Q_g^{\mathcal{Y}}(u) = \inf_{u \in \mathcal{F}(V)} Q_g^{\mathcal{Y}}(\phi u) = \inf_{u \in \mathcal{F}(V)} Q_h^{\mathcal{Y}}(u) = \mathcal{Y}_h(V).$$

□

### 3. EXISTENCE OF SOLUTIONS TO THE LICHNEROWICZ EQUATION

**Theorem 3.1.** *Let  $(M, g)$  be a compact Riemannian manifold with  $g \in W^{2,p}$ ,  $p > n/2$ . Assume that  $\tau \in L^{2p}$  is given. Then the following statements are equivalent:*

- (1) *There exists a solution to (1.1) for all  $A \in L^{2p}$ ,  $A \not\equiv 0$*
- (2) *There exists a solution to (1.1) for at least one  $A \in L^{2p}$ ,  $A \not\equiv 0$ ,*
- (3) *The set  $Z = \tau^{-1}(0)$  satisfies  $\mathcal{Y}_g(Z) > 0$ .*

*Further, the solution to (1.1), when it exists, is unique unless  $\mathcal{Y}_g(M) = 0$  and  $\tau, A \equiv 0$  for which all solutions are proportional one to another.*

It should be noted that the theorem can be applied in particular when  $Z$  has zero Lebesgue measure. This is the case if  $\tau$  never vanishes or if 0 is a regular value for  $\tau$ .

This theorem reproduces results from [12, 13, 15, 16] and references therein (see also [9]) in which several proofs are given according to the sign of  $\mathcal{Y}_g(M)$  and the nullity of  $\tau$ . The main novelty is that the proof establishes a direct link between existence of solutions to the Lichnerowicz equation and the fact that  $\mathcal{Y}_g(Z) > 0$ . We first state a lemma:

**Lemma 3.2.** *Under the assumptions of the theorem, if  $\mathcal{Y}_g(Z) > 0$ , there exists a constant  $K > 0$  such that the operator*

$$u \mapsto -\frac{4(n-1)}{n-2}\Delta u + \text{Scal } u + K\frac{n-1}{n}\tau^2 u$$

*has positive first eigenvalue.*

*Proof.* Assume by contradiction that for all  $k \in \mathbb{N}$ , the first eigenvalue of

$$L_k : u \mapsto -\frac{4(n-1)}{n-2}\Delta u + \text{Scal } u + k\frac{n-1}{n}\tau^2 u$$

is non-positive. We denote it by  $\lambda_k$  and let  $u_k \in W^{2,p/2}$  be the first eigenfunction normalized so that  $u_k \geq 0$  and  $\|u_k\|_{L^2} = 1$ . The sequence  $(\lambda_k)_k$  is increasing since

$$\begin{aligned} \lambda_{k+1} &= \int_M u_{k+1} L_{k+1} u_{k+1} d\mu^g \\ &= \int_M u_{k+1} L_k u_{k+1} d\mu^g + \int_M \frac{n-1}{n} \tau^2 u_{k+1}^2 \\ &\geq \int_M u_{k+1} L_k u_{k+1} d\mu^g \\ &\geq \lambda_k. \end{aligned}$$

We claim that the sequence  $(u_k)_k$  is bounded in  $W^{1,2}$ . Indeed, we have, using the Hölder inequality:

$$\begin{aligned} 0 &\geq \int_M \left[ \frac{4(n-1)}{n-2} |du_k|^2 + \text{Scal } u_k^2 \right] d\mu^g \\ &\geq \frac{4(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - \|\text{Scal}\|_{L^p} \|u_k\|_{L^N}^{\frac{n}{p}} \|u_k\|_{L^2}^{2-\frac{n}{p}} \\ &\geq \frac{4(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - 2\|\text{Scal}\|_{L^p} \left[ \frac{n}{p} \epsilon \|u_k\|_{L^N}^2 + \frac{2p-n}{2p} \|u_k\|_{L^2}^2 \epsilon^{-n/(2p-n)} \right] \\ &\geq \frac{4(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - \|\text{Scal}\|_{L^p} \left[ \frac{2n}{sp} \epsilon \|u_k\|_{W^{1,2}}^2 + \frac{2p-n}{p} \|u_k\|_{L^2}^2 \epsilon^{-n/(2p-n)} \right], \end{aligned}$$

where we used the  $\epsilon$ -Young inequality and the Sobolev inequality (2.7). Assuming that  $\text{Scal} \neq 0$  (if  $\text{Scal} \equiv 0$  the argument is simpler), we choose  $\epsilon$  such that

$$\|\text{Scal}\|_{L^p} \frac{2n}{sp} \epsilon = \frac{2(n-1)}{n-2},$$

so

$$0 \geq \frac{2(n-1)}{n-2} \int_M |du_k|^2 d\mu^g - C \|u_k\|_{L^2}^2,$$

for some explicit constant  $C = C(n, s, p, \|\text{Scal}\|_{L^p})$ . Since  $\|u_k\|_{L^2} = 1$ , this proves the claim that  $(u_k)_k$  is bounded in  $L^2$ .

From Rellich theorem, we now extract a subsequence  $(k_i)_i$  of  $k$  such that

$$u_{k_i} \rightarrow u_\infty \text{ in } L^2$$

for some  $u_\infty \in W^{1,2}$ . In particular,  $\|u_\infty\|_{L^2} = 1$ . We can also assume that

$$u_{k_i} \rightharpoonup u_\infty \text{ in } W^{1,2}.$$

We claim that  $u_\infty \equiv 0$  a.e. on  $M \setminus Z$ . Otherwise,

$$\int_M \tau^2 u_{k_i}^2 d\mu^g \rightarrow \int_M \tau^2 u_\infty^2 d\mu^g \neq 0,$$

so

$$\begin{aligned} \lambda_{k_i} &= \int_M u_{k_i} L_{k_i} u_{k_i} d\mu^g \\ &= \int_M u_{k_i} L_0 u_{k_i} d\mu^g + k_i \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 d\mu^g \\ &\geq \lambda_0 + k_i \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 d\mu^g \\ &\rightarrow_{i \rightarrow \infty} \infty, \end{aligned}$$

contradicting the fact that  $(\lambda_k)_k$  is bounded. Since  $\|u_\infty\|_{L^2} = 1$  and belongs to  $\mathcal{F}(Z)$ , we have a contradiction if  $\mathcal{F}(Z) = \{0\}$ . In the case where  $\mathcal{F}(Z) \neq \{0\}$ , we also get a contradiction since

$$\lambda_{k_i} = G_g(u_{k_i}) + k \frac{n-1}{n} \int_M \tau^2 u_{k_i}^2 \geq G_g(u_{k_i}),$$

so, since  $G_g$  is weakly lower semicontinuous,

$$\liminf_{i \rightarrow \infty} \lambda_{k_i} \geq \liminf_{i \rightarrow \infty} G_g(u_{k_i}) \geq G_g(u_\infty) \geq \lambda_g(Z) > 0.$$

This gives the final contradiction.  $\square$

*Proof of Theorem 3.1.* The statement  $1 \Rightarrow 2$  is obvious. We now prove that  $2 \Rightarrow 3$ . The proof is similar to that of Proposition 2.3. If  $\mathcal{F}(Z) = \{0\}$ , Statement 3 is satisfied since  $\mathcal{Y}_g(Z) = \infty$ . Otherwise, assume given  $A \in L^{2p}$  and  $\phi \in W^{2,p}$  satisfying (1.1). We set  $\hat{g} = \phi^{N-2}g$  and  $\hat{u} = u\phi^{-1}$ . For all  $u \in \mathcal{F}(Z)$ , we have

$$\begin{aligned} G_g(u) &= G_g(\phi \hat{u}) \\ &= \int_M \left[ \frac{4(n-1)}{n-2} \left( \phi^2 |d\hat{u}|_g^2 + \langle \phi d\phi, d(\hat{u}^2) \rangle_g + \hat{u}^2 |d\phi|_g^2 \right) + \text{Scal } \phi^2 \hat{u}^2 \right] d\mu^g \\ &= \int_M \left[ \frac{4(n-1)}{n-2} \left( \phi^2 |d\hat{u}|_g^2 - (\phi \Delta \phi) \hat{u}^2 \right) + \text{Scal } \phi^2 \hat{u}^2 \right] d\mu^g \\ &= \int_M \left[ \frac{4(n-1)}{n-2} \phi^2 |d\hat{u}|_g^2 + \left( \frac{A^2}{\phi^N} - \frac{n-1}{n} \tau^2 \phi^N \right) \hat{u}^2 \right] d\mu^g \\ &= \int_M \left[ \frac{4(n-1)}{n-2} |d\hat{u}|_{\hat{g}}^2 + \left( \frac{A^2}{\phi^{2N}} - \frac{n-1}{n} \tau^2 \right) \hat{u}^2 \right] d\mu^{\hat{g}} \\ &\geq \int_M \left[ \frac{4(n-1)}{n-2} |d\hat{u}|_{\hat{g}}^2 + \frac{A^2}{\phi^{2N}} \hat{u}^2 \right] d\mu^{\hat{g}} \quad (\text{since } \hat{u} \in \mathcal{F}(Z)). \end{aligned}$$

This immediately rules out the possibility that  $\mathcal{Y}_g(Z) < 0$  since  $G_g(u) \geq 0$  for all  $u \in \mathcal{F}(Z)$ . Assume next that  $M \setminus Z$  has positive Lebesgue measure. Then,  $\hat{u} \equiv 0$  on  $M \setminus Z$ . As a consequence, from the Poincaré inequality, there is a constant  $\mu = \mu(g, \tau)$  so that

$$G_g(u) \geq \frac{4(n-1)}{n-2} \int_M |d\hat{u}|_{\hat{g}}^2 d\mu^{\hat{g}} \geq \mu \|\hat{u}\|_{W^{1,2}}^2,$$

(see e.g. [11, Lemma 7.16]) and, hence, from the Sobolev embedding theorem,

$$G_g(u) \geq s\mu \|\hat{u}\|_{L^N}^2.$$

This proves that

$$\mathcal{Y}_g(Z) = s\mu > 0.$$

The only remaining possibility is that  $\tau \equiv 0$  a.e. that is to say  $Z = M$  and  $\mathcal{Y}_g(M) = 0$ . From the proof of Proposition 2.2, there exists a function  $u_\infty \geq 0$ ,  $u_\infty \not\equiv 0$  so that  $G_g(u_\infty) = 0$ . From the inequality

$$G_g(u_\infty) \geq \frac{4(n-1)}{n-2} \int_M |d\hat{u}_\infty|_g^2 d\mu^{\hat{g}},$$

we have  $d\hat{u}_\infty \equiv 0$ :  $\hat{u}_\infty$  is a constant function. This gives a contradiction since

$$0 = G_g(u_\infty) = \int_M \frac{A^2}{\phi^{2N}} \hat{u}_\infty^2 d\mu^{\hat{g}} > 0.$$

We finally prove that 3  $\Rightarrow$  1. The proof goes as usual by the sub- and super-solution method (see e.g. [21, Chapter 14]). Let  $K$  be as in the statement of Lemma 3.2. We let  $u$  denote the solution to

$$(3.1) \quad -\frac{4(n-1)}{n-2} \Delta u + \text{Scal } u + K \frac{n-1}{n} \tau^2 u = A^2.$$

Since the operator on the left hand side is positive, its Green function is positive, so  $u \in W^{2,p}$  is also positive (note that  $u$  is Hölder continuous). We set

$$\begin{cases} u_+ = \lambda_+ u, \\ u_- = \lambda_- u \end{cases}$$

for some positive constants  $\lambda_\pm$  to be chosen later. We want  $u_+$  to be a super-solution to the Lichnerowicz equation (1.1), i.e.  $u_+$  has to satisfy

$$-\frac{4(n-1)}{n-2} \Delta u_+ + \text{Scal } u_+ + \frac{n-1}{n} \tau^2 u_+^{N-1} \geq \frac{A^2}{u_+^{N+1}}.$$

From Equation (3.1), this is equivalent to

$$\frac{n-1}{n} \tau^2 (\lambda_+^{N-1} u^{N-1} - K \lambda_+ u) + \lambda_+ A^2 \geq \frac{A^2}{\lambda_+^{N+1} u^{N+1}}.$$

This inequality holds true if both the following inequalities are fulfilled:

$$\begin{cases} \lambda_+^{N-2} u^{N-2} \geq K, \\ \lambda_+^{N+2} \geq u^{-N-1}. \end{cases}$$

Since  $u$  is bounded from above and away from zero, they are true for large enough  $\lambda_+$ . Calculations for the sub-solution are similar: if  $\lambda_-$  is a small enough positive constant  $u_-$  is a sub-solution to the Lichnerowicz equation (1.1). By the sub- and super-solution argument, we get existence of  $u \in W^{2,p}$  solving (1.1). Uniqueness of  $u$  will be proven in the next proposition.  $\square$

**Proposition 3.3.** *Let  $(M, g)$  be a compact Riemannian manifold with  $g \in W^{2,p}$ ,  $p > n/2$ . Let  $\tau, A \in L^{2p}$  be two given functions. Assume given two positive functions  $\phi_1, \phi_2 \in W^{2,p}$  solving the Lichnerowicz equation (1.1).*

- If  $\tau \not\equiv 0$  or  $A \not\equiv 0$ , we have  $\phi_1 \equiv \phi_2$ ,
- If  $\tau, A \equiv 0$ ,  $\phi_1$  and  $\phi_2$  are proportional.

*Proof.* The proof of this fact is well known, we refer the reader e.g. to [16, Proposition 2] or to [6]. We present here the argument from [4]. Since  $\phi_1$  and  $\phi_2$  are bounded from below, we have that  $\phi_1^2/\phi_2$  and  $\phi_2^2/\phi_1$  both belong to  $W^{2,p}$ . By an integration by parts and some routine calculations, we have

$$(3.2) \quad \begin{aligned} & \int_M \left( -\frac{\Delta\phi_1}{\phi_1} + \frac{\Delta\phi_2}{\phi_2} \right) (\phi_1^2 - \phi_2^2) d\mu^g \\ &= \int_M \left| d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 \right|^2 d\mu^g + \int_M \left| d\phi_2 - \frac{\phi_2}{\phi_1} d\phi_1 \right|^2 d\mu^g. \end{aligned}$$

If we set

$$f(\phi) := \frac{n-2}{4(n-1)} \left[ \frac{A^2}{\phi^{N+2}} - \text{Scal} - \frac{n-1}{n} \tau^2 \phi^{N-2} \right],$$

we have

$$-\frac{\Delta\phi_1}{\phi_1} = f(\phi_1) \quad \text{and} \quad -\frac{\Delta\phi_2}{\phi_2} = f(\phi_2),$$

so the identity (3.2) gives

$$\begin{aligned} & \int_M [f(\phi_1) - f(\phi_2)] (\phi_1^2 - \phi_2^2) d\mu^g \\ &= \int_M \left| d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 \right|^2 d\mu^g + \int_M \left| d\phi_2 - \frac{\phi_2}{\phi_1} d\phi_1 \right|^2 d\mu^g. \end{aligned}$$

Since  $f$  is a decreasing function, we have

$$[f(\phi_1) - f(\phi_2)] (\phi_1^2 - \phi_2^2) \leq 0 \text{ a.e.}$$

This impose that

$$\int_M \left| d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 \right|^2 d\mu^g + \int_M \left| d\phi_2 - \frac{\phi_2}{\phi_1} d\phi_1 \right|^2 d\mu^g = 0.$$

In particular, we have

$$d\phi_1 - \frac{\phi_1}{\phi_2} d\phi_2 = 0 \text{ a.e.} \Leftrightarrow d\left(\frac{\phi_1}{\phi_2}\right) = 0 \text{ a.e.}$$

meaning that  $\phi_1$  and  $\phi_2$  are proportional one another and they are equal unless  $f$  is a constant function at all points of  $M$ , i.e. unless  $\tau, A \equiv 0$ .  $\square$

#### 4. EXISTENCE OF SOLUTIONS TO THE PRESCRIBED SCALAR CURVATURE EQUATION

Our focus in this section is Equation (1.1) with  $A \equiv 0$ , namely

$$(4.1) \quad -\frac{4(n-1)}{n-2} \Delta\phi + \text{Scal} \phi = -f\phi^{N-1},$$

where  $f = \frac{n-1}{n} \tau^2 \geq 0$ . This equation is the well-known prescribed scalar curvature equation (see e.g. [2] for an introduction). The aim of this section is to give a full proof of Theorem 4.1 with an argument that is simpler than the one in [19, 7], following the lines of [8]. One difficulty in the study of Equation (4.1) is to show that  $\phi \neq 0$  since  $\phi \equiv 0$  is a trivial solution to (4.1). This is overcome by studying the asymptotics of  $\phi$  in the non-compact case while here the argument has to be different. The theorem we prove is the following:

**Theorem 4.1.** *Let  $(M, g)$  be a compact Riemannian manifold with  $g \in W^{2,p/2}$ ,  $p > n$ . Assume that  $f \in L^p$ ,  $f \geq 0$ ,  $f \not\equiv 0$ , is given. Then the following statements are equivalent:*

- (1) *There exists a positive solution  $\phi \in W^{2,p}$  to (4.1),*

(2) We have  $\mathcal{Y}_g(M) < 0$  and the set  $Z = f^{-1}(0)$  satisfies  $\mathcal{Y}_g(Z) > 0$ .

Further, the solution to (4.1), when it exists, is unique.

The proof of  $1 \Rightarrow \mathcal{Y}_g(Z) > 0$  is entirely similar to the one given in the proof of Theorem 3.1 so we omit it. Note also that the metric  $h := \phi^{N-2}g$  has scalar curvature  $-f$  so

$$\mathcal{Y}_g(M) = \mathcal{Y}_h(M) \leq Q_h^Y(1) = \frac{G_h(1)}{\text{Vol}_h(M)^{2/N}} < 0.$$

The proof of the converse implication will occupy the remaining of this note. We first prove it assuming that  $f \in L^\infty$  and deduce the general case from this particular case.

We introduce the functional  $F$  defined for all  $\phi \in W^{1,2}$  by

$$(4.2) \quad F(\phi) := \int_M \left[ \frac{4(n-1)}{n-2} |d\phi|^2 + \text{Scal} \phi^2 + \frac{2}{N} f |\phi|^N \right] d\mu^g$$

Note that the assumption that  $f \in L^\infty$  is required in order to ensure that

$$I(\phi) = \int_M f |\phi|^N d\mu^g < \infty$$

for all  $\phi \in W^{1,2}$ . Note that  $\phi \mapsto I(\phi)$  is continuous for the strong topology and convex since  $f \geq 0$ . In particular, it is weakly lower semi-continuous. From Proposition 2.1, we conclude that  $F$  is sequentially weakly lower semi-continuous.

We now show that  $F$  is coercive. This will imply the existence of a minimizer for  $F$ . The proof is similar (yet simpler) than the one given in [8, Proposition 4.8].

**Lemma 4.2.** *Assume that 2 in Theorem 4.1 is satisfied, then the functional  $F$  is coercive.*

*Proof.* We assume, by contradiction, that there exists a constant  $B > 0$  and a sequence of elements  $u_k \in W^{1,2}$  such that, for all  $k$ ,  $F(u_k) \leq B$  while  $\|u_k\|_{W^{1,2}} \rightarrow \infty$ .

We first remark that  $F(|u_k|) = F(u_k)$  so, upon replacing  $u_k$  by  $|u_k|$ , we can suppose that  $u_k \geq 0$ . Let  $q$  be as in the proof of Proposition 2.1. We have

$$\begin{aligned} \frac{4(n-1)}{n-2} \|u_k\|_{W^{1,2}}^2 &\leq F(u_k) + \int_M \left( \frac{4(n-1)}{n-2} + \text{Scal} \right) u_k^2 d\mu^g \\ &\leq B + \left( \text{Vol}_g(M)^{1-2/q} + \|\text{Scal}\|_{L^p} \right) \|u_k\|_{L^q}^2. \end{aligned}$$

This proves that  $\|u_k\|_{L^q} \rightarrow \infty$  and that  $\|u_k\|_{W^{1,2}} \lesssim \|u_k\|_{L^q}$ . We set  $\gamma_k = \|u_k\|_{L^q}$  and  $v_k := \gamma_k^{-1} u_k$  so that the sequence  $(v_k)_k$  is bounded in  $W^{1,2}$  and satisfies  $\|v_k\|_{L^q} = 1$ . We can assume, without loss of generality, that  $v_k$  converges weakly in  $W^{1,2}$  and strongly in  $L^q$  to some  $v \in W^{1,2}$ . Since  $\|v\|_{L^q} = 1$ , we have  $v \neq 0$ .

We now claim that  $v \in \mathcal{F}(Z)$ . Indeed, we have

$$(4.3) \quad B \geq F(u_k) = \gamma_k^2 G(v_k) + \gamma_k^N \int_M f v_k^N d\mu^g = \gamma_k^N \left( \int_M f v_k^N d\mu^g + o(1) \right).$$

If we were able to prove that

$$(4.4) \quad \int_M f v_k^N d\mu^g \rightarrow \int_M f v^N d\mu^g,$$

we would immediately conclude that

$$\int_M f v^N d\mu^g = 0.$$

Yet, convergence of  $(v_k)_k$  to  $v$  is so weak that proving that (4.4) (if true) holds is delicate. We bypass this issue by the following argument. Assume, by contradiction, that  $v \notin \mathcal{F}(Z)$ , then there exist a set  $W \subset M \setminus Z$  with positive measure and an  $\epsilon > 0$  such that  $v \geq \epsilon \mathbb{1}_W$  a.e. (here  $\mathbb{1}_W$  is the indicator function of  $W$ ). Then,

$$\int_M f v^q d\mu^g \geq \epsilon^q \int_M f \mathbb{1}_W^q d\mu^g = \epsilon^q \int_W f d\mu^g > 0.$$

As a consequence, we have, for  $k$  large enough,

$$\int_M f v_k^q d\mu^g \geq \frac{\epsilon^q}{2} \int_W f d\mu^g.$$

From Hölder's inequality, we have

$$\left( \int_M f v_k^N d\mu^g \right)^{q/N} \left( \int_M f d\mu^g \right)^{1-q/N} \geq \int_M f v_k^q d\mu^g \geq \frac{\epsilon^q}{2} \int_W f d\mu^g.$$

This shows that  $\int_M f v_k^N d\mu^g$  is bounded from below by a positive constant. This yields a contradiction with (4.3). As a consequence, we have  $v \in \mathcal{F}(Z)$ .

Due to our assumption on  $Z$ , we have  $G_g(v) \geq \mathcal{Y}_g(V) \|v\|_{L^N}^{2/N} > 0$ . So

$$\liminf_{k \rightarrow \infty} G_g(v_k) \geq G_g(v) > 0.$$

In particular, we have

$$\liminf_{k \rightarrow \infty} F(u_k) \geq \liminf_{k \rightarrow \infty} G_g(u_k) = \liminf_{k \rightarrow \infty} \gamma_k^2 G_g(v_k) = \infty.$$

This contradicts the assumption  $F(u_k) \leq B$ .  $\square$

We have now all the ingredients to conclude that  $F$  admits a minimizer  $\phi$ . Since  $F(|\phi|) = F(\phi)$ , we can assume, without loss of generality, that  $\phi \geq 0$ .  $\phi$  is then a solution in a weak sense to (4.1). By elliptic regularity, we conclude that  $\phi \in W^{2,p}$  and by Harnack's inequality that  $\phi > 0$  provided  $\phi \not\equiv 0$ .

We rule out the possibility that  $\phi \equiv 0$  as follows. Since  $\mathcal{Y}_g(M) < 0$ , there exists  $w \in W^{1,2}$  such that  $G_g(w) < 0$ . For any  $\lambda > 0$  we have

$$F(\lambda w) = \lambda^2 G_g(w) + \lambda^N I(w).$$

In particular, if  $\lambda$  is small enough we have  $F(\lambda w) < 0$ . This shows that the zero function is not a global minimum of  $F$ . This forces  $\phi \not\equiv 0$ .

Uniqueness of  $\phi$  is obtained by applying Proposition 3.3.

We now need to get rid of the assumption  $f \in L^\infty$ . For all  $k > 0$ , we set  $f_k := \min\{f, k\} \in L^\infty$ . Let  $\phi_k$  denote the solution to (4.1) with  $f$  replaced by  $f_k$ . Note that the zero set of  $f_k$  is the same as that of  $f$  so the preceding construction applies. It follows from the maximum principle that  $\phi_{k+1} \leq \phi_k$  for all  $k > 0$  (the argument is similar to the one in the proof of Proposition 3.3). Since  $\phi_1 \in W^{2,p} \subset L^\infty$ , the sequence  $f_k \phi_k^{N-1}$  is uniformly bounded in  $L^p$ . Hence, from elliptic regularity, the sequence  $(\phi_k)_k$  is bounded in  $W^{2,p}$ . By the compactness of the embedding  $W^{2,p} \hookrightarrow L^\infty$  together with elliptic regularity, there exists a subsequence  $(\phi_{\theta(k)})_k$  of  $(\phi_k)_k$  that converges to some  $\phi \in W^{2,p}$ ,  $\phi \geq 0$  solving (4.1). Note that, from Dini's theorem,  $(\phi_k)_k$  converges in  $L^\infty$  to  $\phi$ . All we need to do is to exclude that  $\phi \equiv 0$ .

This can be done as follows. Let  $w \in W^{1,2}$  be, as before, such that  $G_g(w) < 0$ . Since  $W^{2,p}$  is dense in  $W^{1,2}$ , we can assume that  $w \in W^{2,p} \subset L^\infty$ . As before, considering  $u = \lambda w$  in the

functional (4.2), we get existence of  $v$  such that  $F(v) < 0$ . Set

$$F_k(u) := \int_M \left[ \frac{4(n-1)}{n-2} |d\phi|^2 + \text{Scal } \phi^2 + \frac{2}{N} f_k |\phi|^N \right] d\mu^g$$

So we have  $F_k(\phi_k) \leq F_k(v) \leq F(v) < 0$ . Now remark that  $F_k(\phi_k) \rightarrow_{k \rightarrow \infty} F(\phi)$ . This forces  $F(\phi) \leq F(v) < 0$  which shows that  $\phi \neq 0$ . By construction  $\phi \geq 0$  and from Harnack's inequality, we have  $\phi > 0$ . This ends the proof of Theorem 4.1. Uniqueness is obtained from Proposition 3.3.

#### REFERENCES

- [1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314, Springer-Verlag, Berlin, 1996. [2](#)
- [2] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. [9](#)
- [3] R. Bartnik and J. Isenberg, *The constraint equations*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 1–38. [1](#)
- [4] Haïm Brézis and Luc Oswald, *Remarks on sublinear elliptic equations.*, Nonlinear Anal., Theory Methods Appl. **10** (1986), 55–64 (English). [9](#)
- [5] Y. Choquet-Bruhat, *General relativity and the Einstein equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2009. [1](#)
- [6] M. Dahl, R. Gicquaud, and E. Humbert, *A limit equation associated to the solvability of the vacuum Einstein constraint equations by using the conformal method*, Duke Math. J. **161** (2012), no. 14, 2669–2697. [1](#), [9](#)
- [7] J. Dilts and D. Maxwell, *Yamabe classification and prescribed scalar curvature in the asymptotically Euclidean setting*, Comm. Anal. Geom. **26** (2018), no. 5, 1127–1168. [1](#), [2](#), [9](#)
- [8] R. Gicquaud, *Prescribed non positive scalar curvature on asymptotically hyperbolic manifolds with application to the Lichnerowicz equation*, arXiv:1909.05343. [1](#), [2](#), [9](#), [10](#)
- [9] ———, *Solutions to the Einstein constraint equations with a small TT-tensor and vanishing Yamabe invariant*, arXiv:1802.05080v2. [5](#)
- [10] R. Gicquaud and Q.A. Ngô, *A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor*, Class. Quantum Grav. **31** (2014), no. 19, 195014 (20pp). [1](#)
- [11] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. [7](#)
- [12] M. Holst, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett. **100** (2008), no. 16, 161101, 4. [1](#), [5](#)
- [13] ———, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613. [1](#), [5](#)
- [14] J. Isenberg, *Constant mean curvature solutions of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **12** (1995), no. 9, 2249–2274. [1](#)
- [15] D. Maxwell, *Rough solutions of the Einstein constraint equations on compact manifolds*, J. Hyperbolic Differ. Equ. **2** (2005), no. 2, 521–546. [5](#)
- [16] ———, *A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature*, Math. Res. Lett. **16** (2009), no. 4, 627–645. [1](#), [5](#), [9](#)
- [17] T. Ouyang, *On the positive solutions of semilinear equations  $\Delta u + \lambda u + hu^p = 0$  on compact manifolds. II*, Indiana Univ. Math. J. **40** (1991), no. 3, 1083–1141. [1](#)
- [18] ———, *On the positive solutions of semilinear equations  $\Delta u + \lambda u - hu^p = 0$  on the compact manifolds*, Trans. Amer. Math. Soc. **331** (1992), no. 2, 503–527. [1](#)
- [19] A. Rauzy, *Courbures scalaires des variétés d'invariant conforme négatif*, Trans. Amer. Math. Soc. **347** (1995), no. 12, 4729–4745. [1](#), [9](#)
- [20] J. Tang, *Solvability of the equation  $\Delta_g u + \tilde{S}u^\sigma = Su$  on manifolds*, Proc. Amer. Math. Soc. **121** (1994), no. 1, 83–92. [1](#)
- [21] M. E. Taylor, *Partial differential equations III. Nonlinear equations*, second ed., Applied Mathematical Sciences, vol. 117, Springer, New York, 2011. [8](#)

(R. Gicquaud) INSTITUT DENIS POISSON, UFR SCIENCES ET TECHNOLOGIE, FACULTÉ DE TOURS, PARC DE GRANDMONT, 37200 TOURS, FRANCE

E-mail address: [romain.gicquaud@lmpt.univ-tours.fr](mailto:romain.gicquaud@lmpt.univ-tours.fr)