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Quantization of hyper-elliptic curves from isomonodromic systems and topological recursion

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Abstract : We prove that the topological recursion formalism can be used to compute the WKB expansion of solutions of second order differential operators obtained by quantization of any hyper-elliptic curve. We express this quantum curve in terms of spectral Darboux coordinates on the moduli space of meromorphic \mathfrak{sl}_2 -connections on \mathbb{P}^1 and argue that the topological recursion produces a $2g$ -parameter family of associated tau functions, where $2g$ is the dimension of the moduli space considered. We apply this procedure to the 6 Painlevé equations which correspond to $g = 1$ and consider a $g = 2$ example.

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1 Introduction

Since the pioneering works of Kontsevich [30] proving Witten conjecture [35] on intersection numbers on the moduli space of Riemann surfaces, it is known that there is a big interplay between the theory of integrable systems and enumerative geometry, going through mirror symmetry. The original presentation states that there exists a generating function for intersections of ψ classes on $\mathcal{M}_{g,n}$ which is a KdV tau function. However, one can present it in a slightly different way as follows. Let us define a different generating series by

$$\Psi^K(x, \hbar) := \exp \left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{\mathbf{k} \in \mathbb{N}^n} \int_{\mathcal{M}_{g,n}} \prod_{j=1}^n \left[\psi_j^{k_j} \frac{(2k_i - 1)!!}{x^{k_i + \frac{1}{2}}} \right] \right]. \quad (1-1)$$

The Virasoro constraints satisfied by the corresponding KdV tau function is equivalent to the Airy equation

$$\left(\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{x}{4} \right) \Psi^K(x, \hbar) = 0. \quad (1-2)$$

This differential operator is related to our enumerative problem by mirror symmetry. Indeed, $\Psi^K(x, \hbar)$ is a generating series for Gromov-Witten invariants of the point. This Gromov-Witten theory turns out to have a Landau-Ginzburg model defined on the Riemann surface $\{y^2 = \frac{x}{4}\} \subset \mathbb{C}^2$. The Airy equation can thus be interpreted as a quantization of the curve mirror symmetric to the Gromov-Witten theory of the point.

One may wonder if this procedure mixing mirror symmetry and quantization of algebraic curves can be applied to other problems of enumerative geometry. More precisely, given a problem of enumerative geometry, can one build a generating series $\Psi(x, \hbar)$ for the numbers of interest such that it is annihilated by a differential operator

$$\hat{P}\left(x, \hbar \frac{\partial}{\partial x}\right) \Psi(x, \hbar) = 0 \quad (1-3)$$

whose classical limit $P(x, y) := \lim_{\hbar \rightarrow 0} \hat{P}(x, y)$ defines a Riemann surface $\{P(x, y) = 0\} \subset \mathbb{C}^2$ which is mirror to our enumerative problem? The topological recursion formalism [17] developed in the last 10 years and giving a universal solution to semi-simple Gromov-Witten theories [13] is conjectured to give a positive answer to this question in a large setup.

This problem consists in proving that the application of the topological recursion to a classical curve $\Sigma := \{P(x, y) = 0\}$ allows to build a generating series $\Psi(x, \hbar)$ solution to a differential equation $\hat{P}\left(x, \hbar \frac{\partial}{\partial x}\right) \Psi(x, \hbar) = 0$ [6]. This claim has been proved in the case when Σ has genus zero [8, 32] and in a variety of examples (see [33] for a nice review of this topic) when considering variables in \mathbb{C}^* instead of \mathbb{C} as well. Unfortunately, until recently no example with higher genus Riemann surface Σ was worked out. Some attempts in this direction have been made in the context of Painlevé equation where the expected genus is equal to 1. In [18, 27, 28, 29], solutions to Painlevé equations were built using topological recursion starting from a singular genus 0 curve, namely by considering a singular point in a corresponding moduli space of quadratic differentials. These works use the general result of [3] which is valid only for genus 0 Riemann surfaces.

A breakthrough has been made in [26] where the author proved that the topological recursion can be used to quantize the Weierstrass curve $y^2 = x^3 + \alpha x + \beta$ for generic values of α and β . On the way, this allows to provide with a 2-parameter solution of Painlevé 1 equation.

This quantization procedure may not only be used to find solutions to Painlevé type equations but also to compute enumerative invariants as Gromov-Witten invariants or Hurwitz numbers [14] as explained above. The research activity on this topic is nowadays very active for its possible applications to the computation of knot invariants in the context of the volume conjecture [7, 11].

In the present article, we generalize Iwaki's wonderful result to the quantization of any hyper-elliptic curve, paving the way to a possible generalization to any algebraic curve. Let us now summarize how this is done.

In Section 2, we recall the topological recursion formalism in the case of hyper-elliptic curves. Given a meromorphic quadratic differential ϕ_0 on \mathbb{P}^1 , let $\Sigma_{\phi_0} := \{(x, y) \in \mathbb{C}/y^2(dx)^2 = \phi_0\}$ be an associated compact Riemann surface. We explain how the recursion associates a set of multilinear forms $\omega_{h,n}$ on $\Sigma_{\phi_0}^n$ to such a quadratic differential together with a Torelli marking of Σ_{ϕ_0} .

In Section 3, we recall some well-known facts about the moduli space of quadratic differentials and explain how the output $(\omega_{h,n})_{h,n}$ varies when ϕ_0 moves in this space.

After these background sections, we present the first important result of this paper. Given the same data as above, one can collect the result of the topological recursion into a single generating series (see definition 4.3)

$$\psi(x, \hbar) = \exp \left[\sum_{h \geq 0} \sum_{n \geq 1} \frac{\hbar^{2h-2+n}}{n!} \int_{\gamma(x)} \cdots \int_{\gamma(x)} \omega_{h,n} \right] \quad (1-4)$$

where $\gamma(x)$ is a well chosen integration path in Σ_{ϕ_0} with end points in the fiber above a point x in the base curve \mathbb{P}^1 . If this function does not satisfy any differential equation by quantization of Σ_{ϕ_0} , we prove in Theorem 4.1 that it is a solution to a PDE with respect to x and a subset of coordinates on the moduli space of quadratic differentials. The proof of this first important result follows the line of [12] taking into account the additional contributions involved by the non-vanishing genus of Σ_{ϕ_0} .

In order to obtain a function annihilated by a quantum curve, one needs to correct $\psi(x, \hbar)$ by exponentially small corrections in \hbar to build a “non-perturbative” analog. This non-perturbative wave function $\Psi(x, \hbar)$ is built in Definition 5.1 as a Fourier transform of $\psi(x, \hbar)$. We finally prove the

main result of this article in Theorem 5.1. The latter proves that $\Psi(x, \hbar)$ is annihilated by “quantum curve”

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 R(x) \frac{\partial}{\partial x} - \hbar Q(x) - \mathcal{H}(x) \right] \Psi(x, \hbar) = 0. \quad (1-5)$$

In this expression, all functions are rational functions of x with poles at the poles of ϕ_0 together with simple poles at a set of “apparent singularities” $(q_i)_{i=1}^g$ where g is the genus of Σ_{ϕ_0} . Theorem 5.1 provides explicit expressions of these different functions. The proof of this theorem mainly relies on the fact that $\Psi(x, \hbar)$ is a solution to a PDE similar to the one satisfied by $\psi(x, \hbar)$ but that additionally it also has “good” monodromies along non-trivial cycles as explained in Lemma 5.1. This simple property allows to prove that some corresponding Wronskian functions are rational in x eventually leading to the result.

In order to have a better geometrical understanding of the quantum curve thus produced, we linearize the system in Section 6 replacing the quantum curve by a \mathfrak{sl}_2 -connection on \mathbb{P}_1 . We then study the corresponding characteristic variety, the \hbar -deformed spectral curve in Theorem 6.1. From this point of view, the topological recursion produces some flows in the \hbar -direction in a moduli space of quadratic differentials starting from the initial value ϕ_0 . Finally, in Section 7, we embed the result in a corresponding isomonodromic system explaining how our procedure allows to build isomonodromic tau functions.

In Section 8, we apply the procedure to the simplest cases. We thus get 2-parameter solutions to the 6 Painlevé equations as well as corresponding tau functions. We also consider a genus 2 example leading to the second element in Painlevé 2 hierarchy.

The present work gives one possible quantization of a spectral curve and is partly motivated by applications in mathematical physics. Indeed, recent progresses have been made in the computation of isomonodromic tau functions motivated by the possibility of interpreting the latter in terms of conformal blocks in associated Conformal Field Theories [10, 23, 34]. From this perspective, our work only considers a base curve equal to \mathbb{P}^1 and the Lie algebra \mathfrak{sl}_2 . We shall naturally consider a generalization of our work to the higher genus base curves and arbitrary semi-simple Lie algebras. We hope that the present article could generalize nicely to these cases. Indeed, the PDE of Theorem 4.1 can be thought of as a loop equation in the general topological recursion formalism and should possibly be obtained for this general setup. The second step starting from this PDE to the quantum curve only uses the monodromy properties of the non-perturbative wave functions along cycles in the spectral cover. We believe that this step can be adapted to the general setup as well.

On another hand, we build only formal trans-series solutions to differential equations in the present article. It is fundamental to understand if, and when, these formal objects admit some Borel summability properties in order to study their Stokes properties in the spirit of the exact WKB analysis of [24, 25]. We hope to address this issue as well as the related questions of the dependance of the wave functions to a choice of Torelli marking in a future work. This second question can be interpreted as asking if our quantization procedure depends on a choice of polarization. A choice of Torelli marking can indeed probably be interpreted as a choice of real polarization. Changing such a polarization will probably lead to modular properties related to cluster transformations as in the recent works on exact WKB computations mentioned above.

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As this article was being written, Bertrand Eynard and Elba Garcia-Failde informed us that they were working on a similar problem and propose an independent solution using different methods. We would like to thank them for letting us know about their work.

2 Quadratic differentials and topological recursion

In this section, we recall the formalism of topological recursion [17, 5] starting from the data of a quadratic differential on the Riemann sphere.

2.1 Quadratic differentials and initial data

Let ϕ be a meromorphic quadratic differential on \mathbb{P}^1 , namely it reads

$$\phi(x) = f_\phi(x) (dx)^2 \quad (2-1)$$

where $f_\phi(x)$ is a rational function of x . Let us denote by Σ_ϕ the compact Riemann surface:

$$\Sigma_\phi = \overline{\{(x, y) \in \mathbb{C}^2 \mid y^2 = f_\phi(x)\}} \quad (2-2)$$

where $\bar{\cdot}$ denotes the compactification at $x = \infty$. For reasons explained below, we sometimes call Σ_ϕ a *classical spectral curve*.

Definition 2.1 (Admissible curves). *A quadratic differential ϕ is called admissible if Σ_ϕ is a smooth algebraic curve such that, away from $x = \infty$, the poles of ϕ are distinct from the critical values of the map $x : \Sigma_\phi \rightarrow \mathbb{P}^1$.*

Let us denote by P_ϕ the set of poles of ϕ on \mathbb{P}^1 and by \mathcal{P}_ϕ the set composed of the pre-images by x of those poles on Σ_ϕ .

Let us denote by $\mathcal{R}_\phi := \{a_i\}$ the set of finite ramification points of the map x defined by

$$\begin{cases} x(a_i) = u_i \neq \infty \\ dx(a_i) = 0 \end{cases} \quad (2-3)$$

In addition, we will the set $\{u_i = x(a_i)\}$ will be referred to as critical points.

In its original version [17], the topological recursion is a procedure taking as input an algebraic curve together with a Torelli marking. For this purpose, in the present paper, our input is defined in the following way.

Definition 2.2 (Admissible initial data). *An admissible initial data is the data of a pair $(\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)})$ where ϕ_0 is an admissible quadratic differential, $g(\phi_0)$ denotes the genus of Σ_{ϕ_0} and $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)}$ is a symplectic basis of $H_1(\Sigma_{\phi_0}, \mathbb{Z})$.*

To any such initial data, one associates the initial values $\omega_{0,1} \in H^0(\Sigma_{\phi_0} \setminus \mathcal{P}_\phi, K_{\Sigma_{\phi_0} \setminus \mathcal{P}_\phi})$ and $\omega_{0,2} \in H^0(\Sigma_{\phi_0} \times \Sigma_{\phi_0}, (p_1^ K_{\Sigma_{\phi_0}} \otimes p_2^* K_{\Sigma_{\phi_0}})(-2\Delta))$ where p_1 and p_2 the projections $\Sigma_{\phi_0} \times \Sigma_{\phi_0} \rightarrow \Sigma_{\phi_0}$ on the first and second factor respectively and Δ is the diagonal divisor in $\Sigma_{\phi_0}^2$ in the following way.*

By choosing a branch of the square root once and for all, one defines

$$\omega_{0,1}[\phi_0] := [\phi_0]^{\frac{1}{2}} := y dx \quad (2-4)$$

where $y^2 = f_\phi$.

$\omega_{0,2}[\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g]$ is defined as the unique differential on $\Sigma_{\phi_0} \times \Sigma_{\phi_0}$ whose only singularities are double poles without residue on the diagonal and normalized by

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + O(dz_1 \otimes dz_2) \quad (2-5)$$

in any local coordinates as $z_1 \rightarrow z_2$ and

$$\forall i \in \llbracket 1, g \rrbracket : \oint_{z_1 \in \mathcal{A}_i} \omega_{0,2}(z_1, z_2) = 0. \quad (2-6)$$

It is worth noticing that $\omega_{0,1}$ does not depend on the Torelli marking while the latter fixes $\omega_{0,2}$ uniquely.

2.2 Topological recursion

Let us now present the topological recursion formalism in this simple setup. It is an inductive procedure associating to any admissible initial data $(\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)})$, a set of differential forms $\omega_{h,n}[\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)}] \in H^0(\Sigma_{\phi_0}^n, K_{\Sigma_{\phi_0}}^{\otimes n}(-(6h-6+4n)\mathcal{R}^{[n]}))$ where $\mathcal{R}^{[n]} = \bigsqcup_{i=1}^n p_i^{-1}(\mathcal{R})$, $p_i : \Sigma_{\phi_0}^n \rightarrow \Sigma_{\phi_0}$ being the projection along the i^{th} component. It is defined as follows:

Definition 2.3. *For any admissible initial data $(\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)})$, let us define $\omega_{0,1}[\phi_0]$ and $\omega_{0,2}[\phi_0, (\mathcal{A}_i, \mathcal{B}_i)]$ as above and, for $2h-2+n \geq 0$, $\omega_{h,n}[\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)}] \in H^0(\Sigma_{\phi_0}^n, K_{\Sigma_{\phi_0}}^{\otimes n}(-(6g-6+4n)\mathcal{R}^{[n]}))$ is defined inductively by*

$$\begin{aligned} \omega_{h,n}(z_1, \dots, z_n) &:= \sum_{p \in \mathcal{R}} \operatorname{Res}_{z \rightarrow p} \frac{\int_{\sigma(z)}^z \omega_{0,2}(z_1, \cdot)}{2(\omega_{0,1}(z) - \omega_{0,1}(\sigma(z)))} \left[\omega_{h-1, n+1}(z, \sigma(z), z_2, \dots, z_n) \right. \\ &\quad \left. + \sum_{\substack{h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\} \\ (h_1, |A|) \notin \{(0,0), (h, n-1)\}}} \omega_{h_1, |A|+1}(z, A) \omega_{h_2, |B|+1}(z, B) \right] \end{aligned}$$

where $\sigma : \Sigma_{\phi_0} \rightarrow \Sigma_{\phi_0}$ is the hyper-elliptic involution, namely, it is defined by

$$\forall z \in \Sigma_{\phi_0} \setminus \mathcal{R}, \quad x(z) = x(\sigma(z)) \quad \text{and} \quad \sigma(z) \neq z. \quad (2-7)$$

For $h \geq 2$, we define the free energies $\omega_{h,0}[\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\phi_0)}] \in \mathbb{C}$ by

$$\omega_{h,0} := \frac{1}{2-2h} \sum_{p \in \mathcal{R}} \operatorname{Res}_{z \rightarrow p} \omega_{h,1}(z) \int_o^z \omega_{0,1}$$

where $o \in \Sigma_{\phi_0}$ is an arbitrary base point of which $\omega_{h,0}$ is independent.

Finally, for $h \in \{0, 1\}$, we can define $\omega_{0,0}$ and $\omega_{1,0}$. Their explicit expressions being technical and useless for our purpose, we refer the reader to [17] for them. The only important point is that they are defined in such a way that they satisfy the upcoming variational formulas of Lemma 3.1.

In the expression above and in the following, we do not write the dependence of $\omega_{h,n}$ in the initial data except when we want to emphasize it.

3 Space of quadratic differentials and deformations

An important property of the objects built by topological recursion is that they have nice properties under variations of the initial data. One can actually think of $\omega_{h,n}$ as a generating series of n^{th} order derivatives of $\omega_{h,0}$ with respect to an infinite number of parameters in the space of initial data. In order to make this statement more precise, we shall now consider a finite dimensional sub-space of the space of initial data.

3.1 Space of quadratic differentials and coordinates

Let $n \geq 0$ be a positive integer and let $(X_\nu)_{\nu=1}^n$ be a set of distinct points in \mathbb{C} . Let $(r_\nu)_{\nu=1}^n \in (\mathbb{N} \setminus \{0\})^n$ be a set of degrees associated to the points $(X_\nu)_{\nu=1}^n$. Let us denote by $r_\infty \in \mathbb{N} \setminus \{0\}$ a

degree at infinity. Without loss of generality, we shall assume the existence of a pole at infinity in this article to avoid cumbersome notations. One defines by $D = \sum_{\nu=1}^n r_\nu(X_\nu) + r_\infty(\infty)$ the resulting divisor.

Definition 3.1 (Space of quadratic differentials $\mathcal{Q}(\mathbb{P}^1, D, n_\infty)$). *Given a divisor $D = \sum_{\nu=1}^n r_\nu(X_\nu) + r_\infty(\infty)$ and $n_\infty \in \{0, 1\}$, let $\mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ be the moduli space of quadratic differentials on \mathbb{P}^1 such that any $\phi \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ has a pole of order at most $2r_\nu$ at the finite pole $X_\nu \in \mathcal{P}^{finite}$ and a pole of order at most $2r_\infty - n_\infty$ at infinity.*

Note in particular that **the degree of the pole at infinity may be even or odd while for finite poles we assume that the degrees are always even**. This requirement will be more transparent from the integrable systems perspective where we may allow at most only one odd pole to avoid degeneracy. Using a trivial reparametrization, this pole may always be chosen to be ∞ , a widespread convention used in many examples like Painlevé equations.

$\mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ is a finite dimensional space equipped with a Poisson structure (see for exemple [9, 4] for a recent account close to our presentation). In the present paper, we shall be interested only in the formal neighborhood of a point in this space hence we shall not discuss any global property of such spaces. On the contrary, we will now describe local coordinates around a quadratic differential ϕ_0 .

A first set of coordinates is given by the coefficients of the partial fraction decomposition of f_ϕ

$$f_\phi = \sum_{k=0}^{2(r_\infty-2)-n_\infty} H_{\infty,k} x^k + \sum_{\nu=1}^n \sum_{k=1}^{2r_\nu} \frac{H_{\nu,k}}{(x - X_\nu)^k}. \quad (3-1)$$

As a moduli space, this space can be equipped with a Poisson structure in such a way that part of these coefficients are Casimirs (see for example [1, 2]). Fixing them allows restricting to a symplectic leaf of this Poisson manifold. We shall now present and use different coordinates that are natural from the topological recursion perspective, allowing fixing the values of these Casimirs.

Definition 3.2. *Given $\phi \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ the associated one form $\omega_{0,1} := \phi^{\frac{1}{2}}$ is meromorphic on Σ_ϕ with poles along the pre-image of the points in the divisor D . Let us define the pre-images of the poles by $\{b_\nu^+, b_\nu^-\} := x^{-1}(X_\nu)$ for $\nu \in \llbracket 1, n \rrbracket$ and $\{b_\infty^+, b_\infty^-\} := x^{-1}(\infty)$ if $n_\infty = 0$ and $\{b_\infty\} := x^{-1}(\infty)$ if $n_\infty = 1$. Let us denote by $\mathcal{P} := \mathcal{P}^{finite} \sqcup \mathcal{P}^\infty$ the set of poles on Σ_ϕ where $\mathcal{P}^{finite} := \{b_\nu^+, b_\nu^-\}_{\nu=1}^n$ and $\mathcal{P}^\infty := \{b_\infty^+, b_\infty^-\}$ (resp. $\mathcal{P}^\infty := \{b_\infty\}$) if $n_\infty = 0$ (resp. $n_\infty = 1$).*

To obtain a symplectic leaf of our system, one shall fix the singular behavior of $\omega_{0,1}$ around its poles, i.e. its residue and singular type defined below.

Definition 3.3 (Times). *For any $\mathbf{T} \in \mathbb{C}^{r_\infty + \sum_{\nu=1}^n r_\nu - n_\infty}$, where \mathbf{T} has components labeled $T_{\infty,k}$ with $k \in \llbracket 1, r_\infty \rrbracket$ (resp. $k \in \llbracket 2, r_\infty \rrbracket$) if $n_\infty = 0$ (resp. if $n_\infty = 1$) and $T_{\nu,k}$ for $\nu \in \llbracket 1, n \rrbracket$ and $k \in \llbracket 1, r_\nu \rrbracket$, let $\mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T}) \subset \mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ be the space of quadratic differentials (known as the Whitham-Krichever differentials) such that $\omega_{0,1}[\phi]$ has the following Laurent expansions*

— around b_ν^\pm ,

$$\omega_{0,1}[\phi] = \pm \sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x - X_\nu)^k} + O(dx);$$

— around b_∞^\pm ,

$$\omega_{0,1}[\phi] = \pm \sum_{k=1}^{r_\infty} T_{\infty,k} (x^{-1})^{-k} d(x^{-1}) + O(d(x^{-1})) = \mp \sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2} dx + O(x^{-2} dx)$$

if $n_\infty = 0$;

— around b_∞ ,

$$\omega_{0,1}[\phi] = \sum_{k=2}^{r_\infty} T_{\infty,k} x^{k-1} d(x^{-\frac{1}{2}}) + O(d(x^{-\frac{1}{2}})) = - \sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{2} x^{k-\frac{5}{2}} dx + O(d(x^{-\frac{1}{2}}))$$

if $n_\infty = 1$.

For generic values of the times \mathbf{T} , $\mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T})$ is a symplectic manifold of real dimension equal to twice the genus of Σ_ϕ

$$g(\Sigma_\phi) = r_\infty + \sum_{\nu=1}^n r_\nu - 3. \quad (3-2)$$

One can understand this by remarking that fixing the value of \mathbf{T} fixes the values of the coefficients $\{H_{\infty,k}\}_{k=r_\infty-3}^{2r_\infty-4-n_\infty}$ and $\{H_{\nu,k}\}_{k=r_\nu+1}^{2r_\nu}$ for $\nu \in \llbracket 1, n \rrbracket$ unambiguously. The latter are Casimirs of our system. The remaining $g(\Sigma_\phi)$ coefficients $\{H_{\infty,k}\}_{k=0}^{r_\infty-4}$ and $\{H_{\nu,k}\}_{k=1}^{r_\nu}$ for $\nu \in \llbracket 1, n \rrbracket$ are Hamiltonians of the residual integrable system.

We will not use these coefficients to parametrize the space $\mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T})$ but rather homological coordinates given by the periods of $\omega_{0,1}[\phi]$ in the spirit of the original topological recursion [17] and more recently [4, 9, 15].

Definition 3.4 (Periods). *For admissible initial data $(\phi, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\Sigma_\phi)})$, let the period vector $\epsilon \in \mathbb{C}^{g(\Sigma_\phi)}$ be defined by*

$$\forall i \in \llbracket 1, g(\Sigma_\phi) \rrbracket : \epsilon_i := \oint_{\mathcal{A}_i} \omega_{0,1}. \quad (3-3)$$

Remark that, if the times \mathbf{T} depend only on the quadratic differential, the periods depend on a choice of Torelli marking. This choice of Torelli marking can be interpreted as a choice of real polarization [4].

Hence, any $\phi \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ takes the form

$$\phi(\mathbf{T}, \epsilon) = \left[\sum_{k=r_\infty-3}^{2(r_\infty-2)-n_\infty} H_{\infty,k}(\mathbf{T}) x^k + \sum_{k=0}^{r_\infty-4} H_{\infty,k}(\mathbf{T}, \epsilon) x^k + \sum_{\nu=1}^n \left(\sum_{k=r_\nu+1}^{2r_\nu} \frac{H_{\nu,k}(\mathbf{T})}{(x-X_\nu)^k} + \sum_{k=1}^{r_\nu} \frac{H_{\nu,k}(\mathbf{T}, \epsilon)}{(x-X_\nu)^k} \right) \right] \quad (3-4)$$

where \mathbf{T} and ϵ are local coordinates.

3.2 Variational formulas

Fixing a divisor D , the topological recursion taking as initial data a quadratic differential $\phi_0 \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ is a procedure generating functions on a neighborhood of $\phi_0 \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ with value in different spaces of differentials. Using the local coordinates given by the times and periods described above, one can study them as functions of the latter. The general theory developed for the topological recursion provides a nice way to compute the derivatives of such functions [17].

Remark that, because the coefficients of the expansion around b_∞^+ and b_∞^- are not independent, the variational formulas for $T_{\infty,k}$ include residues both at b_∞^+ and b_∞^- . The same subtlety arises for the coefficients of the expansion around b_ν^\pm .

Lemma 3.1. *[Variational formulas]*

Given admissible initial data $(\phi_0, (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g(\Sigma_{\phi_0})})$, the output of the topological recursion $(\omega_{h,n})_{n \geq 0, h \geq 0}$ satisfies

— *For the times associated to ∞ :*

$$\forall k \geq 2, \quad \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\infty,k}} = \operatorname{Res}_{p \rightarrow b_\infty^+} \omega_{h,n+1}(p, \mathbf{z}) \frac{x(p)^{k-1}}{k-1} - \operatorname{Res}_{p \rightarrow b_\infty^-} \omega_{h,n+1}(p, \mathbf{z}) \frac{x(p)^{k-1}}{k-1} \quad (3-5)$$

if $n_\infty = 0$ and

$$\forall k \geq 2, \quad \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\infty,k}} = \operatorname{Res}_{p \rightarrow b_\infty} \omega_{h,n+1}(p, \mathbf{z}) \frac{x(p)^{k-1-\frac{1}{2}}}{2k-3} \quad (3-6)$$

otherwise.

— For the times associated to b_ν^\pm :

$$\forall k \geq 2, \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\nu,k}} = \operatorname{Res}_{p \rightarrow b_\nu^+} \omega_{h,n+1}(p, \mathbf{z}) \frac{(x(p) - X_\nu)^{-k+1}}{k-1} - \operatorname{Res}_{p \rightarrow b_\nu^-} \omega_{h,n+1}(p, \mathbf{z}) \frac{(x(p) - X_\nu)^{-k+1}}{k-1} \quad (3-7)$$

and

$$\frac{\partial \omega_{h,n}(\mathbf{z})}{\partial T_{\nu,1}} = \int_{b_\nu^+}^p \omega_{h,n+1}(\cdot, \mathbf{z}) - \int_{b_\nu^-}^p \omega_{h,n+1}(\cdot, \mathbf{z}). \quad (3-8)$$

— For the periods:

$$\forall j \in \llbracket 1, g \rrbracket : \frac{\partial \omega_{h,n}(\mathbf{z})}{\partial \epsilon_j} = \frac{1}{2\pi i} \oint_{\mathcal{B}_j} \omega_{h,n+1}(\cdot, \mathbf{z}). \quad (3-9)$$

Remark that one can perform variations with respect to any time by considering ϕ_0 as a point in $\mathcal{Q}(\mathbb{P}^1, D, n_\infty)$ for r_ν and r_∞ large enough.

From this point of view, $\omega_{h,n+m}$ can be thought of as a generating function for the m^{th} derivative of $\omega_{h,n}$ with respect to the parameters \mathbf{T} and ϵ . In particular, one can express the coefficients of the expansion of $\omega_{0,1}$ around any of these poles in these terms.

Corollary 3.1. *The expansion of $\omega_{0,1}$ in local coordinates around its poles reads*

— around b_ν^\pm ,

$$\forall l \geq 2 : \omega_{0,1}[\phi] = \pm \sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x - X_\nu)^k} \pm \sum_{k=2}^l \frac{k-1}{2} \frac{\partial \omega_{0,0}}{\partial T_{\nu,k}} (x - X_\nu)^{k-2} dx + O((x - X_\nu)^{l-1} dx)$$

— around b_∞^\pm ,

$$\forall l \geq 2 : \omega_{0,1}[\phi] = \mp \sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2} dx \mp \sum_{k=2}^l \frac{k-1}{2} \frac{\partial \omega_{0,0}}{\partial T_{\infty,k}} x^{-k} dx + O(x^{-l-1} dx)$$

if $n_\infty = 0$;

— around b_∞ ,

$$\forall l \geq 2 : \omega_{0,1}[\phi] = - \sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{2} x^{k-\frac{5}{2}} dx - \sum_{k=2}^l \frac{2k-3}{2} \frac{\partial \omega_{0,0}}{\partial T_{\infty,k}} x^{-k+\frac{1}{2}} dx + O(x^{-l-\frac{1}{2}} dx)$$

if $n_\infty = 1$.

Thanks to this simple corollary, one can get some expressions of the quadratic differential emphasizing the dependence on the times and the periods.

Lemma 3.2. *A quadratic differential $\phi \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T})$ reads*

$$\begin{aligned} f_\phi &= \left[\left(\sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2} \right)^2 \right]_{\infty,+} + \sum_{\nu=1}^n \left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x - X_\nu)^k} \right)^2 \right]_{X_\nu,-} \\ &+ \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial \omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^n \sum_{k \in K_\nu} U_{\nu,k}(x) \frac{\partial \omega_{0,0}}{\partial T_{\nu,k}} \end{aligned} \quad (3-10)$$

if $n_\infty = 0$ and

$$\begin{aligned} f_\phi &= \left[\left(\sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{2} x^{k-\frac{5}{2}} \right)^2 \right]_{\infty,+} + \sum_{\nu=1}^n \left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k} \frac{dx}{(x - X_\nu)^k} \right)^2 \right]_{X_\nu,-} \\ &+ \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial \omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^n \sum_{k \in K_\nu} U_{\nu,k}(x) \frac{\partial \omega_{0,0}}{\partial T_{\nu,k}} \end{aligned} \quad (3-11)$$

if $n_\infty = 1$.

Here, $[f(x)]_{\infty,+}$ (resp. $[f(x)]_{X_\nu,-}$) refers to the positive part of the expansion in x of a function $f(x)$ around ∞ , including the constant term, (resp. the strictly negative part of the expansion in $(x - X_\nu)$ around X_ν) and we have defined

— $K_\infty = \llbracket 2, r_\infty - 2 \rrbracket$ and $\forall k \in K_\infty$:

$$U_{\infty,k}(x) := (k-1) \sum_{l=k+2}^{r_\infty} T_{\infty,l} x^{l-k-2}, \text{ if } n_\infty = 0 \quad (3-12)$$

and

$$U_{\infty,k}(x) := \left(k - \frac{3}{2}\right) \sum_{l=k+2}^{r_\infty} T_{\infty,l} x^{l-k-2}, \text{ if } n_\infty = 1 \quad (3-13)$$

— $K_\nu = \llbracket 2, r_\nu + 1 \rrbracket$ and $\forall k \in K_\nu$:

$$U_{\nu,k}(x) := (k-1) \sum_{l=k-1}^{r_\nu} T_{\nu,l} (x - X_\nu)^{-l+k-2} \quad (3-14)$$

Proof. The proof immediately follows from the partial fraction decomposition of f_ϕ and the expression of the first holomorphic terms of the expansion of $\omega_{0,1}$ around one of its poles using the variational formulas of Lemma 3.1. \square

Remark 3.1. Note that in the expression of Lemma 3.2, ϕ depends on the periods only through $\omega_{0,0}$.

3.3 Symmetries

In addition to the variational formulas, the output of the topological recursion is skew-symmetric under the hyper-elliptic involution σ , i.e.

$$\forall h \geq 0, \forall n \geq 1 : \omega_{h,n}(z_1, \dots, z_n) + \omega_{h,n}(\sigma(z_1), z_2, \dots, z_n) = \delta_{h,0} \delta_{n,2} \frac{dx(z_1) dx(z_2)}{(x(z_1) - x(z_2))^2}. \quad (3-15)$$

One also has

$$\forall (z_1, z_2) \in (\Sigma_\phi)^2 \setminus \Delta, \omega_{0,2}(z_1, z_2) = \omega_{0,2}(\sigma(z_1), \sigma(z_2)). \quad (3-16)$$

One can use these symmetry properties to obtain easily a few equalities that we shall use repetitively in the following:

$$\forall (z_1, z_2) \in \Sigma_\phi^2 \setminus \Delta, \int_{\sigma(z_1)}^{z_1} \omega_{0,2}(z_2, \cdot) = - \int_{\sigma(z_1)}^{z_1} \omega_{0,2}(\sigma(z_2), \cdot) \quad (3-17)$$

which implies, for any ramification point a (thus satisfying $\sigma(a) = a$),

$$\forall (z_1, z_2) \in \Sigma_\phi^2 \setminus \Delta, \int_a^{z_1} \int_{\sigma(z_2)}^{z_2} \omega_{0,2} = - \int_a^{\sigma(z_1)} \int_{\sigma(z_2)}^{z_2} \omega_{0,2}. \quad (3-18)$$

4 The perturbative world

4.1 Perturbative partition function

Given an admissible initial data $(\phi_0, (\mathcal{A}_i, \mathcal{B}_i))$, one can build generating functions collecting the quantities $(\omega_{h,n})_{h \geq 0, n \geq 0}$ defined in Section 2.2. Since the variational formulas allow to think of $\omega_{h,n}$ as generating functions for variations of $\omega_{h,0}$ it makes sense to collect the information obtained from the topological recursion only in the latter. For this reason, one defines a partition function as

Definition 4.1 (Perturbative partition function). *Given an admissible initial data, one defines the perturbative partition function as a function of a formal parameter \hbar and the initial data by*

$$Z^{\text{pert}}(\hbar, \mathbf{T}, \epsilon) := \exp \left(\sum_{h=0}^{\infty} \hbar^{2h-2} \omega_{h,0}(\mathbf{T}, \epsilon) \right). \quad (4-1)$$

It follows from this definition that $Z^{\text{pert}}(\hbar, \mathbf{T}, \epsilon) \exp(-\hbar^{-2} \omega_{0,0})$ is a formal power series in \hbar^2 .

4.2 Perturbative wave functions

In order to quantize the classical spectral curve, we would like to define some wave functions as some particular generating series of the correlators $\omega_{h,n}$ for $n \geq 1$. We first define:

Definition 4.2 (Definition of $(F_{h,n})_{h \geq 0, n \geq 1}$ by integration of the correlators). *For $n \geq 1$ and $h \geq 0$ such that $2h - 2 + n \geq 1$, let us define*

$$F_{h,n}(z_1, \dots, z_n) = \frac{1}{2^n} \int_{\sigma(z_1)}^{z_1} \cdots \int_{\sigma(z_n)}^{z_n} \omega_{h,n}$$

where one integrates each of the n variables along paths linking two Galois conjugate points inside a fundamental domain cut out by the chosen symplectic basis $(\mathcal{A}_j, \mathcal{B}_j)_{1 \leq j \leq g(\Sigma_\phi)}$.

For $(h, n) = (0, 1)$ we define similarly

$$F_{0,1}(z) := \frac{1}{2} \int_{\sigma(z)}^z \omega_{0,1}.$$

Finally, for $(h, n) = (0, 2)$ one cannot define $F_{0,2}$ in the exact same way since $\omega_{0,2}$ has poles on the diagonal Δ . One thus needs to regularize it by removing the polar part. Hence, we define

$$F_{0,2}(z_1, z_2) := \frac{1}{4} \int_{\sigma(z_1)}^{z_1} \int_{\sigma(z_2)}^{z_2} \omega_{0,2} - \frac{1}{2} \ln(x(z_1) - x(z_2))$$

which also reads, in terms of theta functions,

$$F_{0,2}(z_1, z_2) = \frac{1}{4} \ln \left(\frac{\Theta(\mathbf{v}(z_1) - \mathbf{v}(z_2) + \mathbf{c}) \Theta(\mathbf{v}(\sigma(z_1)) - \mathbf{v}(\sigma(z_2)) + \mathbf{c})}{\Theta(\mathbf{v}(z_1) - \mathbf{v}(\sigma(z_2)) + \mathbf{c}) \Theta(\mathbf{v}(\sigma(z_1)) - \mathbf{v}(z_2) + \mathbf{c})} \right) - \frac{1}{2} \ln(x(z_1) - x(z_2))$$

where \mathbf{v} denotes the Abel-Jacobi map and \mathbf{c} is a non-singular half-integer odd characteristic.

Remark 4.1. Note that since $\mathbf{v}(\sigma(z)) = -\mathbf{v}(z)$, and by skew-symmetry of the theta function, $F_{0,2}(z_1, z_2)$ may alternatively be written as

$$F_{0,2}(z_1, z_2) = \frac{1}{2} \ln \left(\frac{\Theta(\mathbf{v}(z_1) - \mathbf{v}(z_2) + \mathbf{c})}{\Theta(\mathbf{v}(z_1) - \mathbf{v}(\sigma(z_2)) + \mathbf{c})} \right) - \frac{1}{2} \ln(x(z_1) - x(z_2)) \quad (4-2)$$

Another useful way to rewrite $F_{0,2}(z_1, z_2)$ is the following lemma.

Lemma 4.1. *For any pair of distinct ramification points (a_i, a_j) , one has*

$$F_{0,2}(z_1, z_2) = - \int_{a_i}^{\sigma(z_1)} \int_{a_j}^{z_2} \omega_{0,2} - \frac{1}{2} \log \left(\frac{(u_i - x(z_2))(x(z_1) - u_j)}{u_i - u_j} \right) \quad (4-3)$$

where we recall that $u_i := x(a_i)$. In particular, this reformulation shows that $F_{0,2}(z, z)$ is well-defined.

Proof. Since (a_i, a_j) are ramification points, they satisfy $\sigma(a_i) = a_i$ and $\sigma(a_j) = a_j$. Then, we have

$$\frac{1}{4} \int_{\sigma(z_1)}^{z_1} \int_{\sigma(z_2)}^{z_2} \omega_{0,2} = \frac{1}{4} \left[\int_{a_i}^{z_1} \int_{\sigma(z_2)}^{z_2} \omega_{0,2} - \int_{a_i}^{\sigma(z_1)} \int_{\sigma(z_2)}^{z_2} \omega_{0,2} \right] \stackrel{((3-18))}{=} - \frac{1}{2} \int_{a_i}^{\sigma(z_1)} \int_{\sigma(z_2)}^{z_2} \omega_{0,2}$$

$$\begin{aligned}
&= -\frac{1}{2} \int_{a_i}^{\sigma(z_1)} \left[\int_{a_j}^{z_2} \omega_{0,2} - \int_{a_j}^{\sigma(z_2)} \omega_{0,2} \right] \\
&\stackrel{z \rightarrow \sigma(z)}{=} -\frac{1}{2} \int_{a_i}^{\sigma(z_1)} \int_{a_j}^{z_2} [\omega_{0,2}(\cdot, \cdot) - \omega_{0,2}(\cdot, \sigma(\cdot))] \\
&\stackrel{((3-15))}{=} -\int_{a_i}^{\sigma(z_1)} \int_{a_j}^{z_2} \omega_{0,2} + \frac{1}{2} \int_{z=a_i}^{\sigma(z_1)} \int_{z'=a_j}^{z_2} \frac{dx(z)dx(z')}{(x(z) - x(z'))^2}
\end{aligned} \tag{4-4}$$

leading to the result. \square

One can now collect functions $(F_{h,n})_{h \geq 0, n \geq 1}$ into generating series.

Definition 4.3 (Definition of the perturbative wave functions). *We define:*

$$\begin{aligned}
S_{-1}^{\pm \text{pert}}(x) &:= \pm F_{0,1}(z(x)) \\
S_0^{\pm \text{pert}}(x) &:= \frac{1}{2} F_{0,2}(z(x), z(x)) \\
\forall k \geq 1, S_k^{\pm \text{pert}}(x) &:= \sum_{\substack{h \geq 0, n \geq 1 \\ 2h-2+n=k}} \frac{(\pm 1)^n}{n!} F_{h,n}(z(x), \dots, z(x))
\end{aligned}$$

where, for any $\lambda \in \mathbb{P}^1$, we define $z(\lambda) \in \Sigma_\phi$ as the unique point such that $x(z(\lambda)) = \lambda$ and $\omega_{0,1}(z(\lambda)) = \sqrt{\phi(\lambda)}$ ³. Remark that the \pm sign refers to the choice of sheet when choosing a point in the pre-image of λ . Note that $S_0^{\pm \text{pert}}$ makes sense since Lemma 4.1 allows the evaluation of $F_{0,2}$ on the diagonal. Eventually, we define the perturbative wave functions ψ_\pm by:

$$\psi_\pm(x, \hbar, \mathbf{T}, \epsilon) := \exp \left(\sum_{k \geq -1} \hbar^k S_k^{\pm \text{pert}}(x) \right)$$

Remark 4.2. The perturbative partition function ψ should actually be a function of the integration path on Σ_{ϕ_0} between two points. Since, for a given $x \in \mathbb{P}^1$ there are only two such paths up to homotopy in a given fundamental domain linking the two points in the fiber above x , we abusively write $\psi(x)$ as a function of x . The \pm index referring to a choice of orientation of the path, i.e. to the choice of one of the two possible integration contours.

By abuse of notations, we shall sometimes denote the perturbative wave functions as $\psi_\pm(x, \hbar)$ forgetting about the dependance on the other parameters. Remark that the perturbative wave functions satisfy

$$\psi_-(x, \hbar) = \psi_+(x, -\hbar). \tag{4-5}$$

Remark 4.3. Definition 4.3 is identical to the one proposed in [12] and used in many papers like [8, 15, 24, 25, 32].

4.3 Properties and PDE satisfied by the perturbative wave functions

The perturbative wave functions have non-trivial monodromies along elements of $H_1(\Sigma_\phi, \mathbb{Z})$ that can be computed explicitly.

Lemma 4.2. *The perturbative wave functions ψ_\pm satisfy the following properties.*

— For $i \in \llbracket 1, g \rrbracket$, the function $\psi_\pm(x, \hbar, \mathbf{T}, \epsilon)$ has a formal monodromy along \mathcal{A}_i given by

$$\psi_\pm(x, \hbar, \mathbf{T}, \epsilon) \mapsto e^{\pm 2\pi i \frac{\epsilon_i}{\hbar}} \psi_\pm(x, \hbar, \mathbf{T}, \epsilon). \tag{4-6}$$

— For $i \in \llbracket 1, g \rrbracket$, the function $\psi_\pm(x, \hbar, \mathbf{T}, \epsilon)$ has a formal monodromy along \mathcal{B}_i given by

$$\psi_\pm(x, \hbar, \mathbf{T}, \epsilon) \mapsto \frac{Z^{\text{pert}}(\hbar, \mathbf{T}, \epsilon \pm \hbar \mathbf{e}_i)}{Z^{\text{pert}}(\hbar, \mathbf{T}, \epsilon)} \psi_\pm(x, \hbar, \mathbf{T}, \epsilon \pm \hbar \mathbf{e}_i) \tag{4-7}$$

where $\mathbf{e}_i \in \mathbb{C}^g$ is the vector with the i^{th} component equal to 1 and all others vanishing.

3. Let us stress that, as an argument of the function, x refers to a point in \mathbb{P}^1 and not the map $x : \Sigma_{\phi_0} \rightarrow \mathbb{P}^1$. We hope that the reader will not be confused by this notation.

Proof. Reminding that the \mathcal{A} -periods of the $\omega_{h,n}$ are vanishing unless for $(h,n) = (0,1)$ where

$$\forall j \in \llbracket 1, g \rrbracket : \epsilon_j = \oint_{\mathcal{A}_j} \omega_{0,1}, \quad (4-8)$$

one immediately gets the first claim.

The second claim follows a simple computation similar to the one for Painlevé 1 written in [26].

The analytic continuation of the perturbative wave function along the cycle \mathcal{B}_j reads

$$\begin{aligned} & \exp \left[\sum_{h \geq 0} \sum_{n \geq 1} \frac{\hbar^{2h-2} (\pm \hbar)^n}{n! 2^n} \sum_{n_1+n_2=n} \binom{n}{n_1} 2^{n_1} \overbrace{\oint_{\mathcal{B}_j} \dots \oint_{\mathcal{B}_j}}^{n_1} \overbrace{\int_{\sigma(z)}^z \dots \int_{\sigma(z)}^z}^{n_2} \omega_{h,n} \right] \\ &= \exp \left[\sum_{h \geq 0} \sum_{n \geq 1} \hbar^{2h-2} (\pm \hbar)^n \sum_{n_1+n_2=n} \frac{1}{2^{n_2} n_1! n_2!} \frac{\partial^{n_1}}{\partial \epsilon_j^{n_1}} \overbrace{\int_{\sigma(z)}^z \dots \int_{\sigma(z)}^z}^{n_2} \omega_{h,n_2} \right]. \end{aligned} \quad (4-9)$$

Factoring out the terms with $n_2 = 0$ gives

$$\exp \left[\sum_{h \geq 0} \sum_{n \geq 1} \frac{\hbar^{2h-2} (\pm \hbar)^n}{n!} \frac{\partial^n}{\partial \epsilon_j^n} \omega_{h,0} \right] \exp \left[\sum_{n_1 \geq 0} \frac{(\pm \hbar)^{n_1}}{n_1!} \frac{\partial^{n_1}}{\partial \epsilon_j^{n_1}} \sum_{h \geq 0} \sum_{n_2 \geq 1} \hbar^{2h-2} (\pm \hbar)^{n_2} \frac{1}{2^{n_2} n_2!} \overbrace{\int_{\sigma(z)}^z \dots \int_{\sigma(z)}^z}^{n_2} \omega_{h,n_2} \right] \quad (4-10)$$

leading to the result. \square

The perturbative wave functions are built to be solutions of a PDE. Indeed, the main theorem of this section states that

Theorem 4.1. *The perturbative wave functions are solutions of the PDE*

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial}{\partial T_{\infty,k}} - \hbar^2 \sum_{\nu=1}^n \sum_{k \in K_{b_\nu}} U_{\nu,k}(x) \frac{\partial}{\partial T_{\nu,k}} - H(x) \right] \psi_\pm(x, \hbar) = 0 \quad (4-11)$$

where

$$H(x) = \left[\hbar^2 \sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial}{\partial T_{\infty,k}} + \hbar^2 \sum_{\nu=1}^n \sum_{k \in K_{b_\nu}} U_{\nu,k}(x) \frac{\partial}{\partial T_{\nu,k}} \right] [\log Z^{\text{pert}}(\hbar) - \hbar^{-2} \omega_{0,0}] + \frac{\phi_0(x)}{(dx)^2}. \quad (4-12)$$

Remark 4.4. *The dependence in x is completely explicit in (4-11) and (4-12). Coefficients in (4-12) may be computed from the knowledge of Z^{pert} given by topological recursion.*

Proof. The proof follows from the combination of a few lemmas. We first prove in Section A.1 that

Lemma 4.3. *For any $h \geq 0$ and $n \geq 1$ satisfying $2h - 2 + n \geq 2$, the combination*

$$\begin{aligned} & d_{z_1} F_{h,n}(z_1, \dots, z_n) + \sum_{j=2}^n \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot) \left[\frac{d_{z_1} F_{h,n-1}(z_1, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_1)} - \frac{d_{z_j} F_{h,n-1}(z_j, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_j)} \right] \\ &+ \frac{1}{2\omega_{0,1}(z_1)} d_{u_1} d_{u_2} \left[F_{h-1, n+1}(u_1, u_2, z_2, \dots, z_n) \right. \\ &+ \left. \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\}}} F_{h_1, |A|+1}(u_1, \mathbf{z}_A) F_{h_2, |B|+1}(u_2, \mathbf{z}_B) \right]_{u_1 = u_2 = z_1} \end{aligned} \quad (4-13)$$

is a holomorphic form in the first variable z_1 . In this expression, $\sum_{\text{stable}} h_1 + h_2 = \hbar$ refers to the sum where $2h_1 - 2 + |A| > 0$ and $2h_2 - 2 + |B| > 0$.
 $A \sqcup B = \{z_2, \dots, z_n\}$

In the same way,

$$d_{z_1} F_{0,3}(z_1, z_2, z_3) + \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot)}{4\omega_{0,1}(z_1)} - \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_2, \cdot)}{4\omega_{0,1}(z_2)} - \frac{\int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_3, \cdot)}{4\omega_{0,1}(z_3)} \quad (4-14)$$

and

$$d_{z_1} F_{1,1}(z_1) - \frac{\omega_{0,2}(z_1, \sigma(z_1))}{2\omega_{0,1}(z_1)} \quad (4-15)$$

are holomorphic forms in the first variable z_1 .

Then, we have the following lemma

Lemma 4.4. *For any holomorphic differential ω on Σ_ϕ , one has*

$$-2 \frac{y(z_1)}{dx(z_1)} \omega(z_1) = \sum_{p \in \mathcal{P}} \text{Res}_{z_2 \rightarrow p} \frac{\omega(z_2) y(z_2)}{x(z_2) - x(z_1)}.$$

Proof. We first remark that for any holomorphic differential ω on Σ_ϕ , one has

$$\sum_{p \in \mathcal{P}} \text{Res}_{z_2 \rightarrow p} \frac{\omega(z_2) y(z_2)}{x(z_2) - x(z_1)} = - \text{Res}_{z_2 \rightarrow z_1, \sigma(z_1)} \frac{\omega(z_2) y(z_2)}{x(z_2) - x(z_1)}.$$

Indeed, writing $\omega(z) = \frac{P(x(z))dx(z)}{y(z)}$ where $P(x)$ is a rational function, one sees that there is no contribution from the boundary of a fundamental domain when moving the integration contour and that $\omega(z) = -\omega(\sigma(z))$. Computing the residue gives

$$\sum_{p \in \mathcal{P}} \text{Res}_{z_2 \rightarrow p} \frac{\omega(z_2) y(z_2)}{x(z_2) - x(z_1)} = \frac{y(z_1)}{dx(z_1)} (\omega(\sigma(z_1)) - \omega(z_1)). \quad (4-16)$$

so that we obtain the proof of Lemma 4.4. \square

Eventually, application of Lemma 4.4 to the holomorphic differentials of Lemma 4.3 before considering the diagonal specialization $z_1 = \dots = z_n = z$ gives, after some elementary but lengthy computations presented in Appendix A.2, the results of Theorem 4.1. \square

5 The non-perturbative world and the quantum curve

In the previous section, we have seen that the perturbative wave functions are not annihilated by the naive quantization of the classical spectral curve. This differential operator in the variable x needs to be corrected by a combination of linear operators in the times. We shall now see that these perturbative wave functions can be corrected by exponentially small corrections as $\hbar \rightarrow 0$ to produce some non-perturbative analogs that are annihilated by a quantization of the classical spectral curve.

5.1 Definitions

Out of the perturbative wave function, one can build a non-perturbative analog that will be the main character of the present article. The latter is defined as a discrete Fourier transform of the perturbative wave function with respect to the \mathcal{A} -periods as originally expected from [16]. In this section, we pick an admissible initial data $(\phi_0, (\mathcal{A}_j, \mathcal{B}_j)_{j=1}^{g(\Sigma_{\phi_0})})$ where $\phi_0 \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T})$. For simplicity, we denote by $g := g(\Sigma_{\phi_0})$ the genus of the classical spectral curve and by $\epsilon = (\epsilon_i)_{i=1}^g$ the associated periods.

Definition 5.1. Let the non-perturbative partition function be the Fourier transform

$$Z(\mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) := \sum_{\mathbf{k} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^g k_j \rho_j} Z^{\text{pert}}(\hbar, \mathbf{T}, \boldsymbol{\epsilon} + \hbar \mathbf{k}).$$

In the same way, one can define the non-perturbative wave function by

$$\Psi_{\pm}(x, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) := \frac{1}{Z(\mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho})} \sum_{\mathbf{k} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^g k_j \rho_j} Z^{\text{pert}}(\hbar, \mathbf{T}, \boldsymbol{\epsilon} + \hbar \mathbf{k}) \psi_{\pm}(x, \hbar, \mathbf{T}, \boldsymbol{\epsilon} + \hbar \mathbf{k})$$

As functions of \hbar , these non-perturbative objects are of very different nature compared to their perturbative counterparts whose logarithms admit a formal series expansion in \hbar . Considering the \hbar expansion of the perturbative partition function, one can observe that

$$\frac{Z^{\text{pert}}(\hbar, \mathbf{T}, \boldsymbol{\epsilon} + \hbar \mathbf{k})}{Z^{\text{pert}}(\hbar, \mathbf{T}, \boldsymbol{\epsilon})} = \exp \left[\frac{2\pi i}{\hbar} \sum_{j=1}^g k_j \phi_j + \sum_{j,l=1}^g \pi i k_l k_j \tau_{l,j} \right] \exp \left[\sum_{\substack{h \geq 0 \\ n \geq \max(1, 3-2h)}} \sum_{\mathbf{i} \in \llbracket 1, g \rrbracket^n} \frac{\hbar^{2h-2+n}}{\text{Aut}(\mathbf{i})} \prod_{j=1}^n k_{i_j} \frac{\partial^n \omega_{\hbar,0}}{\partial \epsilon_{i_1} \dots \partial \epsilon_{i_n}} \right] \quad (5-1)$$

where, denoting by $l(\mathbf{i})$ the number of times l appears in the vector \mathbf{i} , $\text{Aut}(\mathbf{i}) := \prod_l l(\mathbf{i})!$, $\phi_i := \frac{\partial \omega_{0,0}}{\partial \epsilon_i}$

and $\tau_{i,j} := \frac{\partial^2 \omega_{0,0}}{\partial \epsilon_i \partial \epsilon_j}$.

This allows to recombine the Fourier transform under the form

$$\frac{Z(\mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho})}{Z^{\text{pert}}(\hbar, \mathbf{T}, \boldsymbol{\epsilon})} = \sum_{\mathbf{k} \in \mathbb{Z}^g} e^{\frac{2\pi i}{\hbar} \sum_{j=1}^g k_j (\rho_j + \phi_j) + \sum_{l,j=1}^g \pi i k_l k_j \tau_{l,j}} \exp \left[\sum_{\substack{h \geq 0 \\ n \geq \max(1, 3-2h)}} \sum_{\mathbf{i} \in \llbracket 1, g \rrbracket^n} \frac{\hbar^{2h-2+n}}{\text{Aut}(\mathbf{i})} \prod_{j=1}^n k_{i_j} \frac{\partial^n \omega_{\hbar,0}}{\partial \epsilon_{i_1} \dots \partial \epsilon_{i_n}} \right].$$

Hence, it takes the form

$$Z(\mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) = Z^{\text{pert}}(\hbar, \mathbf{T}, \boldsymbol{\epsilon}) \sum_{m=0}^{\infty} \hbar^m \Theta_m(\hbar, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) \quad (5-2)$$

where the coefficients $\Theta_m(\hbar, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho})$ are obtained as finite linear combinations of derivatives of theta functions of the form $\left. \frac{\partial^n \theta(\mathbf{v}, \boldsymbol{\tau})}{\partial v_{i_1} \dots \partial v_{i_n}} \right|_{\mathbf{v} = \frac{\boldsymbol{\phi} + \boldsymbol{\rho}}{\hbar}}$ with

$$\theta(\mathbf{v}, \boldsymbol{\tau}) := \sum_{\mathbf{k} \in \mathbb{Z}^g} e^{2i\pi \sum_{j=1}^g k_j v_j + \sum_{l,j=1}^g \pi i k_l k_j \tau_{l,j}}.$$

In the same way, the non-perturbative wave function takes the form

$$\Psi_{\pm}(x, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho}) = \psi_{\pm}(x, \mathbf{T}, \boldsymbol{\epsilon}) \frac{\sum_{m=0}^{\infty} \hbar^m \Xi_m(x, \hbar, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho})}{\sum_{m=0}^{\infty} \hbar^m \Theta_m(\hbar, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho})} \quad (5-3)$$

where $\Xi_m(x, \hbar, \mathbf{T}, \boldsymbol{\epsilon}, \boldsymbol{\rho})$ are combinations of derivatives of theta functions of the form $\left. \frac{\partial^n \theta(\mathbf{v}, \boldsymbol{\tau})}{\partial v_{i_1} \dots \partial v_{i_n}} \right|_{\mathbf{v} = \frac{\boldsymbol{\phi} + \boldsymbol{\rho} + \boldsymbol{\mu}(x)}{\hbar}}$ with

$$\mu_j(x) := \frac{\partial S_{-1}(x)}{\partial \epsilon_j}. \quad (5-4)$$

It is important to remark that these non-perturbative objects are not formal power series in \hbar but **formal trans-series in \hbar** .

5.2 Properties and quantum curve

One of the main motivations for the definition of the non-perturbative wave functions is the simplicity of its monodromies compared to its perturbative counterpart.

Lemma 5.1. *The non-perturbative wave functions satisfy the following properties.*

— For $j \in \llbracket 1, g \rrbracket$, the function $\Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho)$ has a formal monodromy along \mathcal{A}_j given by

$$\Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho) \mapsto e^{\pm 2\pi i \frac{\epsilon_j}{\hbar}} \Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho). \quad (5-5)$$

— For $j \in \llbracket 1, g \rrbracket$, the function $\Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho)$ has a formal monodromy along \mathcal{B}_j given by

$$\Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho) \mapsto e^{\mp 2\pi i \frac{\rho_j}{\hbar}} \Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho). \quad (5-6)$$

Proof. The proof easily follows from the monodromies of the perturbative wave functions. \square

The main result of this article is that, unlike its perturbative partners, the non-perturbative wave functions are solutions of a second order ODE that quantizes the classical spectral curve. The second order differential operator annihilating both non-perturbative wave functions is thus often referred to as the corresponding *quantum curve*.

Theorem 5.1 (Quantum curve). *The non-perturbative wave functions satisfy*

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 R(x) \frac{\partial}{\partial x} - \hbar Q(x) - \mathcal{H}(x) \right] \Psi_{\pm} = 0 \quad (5-7)$$

with

$$\mathcal{H}(x) = \left[\hbar^2 \sum_{k \in K_{\infty}} U_{\infty, k}(x) \frac{\partial}{\partial T_{\infty, k}} + \hbar^2 \sum_{\nu=1}^n \sum_{k \in K_{b_{\nu}}} U_{\nu, k}(x) \frac{\partial}{\partial T_{\nu, k}} \right] \left[\log Z(\mathbf{T}, \epsilon, \rho) - \hbar^{-2} \omega_{0,0} \right] + \frac{\phi_0(x)}{(dx)^2} \quad (5-8)$$

and

$$R(x) = \frac{\partial \log W(x)}{\partial x} \quad (5-9)$$

where the Wronskian

$$W(x) := \hbar \left[\frac{\partial \Psi_+(x)}{\partial x} \Psi_-(x) - \frac{\partial \Psi_-(x)}{\partial x} \Psi_+(x) \right] \quad (5-10)$$

is a rational function of the form

$$W(x) = w \frac{\prod_{j=1}^g (x - q_j)}{\prod_{\nu=1}^n (x - X_{\nu})^{r_{\nu}}} \quad (5-11)$$

with

$$w := \begin{cases} -2T_{\infty, r_{\infty}} \exp(A_{\infty, 0}^+ + A_{\infty, 0}^-) & \text{if } n_{\infty} = 0 \\ -T_{\infty, r_{\infty}} \exp(A_{\infty, 0}^+ + A_{\infty, 0}^-) & \text{if } n_{\infty} = 1 \end{cases} \quad (5-12)$$

and

$$Q(x) = \sum_{j=1}^g \frac{p_j}{x - q_j} + \frac{\hbar}{2} \left[\sum_{k \in K_{\infty}} U_{\infty, k}(x) \frac{\partial(S_+(x) + S_-(x))}{\partial T_{\infty, k}} \right]_{\infty, +} + \frac{\hbar}{2} \sum_{\nu=1}^n \left[\sum_{k \in K_{\nu}} U_{\nu, k}(x) \frac{\partial(S_+(x) + S_-(x))}{\partial T_{\nu, k}} \right]_{X_{\nu}, -} \quad (5-13)$$

with

$$\forall j \in \llbracket 1, g \rrbracket, p_j := -\hbar \frac{\partial \log \Psi_+(x)}{\partial x} \Big|_{x=q_j} = -\hbar \frac{\partial \log \Psi_-(x)}{\partial x} \Big|_{x=q_j}. \quad (5-14)$$

In addition, the pairs $(q_i, p_i)_{i=1}^g$ satisfy

$$\forall i \in \llbracket 1, g \rrbracket, p_i^2 = \mathcal{H}(q_i) - \hbar p_i \left[\sum_{j \neq i} \frac{1}{q_i - q_j} - \sum_{\nu=1}^n \frac{r_{\nu}}{q_i - X_{\nu}} \right] + \hbar \left[\left(Q(x) - \frac{p_i}{x - q_i} \right) \right]_{x=q_i}. \quad (5-15)$$

Before proving this theorem, let us mention that the pairs $(q_i, p_i)_{i=1}^g$, which depend on \hbar , form a set of Darboux coordinates on some associated symplectic space. We shall discuss this point in Section 6.

Remark 5.1. *The sums involved in (5-8) and (5-13) are finite. Moreover, the dependence in x is explicit in the definition (5-13) of Q . The dependence in x is also explicit in $\mathcal{H}(x)$ though the coefficients involve time derivatives of the partition function. These quantities may be seen in two different ways: either they are obtained by topological recursion (that computes the partition function) or they can be seen as undetermined coefficients (independent of x) that are in one-to-one correspondence with the Hamiltonians and their expressions in terms of the Darboux coordinates $(q_i, p_i)_{i=1}^g$. The second aspect shall be developed below in section 6 and in the examples presented in section 8.*

Proof. The details of the proof are given in Appendix B. Let us give the main steps in the remaining of this section, referring to the appendix for the technical details.

Let us first remark that, as a linear combination of perturbative wave functions, $\Psi_{\pm}(x, \mathbf{T}, \epsilon, \rho)$ are solutions of the PDE eq. (4-11).

We shall prove that the non-perturbative wave functions are solutions of a linear second order differential equation following [26].

Let us first introduce a few useful functions built out of Ψ_{\pm} . We denote the Wronskian with respect to x by

$$W(x) := \hbar \left(\frac{\partial \Psi_+}{\partial x} \Psi_- - \Psi_+ \frac{\partial \Psi_-}{\partial x} \right) \quad (5-16)$$

and the Wronskian with respect to the times by

$$\forall p \in \{\infty\} \cup [1, n], \forall k \in K_p : W_{T_{p,k}}(x) := \hbar \left(\frac{\partial \Psi_+}{\partial T_{p,k}} \Psi_- - \Psi_+ \frac{\partial \Psi_-}{\partial T_{p,k}} \right). \quad (5-17)$$

One can use them to define

$$\forall p \in \{\infty\} \cup [1, n], \forall k \in K_p : R_{p,k} := \frac{W_{T_{p,k}}(x)}{W(x)} \quad \text{and} \quad Q_{p,k} := \hbar^2 \frac{\frac{\partial \Psi_+}{\partial x} \frac{\partial \Psi_-}{\partial T_{p,k}} - \frac{\partial \Psi_-}{\partial x} \frac{\partial \Psi_+}{\partial T_{p,k}}}{W(x)}. \quad (5-18)$$

They are defined in such a way that

$$\forall p \in \{\infty\} \cup [1, n], \forall k \in K_p : \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial \Psi_+}{\partial T_{p,k}} & \hbar \frac{\partial \Psi_-}{\partial T_{p,k}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Q_{p,k} & R_{p,k} \end{pmatrix} \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial \Psi_+}{\partial x} & \hbar \frac{\partial \Psi_-}{\partial x} \end{pmatrix}. \quad (5-19)$$

To simplify notations, we shall identify the set of poles \mathcal{P} with the corresponding index set and write $\mathcal{P} := \{\infty\} \cup [1, n]$ when no confusion may arise.

By linearity, one can consider arbitrary linear combinations of the functions defined above and observe that they satisfy similar equations. In particular, defining

$$R(x) := \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) R_{p,k}(x) \quad \text{and} \quad Q(x) := \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) Q_{p,k}(x), \quad (5-20)$$

one has

$$\begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) \frac{\partial \Psi_+}{\partial T_{p,k}} & \hbar \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) \frac{\partial \Psi_-}{\partial T_{p,k}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Q & R \end{pmatrix} \begin{pmatrix} \Psi_+ & \Psi_- \\ \hbar \frac{\partial \Psi_+}{\partial x} & \hbar \frac{\partial \Psi_-}{\partial x} \end{pmatrix}. \quad (5-21)$$

First of all, the fact that Ψ_{\pm} are linear combinations of solutions of eq. (4-11), and taking into account the contributions of the coefficients of these combinations, one observes that we have

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) \frac{\partial}{\partial T_{p,k}} - \mathcal{H}(x) \right] \Psi_{\pm}(x) = 0 \quad (5-22)$$

where $\mathcal{H}(x)$ is obtained by replacing the perturbative partition function by its non-perturbative partner in $H(x)$.

By the definition of Q and R together, Ψ_+ and Ψ_- are thus solutions to the compatible system

$$\left\{ \begin{array}{l} \left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) \frac{\partial}{\partial T_{p,k}} - \mathcal{H}(x) \right] \Psi = 0 \\ \left[\hbar^2 \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) \frac{\partial}{\partial T_{p,k}} - \hbar Q(x) - \hbar^2 R(x) \frac{\partial}{\partial x} \right] \Psi = 0 \end{array} \right. \quad (5-23)$$

Plugging the second equation in to the first one implies that the non-perturbative wave functions are solutions to the ODE

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 R(x) \frac{\partial}{\partial x} - \hbar Q(x) - \mathcal{H}(x) \right] \Psi_{\pm} = 0. \quad (5-24)$$

This is our quantum curve. We now need to study the functions $R(x)$ and $Q(x)$ that may a priori be multi-valued functions of x with singularities at the zeros of $W(x)$, the poles ∞ and $\{X_{\nu}\}$ as well as the critical values $u_i = x(a_i)$. We shall now see that these functions are rational functions of x without poles at the critical values.

We first prove in Appendix B.1 that one can simplify the expression of $R(x)$.

Lemma 5.2. *The function $R(x)$ reads*

$$R(x) = \frac{1}{W(x)} \frac{\partial W(x)}{\partial x}. \quad (5-25)$$

One can also remark that, by definition,

$$Q(x) = \frac{\hbar \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(x) \left(\frac{\partial W(x)}{\partial T_{p,k}} - \frac{\partial W_{T_{p,k}}(x)}{\partial x} \right)}{2 W(x)}. \quad (5-26)$$

Let us now study the properties of the functions $R_{p,k}$ and $Q_{p,k}$.

First of all, let us remind that Ψ_{\pm} are combinations of terms which are derivatives of theta functions evaluated at $\mathbf{v} = \frac{\phi+\rho}{\hbar} + \mu(x)$. On the other hand, because the combination $\theta(\mathbf{v} + \mathbf{u}, \boldsymbol{\tau})\theta(\mathbf{v} - \mathbf{u}, \boldsymbol{\tau})$ is a theta function of order 2 in \mathbf{v} , it can be decomposed in a basis of squares of theta functions with different characteristics (see for example [19]). Hence, the Wronskians involve only combinations of derivatives of theta functions evaluated at $\mathbf{v} = \frac{\phi+\rho}{\hbar}$, their dependence in x appearing only in the coefficients.

This means that, for all pair (p, k) , $R_{p,k}$ and $Q_{p,k}$ admit an expansion in \hbar of the form

$$R_{p,k}(x, \mathbf{T}, \hbar) = \sum_{m=0}^{\infty} \hbar^m R_{p,k}^{(m)}(x, \mathbf{T}, \hbar) \quad \text{and} \quad Q_{p,k}(x, \mathbf{T}, \hbar) = \sum_{m=0}^{\infty} \hbar^m Q_{p,k}^{(m)}(x, \mathbf{T}, \hbar) \quad (5-27)$$

where the coefficients take the form

$$R_{p,k}^{(m)}(x, \mathbf{T}, \hbar) = \hat{R}_{p,k}^{(m)}(x, \mathbf{T}, \mathbf{v}) \Big|_{\mathbf{v} = \frac{\phi+\rho}{\hbar}} \quad \text{and} \quad Q_{p,k}^{(m)}(x, \mathbf{T}, \hbar) = \hat{Q}_{p,k}^{(m)}(x, \mathbf{T}, \mathbf{v}) \Big|_{\mathbf{v} = \frac{\phi+\rho}{\hbar}}. \quad (5-28)$$

Thanks to Lemma 5.1, one can also check that the monodromies of the non-perturbative wave functions around elements of $H_1(\Sigma_{\phi}, \mathbb{Z})$ ensure that the Wronskians do not have any monodromy and thus

Corollary 5.1. *The Wronskians and the functions $\hat{R}_{p,k}^{(h)}$ and $\hat{Q}_{p,k}^{(h)}$ are rational functions of x .*

Indeed, the essential singularities cancel in the definition of the Wronskians.

We can even go further and prove in Appendix B.2 the following important result.

Lemma 5.3. *The rational functions $R_{p,k}$ and $Q_{p,k}$ have no pole at the ramification points.*

This implies that

Corollary 5.2. *The Wronskians $W(x)$ and $W_{T_{p,K}}(x)$ are rational functions of x with poles only at $p \in \mathcal{P}$.*

Proof. From Lemma 5.2, one knows that

$$R(x) = \partial_x [\log W(x)].$$

Lemma 5.3 ensures that it does not have any pole at the ramification points. On the other hand, from the properties of Ψ_{\pm} , one knows that the RHS is a rational function of x with possible poles only at $p \in \mathcal{P}$ and the critical values. Hence, this combination has no poles at the ramification points and $W(x)$ is a rational function of x with poles only at $p \in \mathcal{P}$.

By definition, one has

$$W_{T_{p,K}}(x) = R_{p,K}(x)W(x). \quad (5-29)$$

The left hand side might have poles at the ramification points and $p \in \mathcal{P}$ only. The right hand side does not have any pole at the ramification points leading to the result. \square

The asymptotic expansions around poles recalled in Appendix C ensure that $W(x)$ takes the form

$$W(x) = w \frac{\prod_{j=1}^g (x - q_j)}{\prod_{\nu=1}^n (x - X_{\nu})^{r_{\nu}}} \quad (5-30)$$

where, using the notations of Appendix C, one defines

$$w := \begin{cases} -2T_{\infty, r_{\infty}} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) & \text{if } n_{\infty} = 0 \\ -T_{\infty, r_{\infty}} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) & \text{if } n_{\infty} = 1 \end{cases}. \quad (5-31)$$

To conclude the proof, let us study the properties of $Q(x)$. By its definition eq. (5-26) and Lemma 5.3, it is a rational function of x with simple poles at the q_i and poles at $p \in \mathcal{P}$ whose degree follows from the asymptotics given in Appendix C. It thus reads

$$Q(x) = \sum_{i=1}^g \frac{Q_{q_i}}{x - q_i} + \sum_{k=0}^{r_{\infty}-4} Q_{\infty,k} x^k + \sum_{\nu=1}^n \sum_{k=1}^{r_{\nu}+1} \frac{Q_{\nu,k}}{(x - X_{\nu})^k}. \quad (5-32)$$

One can compute the coefficients of this partial fraction expansion by expanding the differential equations satisfied by the wave functions around the different poles.

— The leading and subleading orders in the expansion of the quantum curve eq. (5-24) around $x = q_i$, for $i \in \llbracket 1, g \rrbracket$, read

$$\begin{cases} Q_{q_i} = p_i \\ p_i^2 = \mathcal{H}(q_i) - \hbar p_i \left[\sum_{j \neq i} \frac{1}{q_i - q_j} - \sum_{\nu=1}^n \frac{r_{\nu}}{q_i - X_{\nu}} \right] + \hbar \left[\left(Q(x) - \frac{p_i}{x - q_i} \right) \right]_{x=q_i} \end{cases} \quad (5-33)$$

where

$$\forall j \in \llbracket 1, g \rrbracket : p_j := -\hbar \left. \frac{\partial \log \Psi_+(x)}{\partial x} \right|_{x=q_j} = -\hbar \left. \frac{\partial \log \Psi_-(x)}{\partial x} \right|_{x=q_j}. \quad (5-34)$$

As we shall see, the second equation means that each pair $(q_i, p_i)_{i=1}^g$ defines a point in a \hbar -deformation of the classical spectral curve.

- Let us consider the sum of the second equation of the system eq. (5-23) for Ψ_+ and Ψ_- . The expansion of this symmetric version around $x = \infty$ reads

$$[Q(x)]_{\infty,+} = \frac{\hbar}{2} \left[\sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial(S_+(x) + S_-(x))}{\partial T_{\infty,k}} \right]_{\infty,+}. \quad (5-35)$$

- The expansion of the same expression around $x = X_\nu$ reads

$$[Q(x)]_{X_\nu,-} = \frac{\hbar}{2} \left[\sum_{k \in K_\nu} U_{\nu,k}(x) \frac{\partial(S_+(x) + S_-(x))}{\partial T_{\nu,k}} \right]_{X_\nu,-}. \quad (5-36)$$

□

6 Lax representation

After building a quantization of the classical spectral curve, it is natural to study a linearization of the corresponding differential equation. In particular, in this section we associate a $\mathfrak{sl}_2(\mathbb{C})$ connection depending on \hbar to such a quantum curve and describe it as a point in the corresponding moduli space of irregular connections. We compute the corresponding quadratic differential which can be understood as a \hbar -deformation of the initial data ϕ_0 in the moduli space of meromorphic connections $\mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T})$. We finally study the corresponding isomonodromic system when it can be obtained as de-autonomization of an isospectral system.

6.1 \mathfrak{sl}_2 connection from quantum curve

Let us linearize the quantum curve. For this purpose, we define

$$\hat{\Psi}_\pm(x) := \frac{1}{W(x)} \left[P(x)\Psi_\pm(x) + \hbar \frac{\partial \Psi_\pm(x)}{\partial x} \right] \quad (6-1)$$

where $P(x)$ is a rational function of x with poles at $x(p)$ for $p \in \mathcal{P}$ to be fixed.

The ODE given in Theorem 5.1 implies that

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} \quad (6-2)$$

where

$$M(x) = \frac{\hbar \frac{\partial P(x)}{\partial x} - \hbar \frac{\partial \log W(x)}{\partial x} P(x) - P(x)^2 + \hbar Q(x) + \mathcal{H}(x)}{W(x)}. \quad (6-3)$$

This rank 2 differential system defines a connection on \mathbb{P}^1 which has poles only on \mathcal{P} by imposing a simple condition on the function $P(x)$.

Lemma 6.1. *If*

$$\forall i \in \llbracket 1, g \rrbracket : P(q_i) = p_i \quad (6-4)$$

then $M(x)$ is a rational function of x with poles only at $x \in \mathcal{P}$.

Proof. To prove this lemma, one only needs to prove that the $M(x)$ does not have any pole at $x = q_i$ for $i \in \llbracket 1, g \rrbracket$.

Let us compute the two leading terms of $\frac{\hbar \frac{\partial P}{\partial x} - \hbar \frac{\partial \log W r^x(\Psi_+, \Psi_-)}{\partial x} P - P^2 + \hbar Q + \mathcal{H}}{W r^x(\Psi_+, \Psi_-)}$ as $x \rightarrow q_i$. Using eq. (5-33), they read

$$- \hbar P(q_i) + \hbar p_i \quad (6-5)$$

and

$$- \hbar P(q_i) \left[\sum_{j \neq i} \frac{1}{q_i - q_j} - \sum_{\nu=1}^n \frac{r_\nu}{q_i - X_\nu} \right] - P(q_i)^2 + \hbar \left[Q(x) - \frac{P(q_i)}{x - q_i} \right]_{x=q_i} + \mathcal{H}(q_i). \quad (6-6)$$

These two terms, which are the coefficients of the singular part of $\frac{\hbar \frac{\partial P}{\partial x} - \hbar \frac{\partial \log W r^x(\Psi_+, \Psi_-)}{\partial x} P - P^2 + \hbar Q + \mathcal{H}}{W r^x(\Psi_+, \Psi_-)}$ vanish if $P(q_i) = p_i$ thanks to Theorem 5.1. \square

Hence $\begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix}$ is a basis of solutions to the Lax system

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} = L(x) \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} \quad (6-7)$$

where

$$L(x) = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix} \quad (6-8)$$

is a rational function of x with value in \mathfrak{sl}_2 with poles only in D if the degree of the poles of $P(x)$ are kept small enough. It defines a connection $\hbar d - L(x)dx$ on \mathbb{P}^1 that can be considered as a point in the corresponding moduli space of \mathfrak{sl}_2 connections with irregular singularities along D . Fixing the value of \mathbf{T} amounts to fixing its residues and irregular type thus leading to a point in a symplectic leaf of dimension $2g$. The definition of $(q_i, p_i)_{i=1}^g$ by

$$\forall i \in \llbracket 1, g \rrbracket, \begin{cases} W(q_i) = 0 \\ P(q_i) = p_i \end{cases} \quad (6-9)$$

actually identifies them with the spectral Darboux coordinates defined by [2] on this symplectic leaf.

It is important to notice that the Lax matrix $L(x)$ depends on \hbar only through $(q_i, p_i)_{i=1}^g$. The latter being Darboux coordinates, they depend on \hbar only through their time evolution as we will see in the next section.

6.2 Deformed spectral curve

Before explaining how to choose a particular gauge in order to make the explicit computation of examples easier, let us describe the spectral curve of the matrix $L(x)$ defined by

$$0 = \det(ydx - L(x)dx) := y^2(dx)^2 - \phi_\hbar. \quad (6-10)$$

It defines a quadratic differential

$$\phi_\hbar = \left[\hbar \frac{\partial P(x)}{\partial x} - \hbar \frac{\partial \log W(x)}{\partial x} P(x) + \hbar Q(x) + \mathcal{H}(x) \right] (dx)^2. \quad (6-11)$$

We have thus defined a flow along the ∂_{\hbar} direction in the corresponding moduli space of quadratic differentials. Let us present a complete expression for our \hbar -deformed quadratic differential using the expression of $Q(x)$ of Theorem 5.1.

Theorem 6.1. *The \hbar -deformed spectral curve reads*

$$\begin{aligned} \frac{\phi_\hbar}{(dx)^2} &= \mathcal{H}(x) + \hbar \sum_{j=1}^g \frac{p_j}{x - q_j} + \frac{\hbar^2}{2} \left[\sum_{k \in K_\infty} U_{\infty, k}(x) \frac{\partial(S_+(x) + S_-(x))}{\partial T_{\infty, k}} \right]_{\infty, +} \\ &+ \frac{\hbar^2}{2} \sum_{\nu=1}^n \left[\sum_{k \in K_\nu} U_{\nu, k}(x) \frac{\partial(S_+(x) + S_-(x))}{\partial T_{\nu, k}} \right]_{X_{\nu, -}} + \hbar \frac{\partial P(x)}{\partial x} - \hbar \frac{\partial \log W(x)}{\partial x} P(x). \end{aligned} \quad (6-12)$$

Even if this expression seems to have simple poles at $x = q_i$, they cancel due to the condition $P(q_i) = p_i$.

6.3 Gauge choice

The characterization of $P(x)$ by its values at $(q_i)_{i=1}^g$ does not fix it without ambiguity. Fixing a specific form of the rational function P corresponds to a gauge choice for the system under consideration, i.e. a choice of representative of the reduced coadjoint orbit under consideration. Let us give an example of gauge choice in this section. We follow the choice considered in [32]. The gauge depends on whether the point above infinity is a ramification point or not (i.e. if $n_\infty = 0$ or $n_\infty = 1$).

6.3.1 Non-degenerate case: $n_\infty = 0$

If ∞ is not a critical value for the map x , one fixes a gauge by writing the system under the form

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} = L(x) \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} \quad (6-13)$$

where

$$L(x) = \sum_{k \leq r_\infty} L_{\infty, k} x^{k-2} \quad (6-14)$$

as $x \rightarrow \infty$ with

$$L_{\infty, r_\infty} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}. \quad (6-15)$$

As we shall see from the examples, the value of α and β can be computed from the Casimirs. More precisely, for $r_\infty \geq 2$, one has $\alpha = T_{\infty, r_\infty}$ and $\beta = T_{\infty, r_\infty} + \frac{\hbar}{2}$.

This requires that $P(x)$ takes the form

$$P(x) = \frac{\alpha x^{g+1} + \beta x^g + \sum_{l=0}^{g-1} \alpha_l x^l}{\prod_{\nu=1}^n (x - X^\nu)^{r_{b_\nu}}}. \quad (6-16)$$

The last coefficients $(\alpha_l)_{l=0}^{g-1}$ are fixed by the g conditions

$$\forall i \in \llbracket 1, g \rrbracket : P(q_i) = p_i. \quad (6-17)$$

We call this case non-degenerate since the leading order of the Lax matrix $L(x)$ at infinity as full rank in this coadjoint orbit.

6.3.2 Degenerate case: $n_\infty = 1$

This degenerate case corresponds to the case when the leading order of the Lax matrix at infinity is of rank 1 instead of rank 2. As in [32], we look for a representative of the coadjoint orbit satisfying

$$L_{\infty, r_\infty} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L_{\infty, r_\infty - 1} = \begin{pmatrix} 0 & \beta \\ A & 0 \end{pmatrix} \quad (6-18)$$

for arbitrary β . This can be obtained by imposing the form

$$P(x) = \frac{\sum_{l=0}^{g-1} \alpha_l x^l}{\prod_{\nu=1}^n (x - X^\nu)^{r_{b_\nu}}} \quad (6-19)$$

satisfying

$$\forall i \in \llbracket 1, g \rrbracket : P(q_i) = p_i. \quad (6-20)$$

7 Isomonodromic deformations

It is often very useful to consider isomonodromic deformations of our Lax matrix $L(x)$. In particular, it is a way to access a set of equations defining the function $(p_i, q_i)_{i=1}^g$. In examples of Section 8, we use Hamiltonian representations of the corresponding isomonodromic systems to get 2-parameters solutions of the six Painlevé equations. In this section, we recall one way to obtain isomonodromic deformations starting from an isospectral system following the work of Montréal school [21]. This section does not present any new material but should be taken as a guidebook for the computations of the example section.

7.1 Isospectral deformations

Using the classical R -matrix formalism on loop algebras, one can define a Poisson structure on the space of Lax matrices $L(x)$ as before. Indeed, one can define a set of commuting flows generated by spectral invariants. In our case, the set of spectral invariants is generated by the following Hamiltonians:

$$\forall k \in \llbracket 1, r_\infty - 3 \rrbracket : H_{p,k} := \frac{1}{2} \operatorname{Res}_{x \rightarrow \infty} x^{-k} \operatorname{Tr} L(x)^2 dx \quad (7-1)$$

and

$$\forall \nu \in \llbracket 1, n \rrbracket, \forall k \in \llbracket 0, r_\nu - 1 \rrbracket : H_{\nu,k} := \frac{1}{2} \operatorname{Res}_{x \rightarrow X_\nu} (x - X_\nu)^k \operatorname{Tr} L(x)^2 dx. \quad (7-2)$$

The associated Hamilton's equations read

$$\forall p \in \{\infty\} \cup \llbracket 1, n \rrbracket, \forall k : \frac{\partial L(x)}{\partial t_{p,k}} = [L_{t_{p,k}}(x), L(x)] \quad (7-3)$$

where

$$\forall k \in \llbracket 1, r_\infty - 3 \rrbracket : L_{t_{\infty,k}} := [x^{-k} L(x)]_{\infty,+} \quad (7-4)$$

and

$$\forall \nu \in \llbracket 1, n \rrbracket, \forall k \in \llbracket 0, r_\nu - 1 \rrbracket : L_{t_{\nu,k}} := [(x - X_\nu)^k L(x)]_{X_\nu,-}. \quad (7-5)$$

These Hamiltonian flows are isospectral since they preserve the spectrum of $L(x)$.

7.2 Non-autonomous system and isomonodromic deformations

Let us now consider that, in addition, $L(x)$ depends explicitly on $t_{p,k}$ in such a way that

$$\frac{\delta L(x)}{\delta t_{p,k}} = \frac{\partial L_{t_{p,k}}}{\partial x} \quad (7-6)$$

where $\frac{\delta L(x)}{\delta t_{p,k}}$ denotes the variation of $L(x)$ with respect to its explicit dependence on $t_{p,k}$ only. Then, Hamilton's equations are replaced by

$$\frac{\partial L(x)}{\partial t_{p,k}} = [L_{t_{p,k}}(x), L(x)] + \frac{\partial L_{t_{p,k}}}{\partial x}. \quad (7-7)$$

This new equation is equivalent to the commutation relation

$$\left[\frac{\partial}{\partial x} - L(x), \frac{\partial}{\partial t_{p,k}} - L_{t_{p,k}}(x) \right] = 0 \quad (7-8)$$

It ensures that the flow along $\frac{\partial}{\partial t_{p,k}}$ is isomonodromic. For this reason, we refer to eq. (7-6) as the isomonodromic condition. From now on, we assume that it is satisfied in this section for all pairs (p, k) described above⁴.

4. To our knowledge, a general procedure to prove the isomonodromic condition eq. (7-6) from some general isospectral deformations formalism is not known

In such a case, an isomonodromic tau function τ is defined by the condition

$$d \ln \tau = \sum_{p,k} H_{p,k} dt_{p,k}. \quad (7-9)$$

One can also use the isomonodromic times $t_{p,k}$ as new coordinates replacing the spectral times $T_{p,k}$. One can decompose our vector field $\frac{\partial}{\partial T_{\nu,k}}$ in this basis by comparing the corresponding Wronskians, or more precisely, the action of the vector fields on the singular part of the logarithm of the wave functions. The change of basis is more naturally expressed by decomposing the usual isomonodromic flows in terms of our vector fields, leading to

$$\forall \nu \in \llbracket 1, n \rrbracket, \forall l \in \llbracket 0, r_\nu - 1 \rrbracket : \frac{\partial}{\partial t_{\nu,l}} = \sum_{k=2}^{r_\nu-l+1} (k-1) T_{\nu,k+l-1} \frac{\partial}{\partial T_{\nu,k}}, \quad (7-10)$$

$$\forall l \in \llbracket 1, r_\infty - 3 \rrbracket : \frac{\partial}{\partial t_{\infty,l}} = \sum_{k=2}^{r_\infty-l-1} (k-1) T_{\infty,k+l+1} \frac{\partial}{\partial T_{\infty,k}} \text{ if } n_\infty = 0 \quad (7-11)$$

and

$$\forall l \in \llbracket 1, r_\infty - 3 \rrbracket : \frac{\partial}{\partial t_{\infty,l}} = \sum_{k=2}^{r_\infty-l-1} \frac{2k-3}{2} T_{\infty,k+l+1} \frac{\partial}{\partial T_{\infty,k}} \text{ if } n_\infty = 1 \quad (7-12)$$

In addition, one can use this decomposition to express the function $\mathcal{H}(x)$ in terms of the variations with respect to the usual isomonodromic times. One has

$$\sum_{k \in K_\infty} U_{\infty,k}(x) \frac{\partial}{\partial T_{\infty,k}} = \sum_{l=1}^{r_\infty-3} x^{l-1} \frac{\partial}{\partial t_{\infty,l}} \quad (7-13)$$

and

$$\forall \nu \in \llbracket 1, n \rrbracket : \sum_{k \in K_\nu} U_{\nu,k}(x) \frac{\partial}{\partial T_{\nu,k}} = \sum_{l=0}^{r_\nu-1} \frac{1}{(x - X_\nu)^{l+1}} \frac{\partial}{\partial t_{\nu,l}}. \quad (7-14)$$

Using these relations, one can build a tau function in terms of the non-perturbative partition function. Rather than writing a lengthy general formula, we prefer referring the reader to Section 8 for examples.

7.3 Flows of Darboux coordinates

The vector $\begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix}$ is subject to a set of compatible equations with respect to the isomonodromic times (denoted by the generic letter t for clarity in the following equations)

$$\frac{\partial}{\partial t} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} = \begin{pmatrix} P_t(x) & M_t(x) \\ W_t(x) & -P_t(x) \end{pmatrix} \begin{pmatrix} \hat{\Psi}_\pm \\ \Psi_\pm \end{pmatrix} \quad (7-15)$$

where

$$L_t(x) = \begin{pmatrix} P_t(x) & M_t(x) \\ W_t(x) & -P_t(x) \end{pmatrix}. \quad (7-16)$$

The compatibility of the system evaluated at $x = q_i$ imposes that, for an arbitrary isomonodromic time t , one has

$$\forall i \in \llbracket 1, g \rrbracket : \begin{cases} \hbar \frac{\partial p_i}{\partial t} = \hbar \frac{\partial P_t(x)}{\partial x} \Big|_{x=q_i} - \frac{\hbar \frac{\partial W_t(x)}{\partial x} \Big|_{x=q_i} \frac{\partial P(x)}{\partial x} \Big|_{x=q_i} + W_t(q_i) \frac{\partial \phi_\hbar(x)}{\partial x} \Big|_{x=q_i}}{\frac{\partial W(x)}{\partial x} \Big|_{x=q_i}} \\ \hbar \frac{\partial q_i}{\partial t} = - \frac{\hbar \frac{\partial W_t(x)}{\partial x} \Big|_{x=q_i} + 2p_i W_t(q_i)}{\frac{\partial W(x)}{\partial x} \Big|_{x=q_i}} \end{cases}. \quad (7-17)$$

Proof. Given any isomonodromic time t , the compatibility condition between the time derivative and the x derivative implies

$$\begin{cases} \hbar \frac{\partial P(x)}{\partial t} = \hbar \frac{\partial P_t(x)}{\partial x} + M_t(x)W(x) - M(x)W_t(x) \\ \hbar \frac{\partial W(x)}{\partial t} = \hbar \frac{\partial W_t(x)}{\partial x} + 2P(x)W_t(x) - 2P_t(x)W(x) \end{cases}. \quad (7-18)$$

Let us remark that, for any rational function $f(x)$,

$$\frac{\partial f(q_i)}{\partial t} = \frac{\partial f(x)}{\partial t} \Big|_{x=q_i} + \frac{\partial f(x)}{\partial x} \Big|_{x=q_i} \frac{\partial q_i}{\partial t}. \quad (7-19)$$

The evaluation of the compatibility conditions at $x = q_i$ then reads

$$\begin{cases} \hbar \frac{\partial p_i}{\partial t} = \hbar \frac{\partial P(x)}{\partial x} \Big|_{x=q_i} \frac{\partial q_i}{\partial t} + \hbar \frac{\partial P_t(x)}{\partial x} \Big|_{x=q_i} - M(q_i)W_t(q_i) \\ -\hbar \frac{\partial W(x)}{\partial x} \Big|_{x=q_i} \frac{\partial q_i}{\partial t} = \hbar \frac{\partial W_t(x)}{\partial x} \Big|_{x=q_i} + 2P(q_i)W_t(q_i) \end{cases}. \quad (7-20)$$

Combining these equations and the equation of the spectral curve, one obtains the result. \square

Remark that this result is valid even if the coefficients of the elements of the Lax matrix depend explicitly on the isomonodromic time t .

8 Examples

In this section, we present a few examples of application of our general formalism. In each case, we follow the same procedure. We first compute the Wronskian $W(x)$. It always takes the form of eq. (5-11) where the leading order is given by eq. (5-12). The value of the zeros of $W(x)$ gives half of the spectral Darboux coordinates and can be computed using the asymptotics of Appendix C.

One can then compute the quantum curve thanks to Theorem 5.1.

We then compute the element $P(x)$ of the Lax matrix in order to have the gauge considered in Section 6.3. After that, we can compute the \hbar -dependent quadratic differential $\phi_\hbar \in \mathcal{Q}(\mathbb{P}^1, D, n_\infty, \mathbf{T})$. First, we express its coefficients in terms of the times and the partition function thanks to Theorem 6.1 and the asymptotics of Appendix C. On the other hand, we express these coefficients in terms of the spectral Darboux coordinates by writing that any pair $(q_i, p_i)_{i=1}^g$ defines a point on the \hbar -deformed spectral curve. This allows relating the partition function to an isomonodromic tau function. Indeed, in all the cases considered, it is known that the isomonodromic condition eq. (7-6) can be fulfilled (see [20, 22] or [32]).

In each case, we write an identification of times allowing to solve this condition. After this identification, we write the associated Hamilton's equations obtained through the compatibility condition eq. (7-17). In the first cases, we show how it leads to the six Painlevé equations, showing that we have produced 2-parameters solutions for the latter. This also provides us an expression of the coordinates p_i 's in terms of the variations of the q_i 's.

8.1 Painlevé 1

Let us consider the case $r_\infty = 4$, $n_\infty = 1$ and $n = 0$. Let us first observe that the PDE eq. (5-22) imposes that $A_{\infty,1}^+ + A_{\infty,1}^- = 0$.

Hence, the Wronskian reads

$$W(x) = -T_{\infty,4}(x - q) \quad (8-1)$$

where

$$q = -\frac{T_{\infty,3}}{T_{\infty,4}} - (A_{\infty,2}^+ - A_{\infty,2}^-). \quad (8-2)$$

From eq. (5-22), one obtains that

$$A_{\infty,2}^+ - A_{\infty,2}^- = -\frac{1}{2} \frac{\partial A_{\infty,1}^+ - A_{\infty,1}^-}{\partial T_{\infty,2}} = -\hbar^2 \frac{\partial^2 \log Z}{\partial T_{\infty,2}^2}, \quad \text{i.e.} \quad q = -\frac{T_{\infty,3}}{T_{\infty,4}} + \hbar^2 \frac{\partial^2 \log Z}{\partial T_{\infty,2}^2}. \quad (8-3)$$

From Theorem 5.1, one can compute

$$Q(x) = \frac{p}{x - q}, \quad (8-4)$$

and the quantum curve reads

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{x - q} \frac{\partial}{\partial x} - \frac{\hbar p}{x - q} - T_{\infty,4}^2 x^3 - 2T_{\infty,3} T_{\infty,4} x^2 - (T_{\infty,3}^2 + 2T_{\infty,2} T_{\infty,4}) x - H_0 \right] \Psi_{\pm}(x) = 0 \quad (8-5)$$

where

$$H_0 = \frac{T_{\infty,3} T_{\infty,2}}{4} + \hbar^2 \frac{T_{\infty,4}}{2} \frac{\partial \ln Z}{\partial T_{\infty,2}}. \quad (8-6)$$

The diagonal element $P(x)$ of the Lax matrix is a constant so that

$$P(x) = p. \quad (8-7)$$

The \hbar -deformed spectral curve reads

$$\frac{\phi \hbar}{(dx)^2} = \frac{T_{\infty,4}^2}{4} x^3 + \frac{T_{\infty,3} T_{\infty,4}}{2} x^2 + \frac{1}{4} (T_{\infty,3}^2 + 2T_{\infty,2} T_{\infty,4}) x + H_0 \quad (8-8)$$

where the condition on (p, q) implies that

$$H_0 = p^2 - \frac{1}{4} T_{\infty,4}^2 q^3 - \frac{1}{2} T_{\infty,3} T_{\infty,4} q^2 - \frac{1}{4} (T_{\infty,3}^2 + 2T_{\infty,2} T_{\infty,4}) q. \quad (8-9)$$

Breaking the autonomy of the Hamiltonian system with respect to $t_{\infty,1}$ by setting $T_{\infty,2} := \frac{t_{\infty,1}}{2T_{\infty,4}}$ as our isomonodromic time, the compatibility condition eq. (7-17) gives the Hamiltonian system

$$\begin{cases} 2\hbar \frac{\partial q}{\partial t_{\infty,1}} = -\frac{\partial H_0}{\partial p} = -2p \\ 2\hbar \frac{\partial p}{\partial t_{\infty,1}} = \frac{\partial H_0}{\partial q} = -\frac{3}{4} T_{\infty,4}^2 q^2 - T_{\infty,3} T_{\infty,4} q - \frac{T_{\infty,3}^2}{4} - t_{\infty,1} \frac{T_{\infty,4}^2}{2} \end{cases} \quad (8-10)$$

which means that q is solution to

$$\hbar^2 \frac{\partial^2 q}{\partial t_{\infty,1}^2} = \frac{3T_{\infty,4}^2}{8} q^2 + \frac{T_{\infty,3} T_{\infty,4}}{2} q + \frac{T_{\infty,3}^2}{8} + t_{\infty,1} \frac{T_{\infty,4}^2}{4}. \quad (8-11)$$

Setting $T_{\infty,4} = \frac{1}{2}$ and $T_{\infty,3} = 0$, one gets Painlevé 1 equation

$$\hbar^2 \frac{\partial^2 \tilde{q}}{\partial t^2} = 6\tilde{q}^2 + t \quad (8-12)$$

for $\tilde{q} = \frac{q}{4}$ and $t = \frac{t_{\infty,1}}{4}$.

This also gives

$$p = -\hbar \frac{\partial q}{\partial t_{\infty,1}}. \quad (8-13)$$

8.2 Painlevé 2

Let us now consider the case $n = 0$, $n_{\infty} = 0$ and $r_{\infty} = 4$. The Wronskian reads

$$W(x) = -2T_{\infty,4} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) (x - q) \quad (8-14)$$

where

$$q = -\frac{T_{\infty,3}}{T_{\infty,4}} - (A_{\infty,1}^+ + A_{\infty,1}^-) \quad (8-15)$$

From Theorem 5.1, one can compute

$$Q(x) = \frac{p}{x-q} + \frac{\hbar}{2} T_{\infty,4} \frac{\partial A_{\infty,0}^+ + A_{\infty,0}^-}{\partial T_{\infty,2}}, \quad (8-16)$$

and the quantum curve reads

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{x-q} \frac{\partial}{\partial x} - \frac{\hbar p}{x-q} - T_{\infty,4}^2 x^4 - 2T_{\infty,3} T_{\infty,4} x^3 - (T_{\infty,3}^2 + 2T_{\infty,4} T_{\infty,2}) x^2 - [2T_{\infty,3} T_{\infty,2} + 2T_{\infty,4} T_{\infty,1}] x - H_0 - \hbar T_{\infty,4} q \right] \Psi_{\pm}(x) = 0. \quad (8-17)$$

where

$$H_0 = \hbar^2 T_{\infty,4} \frac{\partial \ln \hat{Z}}{\partial T_{\infty,2}} + 2T_{\infty,1} T_{\infty,3} + T_{\infty,2}^2 - \hbar T_{\infty,4} q \quad (8-18)$$

with

$$\hat{Z} := \exp\left(\frac{A_{\infty,0}^+ + A_{\infty,0}^-}{2}\right) Z. \quad (8-19)$$

One has

$$P(x) = T_{\infty,4} x^2 + T_{\infty,3} x + p - T_{\infty,4} q^2 - T_{\infty,3} q. \quad (8-20)$$

So that the \hbar -deformed spectral curve reads, by Theorem 6.1,

$$\frac{\phi_{\hbar}}{(dx)^2} = T_{\infty,4}^2 x^4 + 2T_{\infty,3} T_{\infty,4} x^3 + (T_{\infty,3}^2 + 2T_{\infty,4} T_{\infty,2}) x^2 + \left[2T_{\infty,3} T_{\infty,2} + 2T_{\infty,4} \left(T_{\infty,1} + \frac{\hbar}{2} \right) \right] x + H_0. \quad (8-21)$$

The fact that (p, q) belongs to the \hbar -deformed spectral curve means that

$$H_0 = p^2 - T_{\infty,4}^2 q^4 - 2T_{\infty,3} T_{\infty,4} q^3 - (T_{\infty,3}^2 + 2T_{\infty,4} T_{\infty,2}) q^2 - \left[2T_{\infty,3} T_{\infty,2} + 2T_{\infty,4} \left(T_{\infty,1} + \frac{\hbar}{2} \right) \right] q. \quad (8-22)$$

Using the change of coordinates $2T_{\infty,2} = t_{\infty,1}$, the evolution equations of the spectral Darboux coordinates read

$$2\hbar \frac{\partial p}{\partial t_{\infty,1}} = \frac{\partial H_0}{\partial q} \quad \text{and} \quad 2\hbar \frac{\partial q}{\partial t_{\infty,1}} = -\frac{\partial H_0}{\partial p}. \quad (8-23)$$

One recovers a classical representation of Painlevé 2 by setting $T_{\infty,4} = 1$, $T_{\infty,3} = 0$, $T_{\infty,1} = -\theta$

$$\hbar^2 \frac{\partial^2 q}{\partial t_{\infty,1}^2} = 2q^3 + t_{\infty,1} q + \frac{\hbar}{2} - \theta. \quad (8-24)$$

It also gives

$$p = -\hbar \frac{\partial q}{\partial t_{\infty,1}}. \quad (8-25)$$

8.3 Painlevé 3

Let us now consider the case $n = 1$, $n_{\infty} = 0$, $r_{\infty} = 2$ and $r_1 = 2$. This case being more subtle because $r_{\infty} \leq 2$, we shall describe it in greater details. The Wronskian reads

$$W(x) = w \frac{(x-q)}{(x-X_1)^2} = \frac{w}{x-X_1} + \frac{w(X_1-q)}{(x-X_1)^2} \quad (8-26)$$

with

$$\begin{aligned} w &= -2T_{\infty,2} \exp(A_{\infty,0}^+ + A_{\infty,0}^-), \\ (X_1 - q)w &= 2T_{1,2} \exp(A_{1,0}^+ + A_{1,0}^-). \end{aligned} \quad (8-27)$$

Let us work with X_1 fixed and set its value to $X_1 = 0$.

We have also

$$q = 2 - \frac{T_{\infty,1}}{T_{\infty,2}} - (A_{\infty,1}^+ + A_{\infty,1}^-) \quad \text{and} \quad w = -wq \frac{T_{1,1}}{T_{1,2}} - wq (A_{1,1}^+ + A_{1,1}^-). \quad (8-28)$$

This allows expressing

$$\begin{cases} A_{1,0}^+ + A_{1,0}^- = \log\left(-\frac{qw}{2T_{1,2}}\right) \\ A_{1,1}^+ + A_{1,1}^- = -\frac{1}{q} - \frac{T_{1,1}}{T_{1,2}} \end{cases}. \quad (8-29)$$

Working with fixed value of X_1 means that the action of the vector field $\frac{\partial}{\partial X_1}$ is vanishing. This can be translated into the vanishing of the vector field $T_{1,1} \frac{\partial}{\partial T_{1,2}} + T_{1,2} \frac{\partial}{\partial T_{1,3}}$ thanks to eq. (7-10).

From Theorem 5.1, one has

$$Q(x) = \frac{p}{x-q} + \frac{\hbar T_{1,2}}{2x^2} \frac{\partial A_{1,0}^+ + A_{1,0}^-}{\partial T_{1,2}} + \frac{\hbar}{2x} T_{1,2} \frac{\partial A_{1,1}^+ + A_{1,1}^-}{\partial T_{1,2}}. \quad (8-30)$$

On the other hand eq. (5-26) and the asymptotics from Appendix C give that $Q(x) = O(x^{-2})$ as $x \rightarrow \infty$. Thus

$$p = -\frac{\hbar}{2} T_{1,2} \frac{\partial A_{1,1}^+ + A_{1,1}^-}{\partial T_{1,2}}, \quad (8-31)$$

i.e.

$$p = -\frac{\hbar T_{1,2}}{2q^2} \frac{\partial q}{\partial T_{1,2}} - \frac{\hbar T_{1,1}}{2 T_{1,2}} \quad (8-32)$$

and

$$Q(x) = \frac{p}{x-q} + \frac{1}{x^2} \left[\frac{\hbar T_{1,2}}{2q} \frac{\partial q}{\partial T_{1,2}} + \frac{\hbar T_{1,2}}{2} \frac{\partial \log w}{\partial T_{1,2}} - \frac{\hbar}{2} \right] - \frac{p}{x}. \quad (8-33)$$

The quantum curve thus reads

$$\left\{ \hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \left[\frac{1}{x-q} - \frac{2}{x} \right] \frac{\partial}{\partial x} - \frac{\hbar p}{x-q} - T_{\infty,2}^2 - \frac{C_1}{x} - \frac{C_2}{x^2} - \frac{2T_{1,1}T_{1,2}}{x^3} - \frac{T_{1,2}^2}{x^4} \right\} \Psi_{\pm}(x) = 0 \quad (8-34)$$

where

$$C_1 := 2T_{\infty,1}T_{\infty,2} - \hbar p \quad (8-35)$$

and

$$C_2 := \hbar^2 T_{1,2} \frac{\partial \log\left(w^{\frac{1}{2}} Z\right)}{\partial T_{1,2}} + T_{1,1}^2 - \hbar p q - \frac{\hbar}{2} - \frac{\hbar^2 q T_{1,1}}{2 T_{1,2}}. \quad (8-36)$$

One can now compute $P(x)$. As in all cases where $r_{\infty} \leq 2$, its computation is slightly more complicated. Let us write it under the form

$$P(x) = \frac{ax^2 + bx + c}{x^2}. \quad (8-37)$$

The constraint $P(q) = p$ implies that

$$c = pq^2 - aq^2 - bq. \quad (8-38)$$

In terms of these coefficients, the \hbar -deformed spectral curve reads

$$\frac{\phi_{\hbar}}{(dx)^2} = T_{\infty,2}^2 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{2T_{1,1}T_{1,2}}{x^3} + \frac{T_{1,2}^2}{x^4} + \hbar \frac{a+p}{x} + \hbar \frac{(p-a)q}{x^2}. \quad (8-39)$$

On the other hand, $\frac{\phi_{\hbar}}{(dx)^2} = P(x)^2 + M(x)W(x)$ implies that

$$\frac{\phi_{\hbar}}{(dx)^2} = a^2 + \frac{2ab}{x} + O(x^{-2}) \quad (8-40)$$

since both $M(x)$ and $W(x)$ behave as x^{-1} as $x \rightarrow \infty$. Equating the two leading orders gives

$$\begin{cases} a = T_{\infty,2} \\ b = T_{\infty,1} + \frac{\hbar}{2} \end{cases} . \quad (8-41)$$

Hence the \hbar -deformed spectral curve reads

$$\frac{\phi_{\hbar}}{(dx)^2} = T_{\infty,2}^2 + \frac{H_{-1}}{x} + \frac{H_{-2}}{x^2} + \frac{2T_{1,1}T_{1,2}}{x^3} + \frac{T_{1,2}^2}{x^4} \quad (8-42)$$

where

$$H_{-1} := 2T_{\infty,2} \left(T_{\infty,1} + \frac{\hbar}{2} \right) \quad \text{and} \quad H_{-2} := \hbar^2 T_{1,2} \frac{\partial \log \left(w^{\frac{1}{2}} Z \right)}{\partial T_{1,2}} + T_{1,1}^2 - \frac{\hbar}{2} - \hbar T_{\infty,2} q - \hbar^2 \frac{q T_{1,1}}{2T_{1,2}}. \quad (8-43)$$

The fact that the spectral Darboux coordinates belong to the \hbar -deformed spectral curve implies that

$$H_{-2} = p^2 q^2 - T_{\infty,2}^2 q^2 - H_{-1} q - \frac{2T_{1,1}T_{1,2}}{q} - \frac{T_{1,2}^2}{q^2}. \quad (8-44)$$

Let us now explain how to get an isomonodromic system in this case. For this purpose, one has to impose an explicit dependence of the Casimirs on the isomonodromic time $t := t_{1,1}$ by identifying our Lax matrix with the one of [28]. This is done by comparing the elements $P(x)$ and $W(x)$ of our Lax matrix with the ones of [28] as well as the coefficients of x^{-3} and x^{-4} of their determinant. One obtains the identification

$$\begin{cases} T_{\infty,2} := \frac{t}{2} \\ T_{\infty,1} + \frac{\hbar}{2} := -\frac{\theta_{\infty}}{2} \\ T_{1,1} := -\frac{\theta_0}{2} \\ T_{1,2} := \frac{t}{2} \end{cases} . \quad (8-45)$$

Let us now verify that one gets a Hamiltonian system driving the time evolution of the spectral Darboux coordinates. Identifying the value of the zero of $W(x)$ in both matrices as well as the value of $P(x)$ at the corresponding point, one gets

$$\begin{cases} q = -\frac{1}{q_{\text{IMS}}} \\ p = \frac{t}{2} - \frac{\theta_{\infty}}{2q} + (p_{\text{IMS}} - \frac{t}{2}) \frac{1}{q^2} \end{cases} \quad (8-46)$$

where $(q_{\text{IMS}}, p_{\text{IMS}})$ refers to the variables denoted by (q, p) in [28]. Under this identification, they obtained an isomonodromic system with Hamiltonian

$$H^{\text{IMS}} := \frac{1}{t} \left[2q_{\text{IMS}}^2 p_{\text{IMS}}^2 + 2p_{\text{IMS}} (-tq_{\text{IMS}}^2 + \theta_{\infty} q_{\text{IMS}} + t) - (\theta_0 + \theta_{\infty}) t q_{\text{IMS}} - t^2 - \frac{1}{4} (\theta_0^2 - \theta_{\infty}^2) \right]. \quad (8-47)$$

After the identification, the latter reads in our notations

$$H^{\text{IMS}} = \frac{2}{t} H_{-2} - \frac{t}{2} - \frac{\theta_0^2 + \theta_{\infty}^2}{4t}. \quad (8-48)$$

From [28], one has $\hbar \frac{\partial q_{\text{IMS}}}{\partial t} = \frac{\partial H^{\text{IMS}}}{\partial p_{\text{IMS}}}$ and $\hbar \frac{\partial p_{\text{IMS}}}{\partial t} = -\frac{\partial H^{\text{IMS}}}{\partial q_{\text{IMS}}}$. Since

$$dp_{\text{IMS}} \wedge dq_{\text{IMS}} = -dp \wedge dq, \quad (8-49)$$

this just amounts to a change of Darboux coordinates and one has $\hbar \frac{\partial q}{\partial t} = -\frac{\partial H_{-2}}{\partial p}$ and $\hbar \frac{\partial p}{\partial t} = \frac{\partial H_{-2}}{\partial q}$.

This implies that, up to a simple shift, $\omega^{-\frac{1}{2}} Z$ is a tau-function.

Finally, $q_{\text{IMS}} = -q^{-1}$ is solution to Painlevé 3 equation

$$\hbar^2 \frac{\partial^2 q_{\text{IMS}}}{\partial t^2} = \frac{\hbar^2}{q_{\text{IMS}}} \left(\frac{\partial q_{\text{IMS}}}{\partial t} \right)^2 - \frac{\hbar^2}{t} \frac{\partial q_{\text{IMS}}}{\partial t} + \frac{4}{t} (-2T_{1,1} q_{\text{IMS}}^2 - T_{\infty,1}) + 4q_{\text{IMS}}^3 - \frac{4}{q_{\text{IMS}}}. \quad (8-50)$$

8.4 Painlevé 4

Let us now consider the case $n = 1$, $n_\infty = 0$, $r_1 = 1$ and $r_\infty = 3$. The Wronskian reads

$$W(x) = w \frac{(x-q)}{(x-X_1)} = w + \frac{w(X_1-q)}{x-X_1} \quad (8-51)$$

with

$$\begin{aligned} w &= -2T_{\infty,3} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) \\ w(X_1-q) &= 2T_{1,1} \exp(A_{1,0}^+ + A_{1,0}^-) \end{aligned} \quad (8-52)$$

We have also

$$q = X_1 - \frac{T_{\infty,2}}{T_{\infty,3}} - (A_{\infty,1}^+ + A_{\infty,1}^-). \quad (8-53)$$

Theorem 5.1 gives

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \left[\frac{1}{x-q} - \frac{1}{x} \right] \frac{\partial}{\partial x} - \frac{\hbar p}{x-q} - T_{\infty,3}^2 x^2 - 2T_{\infty,2} T_{\infty,3} x - T_{\infty,2}^2 - 2T_{\infty,1} T_{\infty,3} \right. \\ \left. - \frac{H_{-1}}{x} - \frac{T_{1,1}^2}{x^2} - \hbar \frac{T_{\infty,3} q}{x} \right] \Psi_{\pm}(x) = 0 \quad (8-54)$$

where

$$H_{-1} = \hbar^2 \frac{\partial \ln \hat{Z}}{\partial t_{1,0}} + 2T_{\infty,1} T_{\infty,2} - \hbar T_{\infty,3} q \quad (8-55)$$

with

$$\hat{Z} := \exp\left(\frac{A_{1,0}^+ + A_{1,0}^-}{2}\right) Z. \quad (8-56)$$

One can compute

$$P(x) = \frac{T_{\infty,3} x^2 + T_{\infty,2} x + (q - X_1) p - T_{\infty,3} q^2 - T_{\infty,2} q}{(x - X_1)}. \quad (8-57)$$

The \hbar -deformed spectral curve reads

$$\frac{\phi_{\hbar}}{(dx)^2} = T_{\infty,3}^2 x^2 + 2T_{\infty,2} T_{\infty,3} x + T_{\infty,2}^2 + 2 \left(T_{\infty,1} + \frac{\hbar}{2} \right) T_{\infty,3} + \frac{H_{-1}}{x - X_1} + \frac{T_{1,1}^2}{(x - X_1)^2} \quad (8-58)$$

and the constraint on the spectral Darboux coordinates reads

$$H_{-1} = (q - X_1) \left[p^2 - T_{\infty,3}^2 q^2 - 2T_{\infty,2} T_{\infty,3} q - T_{\infty,2}^2 - 2 \left(T_{\infty,1} + \frac{\hbar}{2} \right) T_{\infty,3} - \frac{T_{1,1}^2}{(q - X_1)^2} \right]. \quad (8-59)$$

To recover the Hamiltonian structure more simply, let us select a time t defined by the corresponding vector field

$$\frac{\partial}{\partial t} := \frac{\partial}{\partial x} + \frac{\partial}{\partial t_{1,0}} \quad (8-60)$$

and set $X_1 = 0$ and $T_{\infty,2} = t$ in order to have a solution to the isomonodromicity condition. The compatibility of the system evaluated at $x = q$ recovers the Hamiltonian representation of the evolution equations

$$\begin{cases} \hbar \frac{\partial q}{\partial t} = \frac{\partial H_{-1}}{\partial p} \\ \hbar \frac{\partial p}{\partial t} = -\frac{\partial H_{-1}}{\partial q} \end{cases} \quad (8-61)$$

leading to the fact that \hat{Z} is a corresponding isomonodromic tau function.

This leads to the differential equation

$$q\hbar^2 \frac{\partial^2 q}{\partial t^2} = \frac{1}{2} \left(\hbar \frac{\partial q}{\partial t} \right)^2 - 2T_{1,1} + 2 \left(t^2 + 2 \left(T_{\infty,1} + \frac{\hbar}{2} \right) T_{\infty,3} \right) q^2 + 8tT_{\infty,3}q^3 + 6T_{\infty,3}^2q^4. \quad (8-62)$$

One recovers a known representation of Painlevé 4 by setting $T_{\infty,3} = 1$, $T_{\infty,2} = t$, $T_{\infty,1} + \frac{\hbar}{2} = \theta_\infty$ and $T_{1,1} = \theta_0$.

This also gives

$$p = \frac{\hbar}{2q} \frac{\partial q}{\partial t_{1,0}}. \quad (8-63)$$

8.5 Painlevé 5

Let us now consider the case $n = 2$, $n_\infty = 0$, $r_1 = r_2 = 1$ and $r_\infty = 2$. For simplicity, we consider $X_1 = 0$ and $X_2 = 1$. The Wronskian reads

$$W(x) = w \frac{(x-q)}{x(x-1)} = \frac{wq}{x} + \frac{w(q-1)}{x-1} \quad (8-64)$$

with

$$\begin{aligned} w &= -2T_{\infty,2} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) \\ qw &= 2T_{1,1} \exp(A_{1,0}^+ + A_{1,0}^-) \\ (q-1)w &= 2T_{2,1} \exp(A_{2,0}^+ + A_{2,0}^-) \end{aligned} \quad (8-65)$$

From Theorem 5.1, one has

$$\begin{aligned} Q(x) &= \frac{p}{x-q} + \frac{\hbar T_{1,1}}{x} \frac{2}{2} \frac{\partial(A_{1,0}^+ + A_{1,0}^-)}{\partial T_{1,2}} + \frac{\hbar T_{2,1}}{x-1} \frac{2}{2} \frac{\partial(A_{2,0}^+ + A_{2,0}^-)}{\partial T_{2,2}} \\ &= \frac{p}{x-q} + \frac{\hbar T_{1,1}}{x} \frac{2}{2} \frac{\partial \log[qw]}{\partial T_{1,2}} + \frac{\hbar T_{2,1}}{x-1} \frac{2}{2} \frac{\partial \log[(q-1)w]}{\partial T_{2,2}}. \end{aligned} \quad (8-66)$$

From eq. (5-26) and Appendix C, one knows that

$$Q(x) = \frac{\hbar}{x} T_{1,1} \frac{\partial \log[w]}{\partial T_{1,2}} + \frac{\hbar}{x} T_{2,1} \frac{\partial \log[w]}{\partial T_{2,2}} + O(x^{-2}) \quad (8-67)$$

as $x \rightarrow \infty$.

One thus have

$$p = -\hbar \frac{T_{1,1}}{2} \frac{\partial \log \frac{q}{w}}{\partial T_{1,2}} - \hbar \frac{T_{2,1}}{2} \frac{\partial \log \frac{(q-1)}{w}}{\partial T_{2,2}} \quad (8-68)$$

and the quantum curve reads

$$\left\{ \hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \left[\frac{1}{x-q} - \frac{1}{x} - \frac{1}{x-1} \right] \frac{\partial}{\partial x} - \frac{\hbar p}{x-q} - T_{\infty,2}^2 - \frac{C_0}{x} - \frac{C_1}{x-1} - \frac{T_{1,1}^2}{x^2} - \frac{T_{2,1}^2}{(x-1)^2} \right\} \Psi_\pm(x) = 0 \quad (8-69)$$

where the coefficients of x^{-1} and x^{-2} in the expansion around $x = \infty$ give

$$C_0 + C_1 := 2T_{\infty,1}T_{\infty,2} - \hbar p + \hbar^2 T_{1,1} \frac{\partial \log[w]}{\partial T_{1,2}} + \hbar^2 T_{2,1} \frac{\partial \log[w]}{\partial T_{2,2}} \quad (8-70)$$

and

$$C_1 = T_{\infty,1}^2 - T_{1,1}^2 - T_{2,1}^2 + \hbar T_{2,1} \frac{\partial \log[(q-1)^{-\frac{1}{2}} w^{-\frac{1}{2}} Z]}{\partial T_{2,2}}. \quad (8-71)$$

Let us write

$$P(x) = \frac{ax^2 + bx + c}{x(x-1)} \quad (8-72)$$

where $P(q) = p$ imposes that

$$c = q(q-1)p - aq^2 - bq. \quad (8-73)$$

In terms of these coefficients, the \hbar -deformed spectral curve reads

$$\frac{\phi_\hbar}{(dx)^2} = T_{\infty,2}^2 + \frac{C_0}{x} + \frac{C_1}{x-1} + \frac{T_{1,1}^2}{x^2} + \frac{T_{2,1}^2}{(x-1)^2} - \hbar \frac{(p-a)q}{x} + \hbar \frac{p(1+q) + a(1-q)}{x-1}. \quad (8-74)$$

The expansion around $x = \infty$ gives

$$a = T_{\infty,2} \quad \text{and} \quad b = -T_{\infty,2} + T_{\infty,1} + \frac{\hbar}{2} + \frac{\hbar^2}{2T_{\infty,2}} \left[T_{1,1} \frac{\partial \log w}{\partial T_{1,2}} + T_{2,1} \frac{\partial \log w}{\partial T_{2,2}} \right] \quad (8-75)$$

so that

$$\frac{\phi_\hbar}{(dx)^2} = T_{\infty,2}^2 + \frac{H_0}{x} + \frac{H_1}{x-1} + \frac{T_{1,1}^2}{x^2} + \frac{T_{2,1}^2}{(x-1)^2} \quad (8-76)$$

with

$$\begin{cases} H_0 + H_1 = 2T_{\infty,2} \left(T_{\infty,1} + \frac{\hbar}{2} \right) + \hbar^2 \left[T_{1,1} \frac{\partial \log w}{\partial T_{1,2}} + T_{2,1} \frac{\partial \log w}{\partial T_{2,2}} \right] \\ H_1 = T_{\infty,1}^2 - T_{1,1}^2 - T_{2,1}^2 + \hbar T_{2,1} \frac{\partial \log[(q-1)^{-\frac{1}{2}} w^{-\frac{1}{2}} Z]}{\partial T_{2,2}} + \hbar p(1+q) + T_{\infty,2}(1-q) \end{cases}. \quad (8-77)$$

On the other hand, one can replace the second equation by the fact that (p, q) belongs to the spectral curve:

$$p^2 = T_{\infty,2}^2 + \frac{H_0}{q} + \frac{H_1}{q-1} + \frac{T_{1,1}^2}{q^2} + \frac{T_{2,1}^2}{(q-1)^2}. \quad (8-78)$$

Let us now recover the associated isomonodromic system. For this purpose, we can identify our Lax matrix with the one of [20] by setting $T_{\infty,2} := t$, $T_{1,1} := -\frac{\kappa_1}{2}$, where κ_1 is the monodromy at 0, $T_{2,1} := -\frac{\kappa_2}{2}$, where κ_2 is the monodromy at 1 and $H_0 + H_1 = a$ is related to the monodromy at ∞ . In this identification, $\hbar^2 \left[T_{1,1} \frac{\partial \log w}{\partial T_{1,2}} + T_{2,1} \frac{\partial \log w}{\partial T_{2,2}} \right]$ has to vanish because we are working on a reduced phase space (see [20]) for details so that $H_0 + H_1 = 2T_{\infty,2} \left(T_{\infty,1} + \frac{\hbar}{2} \right) = a$.

After this identification, [20] proves that the isomonodromy condition is fulfilled and one has an Hamiltonian system with Hamiltonian H_0 so that $u = \frac{1-q}{1+q}$ is solution to Painlevé 5 equation

$$\hbar^2 \frac{\partial^2 u}{\partial t^2} = \hbar^2 \left(\frac{1}{2u} + \frac{1}{u-1} \right) \left(\frac{\partial u}{\partial t} \right)^2 - \frac{\hbar^2}{t} \frac{\partial u}{\partial t} + \frac{(\alpha u^2 + \beta)(u-1)^2}{t^2 u} + \frac{\gamma u}{t} + \frac{\delta u(u+1)}{u-1} \quad (8-79)$$

where $\alpha = 2T_{2,1}^2$, $\beta = -2T_{1,1}^2$, $\gamma = 2T_{\infty,2}T_{\infty,1}$ and $\delta = -2$.

8.6 Painlevé 6

Let us consider the case $n = 3$, $n_\infty = 0$, $r_\infty = r_1 = r_2 = r_3 = 1$, $X_1 = 0$ and $X_2 = 1$. In this Fuchsian system, our isomonodromic time will be $X_3 = t = t_{3,0}$. The Wronskian reads

$$W(x) = w \frac{x-q}{x(x-1)(x-t)} = w \left(-\frac{q}{tx} + \frac{q-1}{(t-1)(x-1)} - \frac{q-t}{t(t-1)(x-t)} \right) \quad (8-80)$$

where using Appendix C:

$$\begin{aligned} w &= -2T_{\infty,1} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) \\ -\frac{q}{t} w &= 2T_{1,1} \exp(A_{1,0}^+ + A_{1,0}^-) \\ \frac{q-1}{(t-1)} w &= 2T_{2,1} \exp(A_{2,0}^+ + A_{2,0}^-) \\ -\frac{q-t}{t(t-1)} w &= 2T_{3,1} \exp(A_{3,0}^+ + A_{3,0}^-) \end{aligned} \quad (8-81)$$

and

$$q = 1 + t - \frac{A_{\infty,1}^+ - A_{\infty,1}^-}{2T_{\infty,1}} - A_{\infty,1}^+ + A_{\infty,1}^-. \quad (8-82)$$

From Theorem 5.1, one has

$$Q(x) = \frac{p}{x-q} + \frac{\hbar}{2(x-t)} \frac{\partial}{\partial t} \left[\log \left(\frac{(q-t)w}{t(t-1)} \right) \right]. \quad (8-83)$$

On the other hand, Appendix C gives that $Q(x) = \frac{\hbar}{x} \frac{\partial \log w}{\partial t} + O(x^{-2})$, leading to

$$p = -\frac{\hbar}{2(x-t)} \frac{\partial}{\partial t} \left[\log \left(\frac{(q-t)}{t(t-1)w} \right) \right]. \quad (8-84)$$

On the other hand, using the notations of eq. (7-10), the classical spectral curve reads

$$\frac{\phi_0}{(dx)^2} = \frac{T_{1,1}^2}{x^2} + \frac{T_{2,1}^2}{(x-1)^2} + \frac{T_{3,1}^2}{(x-t)^2} + \frac{\hbar^2}{x} \frac{\partial \omega_{0,0}}{\partial t_{1,0}} + \frac{\hbar^2}{x-1} \frac{\partial \omega_{0,0}}{\partial t_{2,0}} + \frac{\hbar^2}{x-t} \frac{\partial \omega_{0,0}}{\partial t}. \quad (8-85)$$

The asymptotic expansion at infinity $\frac{\phi_0}{(dx)^2} = T_{\infty,1}^2 x^{-2} + O(x^{-3})$ leads to the conditions

$$\hbar^2 \frac{\partial \omega_{0,0}}{\partial t_{1,0}} + \hbar^2 \frac{\partial \omega_{0,0}}{\partial t_{2,0}} + \hbar^2 \frac{\partial \omega_{0,0}}{\partial t} = 0 \quad (8-86)$$

and

$$T_{1,1}^2 + T_{2,1}^2 + T_{3,1}^2 + \hbar^2 \frac{\partial \omega_{0,0}}{\partial t_{2,0}} + t \hbar^2 \frac{\partial \omega_{0,0}}{\partial t} = T_{\infty,1}^2. \quad (8-87)$$

More generally, as vector fields in $\mathcal{Q}(\mathbb{P}^1, -\infty - 0 - 1 - t, 0, \mathbf{T})$, the vector fields $\hbar^2 \frac{\partial}{\partial t_{1,0}} + \hbar^2 \frac{\partial}{\partial t_{2,0}} + \hbar^2 \frac{\partial}{\partial t}$ and $\hbar^2 t \frac{\partial}{\partial t_{2,0}} + \hbar^2 t \frac{\partial}{\partial t}$ are vanishing. The quantum curve reads

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \left[\frac{1}{x-q} - \frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-t} \right] \frac{\partial}{\partial x} - \frac{\hbar p}{x-q} - \frac{C_0}{x} - \frac{C_1}{x-1} - \frac{C_t}{x-t} - \frac{T_{1,1}^2}{x^2} - \frac{T_{2,1}^2}{(x-1)^2} - \frac{T_{3,1}^2}{(x-t)^2} \right] \Psi_{\pm} = 0 \quad (8-88)$$

where

$$\begin{cases} C_0 = \hbar^2 \frac{\partial \log Z}{\partial t_{1,0}} \\ C_1 = \hbar^2 \frac{\partial \log Z}{\partial t_{2,0}} \\ C_t = \hbar^2 \frac{\partial \log[wZ]}{\partial t} - \hbar p \end{cases}. \quad (8-89)$$

Let us write

$$P(x) = \frac{ax^2 + bx + c}{x(x-1)(x-t)}. \quad (8-90)$$

In terms of these coefficients, the \hbar -deformed spectral curve reads

$$\frac{\phi_{\hbar}}{(dx)^2} = \frac{H_0}{x} + \frac{H_1}{x-1} + \frac{H_t}{x-t} + \frac{T_{1,1}^2}{x^2} + \frac{T_{2,1}^2}{(x-1)^2} + \frac{T_{3,1}^2}{(x-t)^2} \quad (8-91)$$

where

$$\begin{cases} H_0 = \hbar^2 \frac{\partial \log Z}{\partial t_{1,0}} + \hbar \frac{pt - q(a+p(1+t))}{t} \\ H_1 = \hbar^2 \frac{\partial \log Z}{\partial t_{2,0}} + \hbar \frac{a(1-q) - qp(1+t)}{1-t} \\ H_t = \hbar^2 \frac{\partial \log[wZ]}{\partial t} - \hbar p + \hbar \frac{a(t-q) - qp(1+t)}{t(t-1)} \end{cases} \quad (8-92)$$

satisfy the conditions

$$\begin{cases} H_0 + H_1 + H_t = 0 \\ H_1 + tH_t + T_{1,1}^2 + T_{2,1}^2 + T_{3,1}^2 = \hbar a \end{cases}. \quad (8-93)$$

The expansion around $x = \infty$ imposes that a is solution to

$$a^2 = T_{\infty,1}^2 + \hbar a. \quad (8-94)$$

This means that we have a Fuchsian system with monodromies at 0, 1, t and ∞ given by $\theta_0 = 2T_{1,1}$, $\theta_1 = 2T_{2,1}$, $\theta_t = 2T_{3,1}$ and $\theta_\infty = 2a$ and q satisfies the associated Painlevé 6 equation

$$\begin{aligned} \hbar^2 \frac{\partial^2 q}{\partial t^2} &= \frac{\hbar^2}{2} \left[\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right] \left(\frac{\partial q}{\partial t} \right)^2 - \hbar^2 \left[\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right] \frac{\partial q}{\partial t} \\ &\quad + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left[2T_{\infty,1}^2 + \frac{\hbar^2}{2} - 2T_{1,1}^2 \frac{t}{q^2} + 2T_{2,1}^2 \frac{t-1}{(q-1)^2} - \left(2T_{3,1} - \frac{\hbar^2}{2} \right) \frac{t(t-1)}{(q-t)^2} \right]. \end{aligned} \quad (8-95)$$

8.7 Second equation of the Painlevé 2 hierarchy

Let us consider the case $n = 0$, $r_\infty = 5$ and $n_\infty = 0$. The Wronskian takes the form

$$W(x) = w(x - q_1)(x - q_2) \quad (8-96)$$

where

$$\begin{aligned} w &= -2T_{\infty,5} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) \\ q_1 + q_2 &= -(A_{\infty,1}^+ + A_{\infty,1}^-) - \frac{T_{\infty,4}}{T_{\infty,5}} \\ q_1 q_2 &= (A_{\infty,2}^+ + A_{\infty,2}^-) + \frac{1}{2}(A_{\infty,1}^+ + A_{\infty,1}^-)^2 + \frac{T_{\infty,4}}{T_{\infty,5}}(A_{\infty,1}^+ + A_{\infty,1}^-) + \frac{T_{\infty,3}}{T_{\infty,5}} \end{aligned} \quad (8-97)$$

The quantum curve reads

$$\begin{aligned} &\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 \left[\sum_{i=1}^2 \frac{1}{x - q_i} \right] \frac{\partial}{\partial x} - \sum_{i=1}^2 \frac{\hbar p_i}{x - q_i} - T_{\infty,5}^2 x^6 - 2T_{\infty,5} T_{\infty,4} x^5 \right. \\ &\quad \left. - (2T_{\infty,5} T_{\infty,3} + T_{\infty,4}^2) x^4 - (2T_{\infty,5} T_{\infty,2} + 2T_{\infty,4} T_{\infty,3}) x^3 - (2T_{\infty,5} T_{\infty,1} + 2T_{\infty,4} T_{\infty,2} + T_{\infty,3}^2) x^2 \right. \\ &\quad \left. - C_1 x - C_0 \right] \Psi_\pm = 0 \end{aligned}$$

where

$$\begin{cases} C_1 = \hbar^2 T_{\infty,5} \frac{\partial}{\partial T_{\infty,2}} \log \left(w^{\frac{1}{2}} Z \right) + 2T_{\infty,4} T_{\infty,1} + 2T_{\infty,3} T_{\infty,2} \\ C_0 = \hbar^2 T_{\infty,4} \frac{\partial}{\partial T_{\infty,2}} \log \left(w^{\frac{1}{2}} Z \right) + 2\hbar^2 T_{\infty,5} \frac{\partial}{\partial T_{\infty,3}} \log \left(w^{\frac{1}{2}} Z \right) + \frac{\hbar^2}{2} T_{\infty,5} \frac{\partial \log(q_1 + q_2)}{\partial T_{\infty,2}} + T_{\infty,2}^2 + 2T_{\infty,3} T_{\infty,1} \end{cases} \quad (8-98)$$

The diagonal term of the Lax matrix reads

$$P(x) = T_{\infty,5} x^3 + T_{\infty,4} x^2 + \alpha x + \beta \quad (8-99)$$

where

$$\begin{aligned} \alpha &= \frac{p_1 - p_2}{q_1 - q_2} - T_{\infty,5}(q_2^2 + q_1^2 + q_1 q_2) - T_{\infty,4}(q_1 + q_2), \\ \beta &= \frac{p_2 q_1 - p_1 q_2}{q_1 - q_2} + T_{\infty,5} q_1 q_2 (q_1 + q_2) + T_{\infty,4} q_1 q_2. \end{aligned}$$

The \hbar -deformed spectral curve reads

$$\frac{\phi_\hbar}{(dx)^2} = T_{\infty,5}^2 x^6 + 2T_{\infty,5} T_{\infty,4} x^5 + (2T_{\infty,5} T_{\infty,3} + T_{\infty,4}^2) x^4 + (2T_{\infty,5} T_{\infty,2} + 2T_{\infty,4} T_{\infty,3}) x^3$$

$$+ (2T_{\infty,5}T_{\infty,1} + 2T_{\infty,4}T_{\infty,2} + T_{\infty,3}^2) x^2 + H_1 x + H_0 \quad (8-100)$$

where

$$\begin{cases} H_1 = C_1 - \hbar T_{\infty,5}(q_1 + q_2) \\ H_0 = C_0 - \hbar T_{\infty,5}(q_1^2 + q_2^2) - \hbar T_{\infty,4}(q_1 + q_2) \end{cases} \quad (8-101)$$

On the other hand, the fact that the spectral Darboux coordinates belong to the \hbar -deformed spectral curve implies that H_0 and H_1 are subject to the constraint

$$\begin{aligned} H_1 q_i + H_0 &= p_i^2 - T_{\infty,5}^2 q_i^6 - 2T_{\infty,5}T_{\infty,4} q_i^5 - (2T_{\infty,5}T_{\infty,3} + T_{\infty,4}^2) q_i^4 - (2T_{\infty,5}T_{\infty,2} + 2T_{\infty,4}T_{\infty,3}) q_i^3 \\ &\quad - (2T_{\infty,5}T_{\infty,1} + 2T_{\infty,4}T_{\infty,2} + T_{\infty,3}^2) q_i^2 \end{aligned} \quad (8-102)$$

for $i \in \{1, 2\}$. This means that

$$\begin{aligned} H_1 &= \frac{p_1^2 - p_2^2}{q_1 - q_2} - T_{\infty,5}^2 (q_1 + q_2)(q_1^4 + q_1^2 q_2^2 + q_2^4) - 2T_{\infty,5}T_{\infty,4}(q_1^4 + q_1^3 q_2 + q_1^2 q_2^2 + q_1 q_2^3 + q_2^4) \\ &\quad - (2T_{\infty,5}T_{\infty,3} + T_{\infty,4}^2)(q_1 + q_2)(q_1^2 + q_2^2) - (2T_{\infty,5}T_{\infty,2} + 2T_{\infty,4}T_{\infty,3})(q_1^2 + q_1 q_2 + q_2^2) \\ &\quad - (2T_{\infty,5}T_{\infty,1} + 2T_{\infty,4}T_{\infty,2} + T_{\infty,3}^2)(q_1 + q_2), \\ H_0 &= \frac{q_1 p_2^2 - q_2 p_1^2}{q_1 - q_2} + q_1 q_2 \left[T_{\infty,5}^2 (q_1^4 + q_1^3 q_2 + q_1^2 q_2^2 + q_1 q_2^3 + q_2^4) + 2T_{\infty,5}T_{\infty,4}(q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3) \right. \\ &\quad \left. + (T_{\infty,4}^2 + 2T_{\infty,5}T_{\infty,3})(q_1^2 + q_1 q_2 + q_2^2) + (2T_{\infty,4}T_{\infty,3} + 2T_{\infty,5}T_{\infty,2})(q_1 + q_2) + T_{\infty,3}^2 \right. \\ &\quad \left. + 2T_{\infty,4}T_{\infty,2} + 2T_{\infty,5}T_{\infty,1} \right]. \end{aligned}$$

As explained for example in Section 6.3.1 of [32], it is possible to introduce an explicit dependence in the times $t_{\infty,1}$ and $t_{\infty,2}$ in such a way that they give rise to isomonodromic deformations by solving eq. (7-6).

This implies that, after having made the Lax matrix $L(x)$ explicitly dependent in these times, one obtains Hamilton's equations

$$\forall i \in \{1, 2\}, \begin{cases} 2\hbar \frac{\partial q_i}{\partial t_{\infty,1}} = -\frac{\partial H_0}{\partial p_i} \\ 2\hbar \frac{\partial q_i}{\partial t_{\infty,2}} = -\frac{\partial H_1}{\partial q_i} \\ 2\hbar \frac{\partial p_i}{\partial t_{\infty,1}} = \frac{\partial H_0}{\partial q_i} \\ 2\hbar \frac{\partial p_i}{\partial t_{\infty,2}} = \frac{\partial H_1}{\partial p_i} \end{cases} \quad (8-103)$$

The second equation gives

$$\begin{cases} p_1 = -(q_1 - q_2) \frac{\partial q_1}{\partial t_{\infty,2}} \\ p_2 = -(q_1 - q_2) \frac{\partial q_2}{\partial t_{\infty,2}} \end{cases} \quad (8-104)$$

9 References

- [1] M.R. Adams, J. Harnad, J. Hurtubise, "Isospectral Hamiltonian Flows in Finite and Infinite Dimensions", *Communications in Mathematical Physics*, Vol. **134**, 555-585, 1990.
- [2] M.R. Adams, J. Harnad, J. Hurtubise, "Spectral Darboux Coordinates and Liouville-Arnold Integration in Loop Algebras", *Communications in Mathematical Physics*, Vol. **155**, Issue 2, 385-413, 1993.
- [3] R. Belliard, B. Eynard, O. Marchal, "Integrable differential systems of topological type and reconstruction by the topological recursion", *Annales Henri Poincaré*, Vol. **18**, Issue 10, 3193-3248, 2017.
- [4] M. Bertola, D. Korotkin and C. Norton, "Symplectic geometry of the moduli space of projective structures in homological coordinates", *Inventiones mathematicae*, Vol. **210**, Issue 3, 759-814, 2015.
- [5] G. Borot, "Lecture notes on topological recursion and geometry", *arxiv:1705.09986*, 2017.

- [6] G. Borot, B. Eynard, “Geometry of spectral curves and all order dispersive integrable system”, *Symmetry, Integrability and Geometry: Methods and Applications*, Vol. **8**, 2012.
- [7] G. Borot, B. Eynard, “All-order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials”, *Quantum Topology*, Vol. **6**, Issue 1, 39-138, 2015.
- [8] V. Bouchard, B. Eynard, “Reconstructing WKB from topological recursion”, *Journal de l’Ecole Polytechnique - Mathématiques*, Vol. **4**, 845-908, 2017.
- [9] T. Bridgeland and I. Smith, “Quadratic differentials as stability conditions”, *Publications mathématiques de l’IHES*, Vol. **121**, Issue 1, 155-278.
- [10] M. Cafasso, P. Gavrylenko, O. Lisovyy, “Tau functions as Widom constants”, *Communications in Mathematical Physics*, Vol. **365**, Issue 2, 1-32, 2019.
- [11] R. Dijkgraaf, H. Fuji and M. Manabe, “The Volume Conjecture, Perturbative Knot Invariants, and Recursion Relations for Topological Strings”, *Nuclear Physics B*, Vol. **849**, 166, 2011.
- [12] O. Dumitrescu, M. Mulase, “Quantum Curves for Hitchin Fibrations and the Eynard-Orantin Theory”, *Letters in Mathematical Physics*, Vol. **104**, Issue 6, 635-671, 2014.
- [13] P. Dunin-Barkowski, N. Orantin, S. Shadrin, L. Spitz, “Identification of the Givental formula with the spectral curve topological recursion procedure”, *Communications in Mathematical Physics*, Vol. **328**, 669-700, 2014.
- [14] P. Dunin-Barkowski, M. Mulase, P. Norbury, A. Popolitov and S. Shadrin, “Quantum spectral curve for the Gromov-Witten theory of the complex projective line”, *Journal für die reine und angewandte Mathematik*, Published Online, 2014.
- [15] B. Eynard, “The Geometry of integrable systems. Tau functions and homology of Spectral curves. Perturbative definition”, *arxiv:1706.04938*, 2017.
- [16] B. Eynard and M. Mariño, “A holomorphic and background independent partition function for matrix models and topological strings”, *Journal of Geometry and Physics*, Vol. **61**, 1181-1202, 2011.
- [17] B. Eynard, N. Orantin, “Invariants of algebraic curves and topological expansion”, *Communications in Number Theory and Physics*, Vol. **1**, 347-452, 2007.
- [18] B. Eynard, “Counting surfaces: CRM Aisenstadt Chair Lectures”, *Progress in Mathematical Physics*, Vol. **70**, Birkhauser Springer, 2016.
- [19] J.D. Fay, “Theta Functions on Riemann Surfaces”, *Lecture Notes in Mathematics*, Vol. **352**, Springer-Verlag Berlin Heidelberg, 1973.
- [20] J. Harnad, M. Routhier, “R-matrix construction of electromagnetic models for the Painlevé transcendents”, *Journal of Mathematical Physics*, Vol. **36**, Issue 9, 1998.
- [21] J. Harnad, M.A. Wisse “Moment Maps to Loop Algebras, Classical R-Matrix and Integrable Systems”, *Quantum Groups Integrable Models and Statistical Systems*, 1992.
- [22] J. Harnad, M. A. Wisse, “Loop Algebra Moment Maps and Hamiltonian Models for the Painlevé Transcendants”, *Fields Institute Communications*, Vol. **7**, 155-169, 1996.
- [23] N. Iorgov, O Lisovyy, J. Teschner, “Isomonodromic Tau-Functions from Liouville Conformal Blocks”, *Communications in Mathematical Physics*, Vol. **336**, Issue 2, 671-694, 2015.
- [24] K. Iwaki, T. Nakanishi, “Exact WKB analysis and cluster algebras”, *Journal of Physics A: Mathematical and Theoretical*, Vol. **47**, Issue 47, 2014.
- [25] K. Iwaki, T. Nakanishi, “Exact WKB Analysis and Cluster Algebras II: Simple Poles, Orbifold Points, and Generalized Cluster Algebras”, *International Mathematical Research Notices*, Vol. **14**, 4375-4417, 2016.
- [26] K. Iwaki, “2-parameter τ -function for the first Painlevé equation :Topological recursion and direct monodromy problem via exact WKB analysis”, *Communications in Mathematical Physics*, Vol. **377**, 1047-1098, 2020.
- [27] K. Iwaki, O. Marchal, “Painlevé 2 equation with arbitrary monodromy parameter, topological recursion and determinantal formulas”, *Annales Institut Henri Poincaré*, Vol. **18**, Issue 8, 2581-2620, 2017.

- [28] K. Iwaki, O. Marchal, A. Saenz, “Painlevé equations, topological type property and reconstruction by the topological recursion”, *Journal of Geometry and Physics*, Vol. **124**, 16-54, 2016.
- [29] K. Iwaki, A. Saenz, “Quantum Curve and the First Painlevé Equation”, *Symmetry, Integrability and Geometry: Methods and Applications*, Vol. **12**, 2016.
- [30] M. Konsevich, “Intersection theory on the moduli space of curves and the matrix Airy function”, *Communications in Mathematical Physics*, Vol. **147**, 1-23, 1992.
- [31] X. Liu, M. Mulase, and A. Sorkin, “Quantum curves for simple Hurwitz numbers of an arbitrary base curve”, *arxiv:1304.0015*, 2013.
- [32] O. Marchal, N. Orantin, “Isomonodromic deformations of a rational differential system and reconstruction with the topological recursion: the \mathfrak{sl}_2 case”, *Journal of Mathematical Physics*, Vol. **61**, 2020.
- [33] P. Norbury, “Quantum curves and topological recursion”, *Proceedings of Symposia in Pure Mathematics*, Vol. **93**, 41-65, 2016.
- [34] J. Teschner, “Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I”, *Advances in Theoretical and Mathematical Physics*, Vol. **15**, Issue 2, 471-564, 2011.
- [35] E. Witten, “Two-dimensional gravity and intersection theory on moduli space”, *Surveys in differential geometry* (Cambridge, MA), 1, Bethlehem, PA: Lehigh Univ., 243-310, 1990.

A PDE for the perturbative wave functions

This section is devoted to the proof of Theorem 4.1.

A.1 Proof of Lemma 4.3

In this section, we prove Lemma 4.3. For this purpose, we generalize the procedure of [12] used also in [26]. This is simply obtained by writing the integration of the RHS of the topological recursion formula along the boundary of a fundamental domain \mathcal{D} obtained by cutting along the \mathcal{A} and \mathcal{B} cycles considered.

A.1.1 Case $2h - 2 + n \geq 2$.

In this case, it reads, after integrating the variables z_2, \dots, z_n along paths from z_i to $\sigma(z_i)$ as (since the poles of the integrand are either ramification points or coinciding points)

$$\begin{aligned} \frac{1}{2\pi i} \oint_{z \in \delta \mathcal{D}} K(z_1, z) R_{h,n}(z, z_2, \dots, z_n) &= \sum_{a \in \mathcal{R}} \operatorname{Res}_{z \rightarrow a} K(z_1, z) R_{h,n}(z, z_2, \dots, z_n) \\ &+ \sum_{i=1}^n \operatorname{Res}_{z \rightarrow z_i, \sigma(z_i)} K(z_1, z) R_{h,n}(z, \dots, z_n) \end{aligned} \quad (\text{A.1})$$

where, for $2h - 2 + n \geq 1$

$$\begin{aligned} R_{h,n}(z_1, \dots, z_n) &:= d_{u_1} d_{u_2} [F_{h-1, n+1}(u_1, u_2, z_2, \dots, z_n) \\ &+ \left. \begin{aligned} &\sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\}}} F_{h_1, |A|+1}(u_1, A) F_{h_2, |B|+1}(u_2, B) \end{aligned} \right] \Bigg|_{u_1=z_1, u_2=\sigma(z_1)} \\ &+ \sum_{j=2}^n \frac{1}{2} \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot) d_{\sigma(z_1)} F_{h, n-1}(\sigma(z_1), \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}}) \\ &+ \sum_{j=2}^n \frac{1}{2} \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(\sigma(z_1), \cdot) d_{z_1} F_{h, n-1}(z_1, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}}) \end{aligned} \quad (\text{A.2})$$

where d_u refers to the exterior derivative with respect to the variable u (which has nothing to do with a local coordinate),

$$K(z_1, z) := \frac{\int_{\sigma(z)}^z \omega_{0,2}(z_1, \cdot)}{2(\omega_{0,1}(z) - \omega_{0,1}(\sigma(z)))} \quad (\text{A.3})$$

and $d_u d_v F_{0,2}(u, v) := \omega_{0,2}(u, v)$. In order to derive this expression, one has used that for $(h, n) \neq (0, 2)$

$$d_{z_1} F_{h,n}(z_1, \dots, z_n) = \frac{1}{2^{n-1}} \int_{\sigma(z_2)}^{z_2} \dots \int_{\sigma(z_n)}^{z_n} \omega_{h,n}(z_1, \cdot, \dots, \cdot). \quad (\text{A.4})$$

The first term of the right hand side is the recursive definition of $d_{z_1} F_{h,n}(z_1, \dots, z_n)$.

The other terms get contributions only from the poles of $\omega_{0,2}$. First of all, thanks to eq. (3-15) and eq. (3-17), one can observe that

$$K(z, z_1) R_{h,n}(z, z_2, \dots, z_n) = K(\sigma(z), z_1) R_{h,n}(\sigma(z), z_2, \dots, z_n) \quad (\text{A.5})$$

meaning that, for $i \in \llbracket 1, n \rrbracket$,

$$\operatorname{Res}_{z \rightarrow z_i, \sigma(z_i)} K(z_1, z) R_{g,n}(z, \dots, z_n) = 2 \operatorname{Res}_{z \rightarrow z_i} K(z_1, z) R_{g,n}(z, \dots, z_n). \quad (\text{A.6})$$

The same properties imply that

$$\begin{aligned}
R_{h,n}(z_1, \dots, z_n) &:= -d_{u_1} d_{u_2} \left[F_{h-1, n+1}(u_1, u_2, z_2, \dots, z_n) + \right. \\
&\quad \left. + \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\}}} F_{h_1, |A|+1}(u_1, A) F_{h_2, |B|+1}(u_2, B) \right] \Big|_{u_1 = u_2 = z_1} \\
&\quad - \sum_{j=2}^n \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot) d_{z_1} F_{h, n-1}(z_1, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})
\end{aligned} \tag{A.7}$$

One has a simple pole as $z \rightarrow z_1$ giving

$$\begin{aligned}
\text{Res}_{z \rightarrow z_1, \sigma(z_1)} K(z_1, z) R_{g,n}(z, \dots, z_n) &= \frac{1}{2\omega_{0,1}(z_1)} d_{u_1} d_{u_2} \left[F_{h-1, n+1}(u_1, u_2, z_2, \dots, z_n) + \right. \\
&\quad \left. + \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\}}} F_{h_1, |A|+1}(u_1, A) F_{h_2, |B|+1}(u_2, B) \right] \Big|_{u_1 = u_2 = z_1} \\
&\quad + \sum_{j=2}^n \frac{\int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot)}{2\omega_{0,1}(z_1)} d_{z_1} F_{h, n-1}(z_1, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})
\end{aligned} \tag{A.8}$$

where $\mathbf{z}_{\{2, \dots, n\} \setminus \{j\}} = \{z_2, \dots, z_n\} \setminus \{z_j\}$.

One can further compute

$$\begin{aligned}
\text{Res}_{z \rightarrow z_j, \sigma(z_j)} K(z_1, z) R_{h,n}(z, \dots, z_n) &= 2 \text{Res}_{z \rightarrow z_j} K(z_1, z) R_{h,n}(z, \dots, z_n) \\
&= -\frac{\int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot)}{2\omega_{0,1}(z_j)} d_{z_j} F_{h, n-1}(z_j, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})
\end{aligned} \tag{A.9}$$

Combining all this, one gets

$$\begin{aligned}
&\frac{1}{2\pi i} \oint_{z \in \delta \mathcal{D}} K(z_1, z) R_{h,n}(z, \dots, z_n) = d_{z_1} F_{h,n}(z_1, \dots, z_n) \\
&+ \sum_{j=2}^n \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot) \left[\frac{d_{z_1} F_{h, n-1}(z_1, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_1)} - \frac{d_{z_j} F_{h, n-1}(z_j, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_j)} \right] \\
&+ \frac{1}{2\omega_{0,1}(z_1)} d_{u_1} d_{u_2} \left[F_{h-1, n+1}(u_1, u_2, z_2, \dots, z_n) \right. \\
&+ \left. \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\}}} F_{h_1, |A|+1}(u_1, \mathbf{z}_A) F_{h_2, |B|+1}(u_2, \mathbf{z}_B) \right] \Big|_{u_1 = u_2 = z_1}.
\end{aligned} \tag{A.10}$$

By Riemann bilinear identity, the left hand side is an holomorphic form in z_1 , thus concluding the proof.

A.1.2 Case $2h - 2 + n = 1$.

Let us now consider the case $(h, n) = (0, 3)$. One has

$$\frac{1}{2\pi i} \oint_{z \in \delta \mathcal{D}} K(z_1, z) R_{0,3}(z, z_2, z_3) = \sum_{a \in \mathcal{R}^{\text{finite}}} \operatorname{Res}_{z \rightarrow a} K(z_1, z) R_{0,3}(z, z_2, z_3) + \sum_{i=1}^3 \operatorname{Res}_{z \rightarrow z_i, \sigma(z_i)} K(z_1, z) R_{0,3}(z, z_2, z_3) \quad (\text{A.11})$$

where

$$\begin{aligned} R_{0,3}(z_1, z_2, z_3) &:= \frac{1}{4} \left(\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(\sigma(z_1), \cdot) + \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_2)}^{z_2} \omega_{0,2}(\sigma(z_1), \cdot) \right) \\ &= -\frac{1}{2} \int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot), \end{aligned} \quad (\text{A.12})$$

the second equality follows from eq. (3-17).

Once again, the left hand side is holomorphic while the first term on the right hand side is the recursive definition of $d_{z_1} F_{0,3}(z_1, z_2, z_3)$.

Evaluation of the residues gives

$$\begin{aligned} \frac{1}{2\pi i} \oint_{z \in \delta \mathcal{D}} K(z_1, z) R_{0,3}(z, z_2, z_3) &= d_{z_1} F_{0,3}(z_1, z_2, z_3) + \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot)}{4\omega_{0,1}(z_1)} \\ &\quad - \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_2, \cdot)}{4\omega_{0,1}(z_2)} - \frac{\int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_3, \cdot)}{4\omega_{0,1}(z_3)}. \end{aligned} \quad (\text{A.13})$$

Finally, for $(h, n) = (1, 1)$, one only has contributions from poles at the ramification points, z_1 and $\sigma(z_1)$. This gives

$$\frac{1}{2\pi i} \oint_{z \in \delta \mathcal{D}} K(z_1, z) R_{1,1}(z) = d_{z_1} F_{1,1}(z_1) - \frac{\omega_{0,2}(z_1, \sigma(z_1))}{2\omega_{0,1}(z_1)}. \quad (\text{A.14})$$

A.2 Conclusion of the proof of Theorem 4.1

A.2.1 Case $2h - 2 + n \geq 2$.

Let us apply eq. (4.4) to the holomorphic differential of eq. (4-13).

To simplify the expression obtained, let us first note that

$$\sum_{p \in \mathcal{P}} \operatorname{Res}_{z_2 \rightarrow p} \frac{\omega(z_2) y(z_2)}{x(z_2) - x(z_1)} \quad (\text{A.15})$$

is vanishing for $\omega(z) = \frac{\Omega(z)}{\omega_{0,1}(z)}$ where $\Omega(z)$ is a quadratic differential holomorphic at the p 's. This allows getting rid of the contributions of this residue for all the terms proportional to $\frac{1}{\omega_{0,1}(z_1)}$ in eq. (4-13).

One thus gets

$$\begin{aligned} &d_{z_1} F_{h,n}(z_1, \dots, z_n) + \sum_{j=2}^n \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot) \left[\frac{d_{z_1} F_{h,n-1}(z_1, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_1)} - \frac{d_{z_j} F_{h,n-1}(z_j, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_j)} \right] \\ &+ \frac{1}{2\omega_{0,1}(z_1)} d_{u_1} d_{u_2} \left[F_{h-1, n+1}(u_1, u_2, z_2, \dots, z_n) \right. \\ &\quad \left. + \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ A \sqcup B = \{z_2, \dots, z_n\}}} F_{h_1, |A|+1}(u_1, \mathbf{z}_A) F_{h_2, |B|+1}(u_2, \mathbf{z}_B) \right]_{u_1 = u_2 = z_1} \\ &= -\frac{dx(z_1)}{2y(z_1)} \sum_{p \in \mathcal{P}} \operatorname{Res}_{z \rightarrow p} \frac{y(z)}{x(z) - x(z_1)} \left[d_z F_{h,n}(z, z_2, \dots, z_n) - \sum_{j=2}^n \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z, \cdot) \frac{d_{z_j} F_{h,n-1}(z_j, \mathbf{z}_{\{2, \dots, n\} \setminus \{j\}})}{2\omega_{0,1}(z_j)} \right] \end{aligned} \quad (\text{A.16})$$

Using that, for any function f ,

$$\lim_{z_1 \rightarrow z_j} \int_{\sigma(z_j)}^{z_j} \omega_{0,2}(z_1, \cdot) [f(z_1) - f(z_j)] = d_{z_j} f(z_j), \quad (\text{A.17})$$

the diagonal specialization $z = z_1 = z_2 = \dots = z_n$ gives

$$\begin{aligned} & \frac{1}{n} dF_{h,n}(z, \dots, z) + (n-1) d_{z_1} \left[\frac{1}{2\omega_{0,1}(z_1)} d_{z_1} F_{h,n-1}(z_1, z, z, \dots, z) \right]_{z_1=z} \\ & + \frac{1}{2\omega_{0,1}(z)} d_{u_1} d_{u_2} [F_{h-1,n+1}(u_1, u_2, z, \dots, z)]_{u_1=u_2=z} \\ & + \frac{1}{2\omega_{0,1}(z)} \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ n_1 + n_2 = n-1}} \frac{(n-1)!}{n_1! n_2!} \frac{dF_{h_1, n_1+1}(z, \dots, z)}{n_1+1} \frac{dF_{h_2, n_2+1}(z, \dots, z)}{n_2+1} \\ & = -\frac{dx(z)}{2y(z)} \sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} d_{z'} F_{h,n}(z', z, \dots, z) + \frac{d_z F_{h,n-1}(z, \dots, z)}{4y(z)^2} \sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot). \end{aligned} \quad (\text{A.18})$$

which, multiplying by $\frac{2y(z)}{(n-1)! dx(z)}$, can be written

$$\begin{aligned} & 2y(z) \frac{d}{dx(z)} \left[\frac{F_{h,n}(z, \dots, z)}{n!} \right] + \left(\frac{d}{dx(z)} \right)^2 \left[\frac{F_{h,n-1}(z, z, z, \dots, z)}{(n-1)!} \right] \\ & + \sum_{\substack{\text{stable} \\ h_1 + h_2 = h \\ n_1 + n_2 = n-1}} \frac{d}{dx(z)} \left[\frac{F_{h_1, n_1+1}(z, \dots, z)}{(n_1+1)!} \right] \frac{d}{dx(z)} \left[\frac{F_{h_2, n_2+1}(z, \dots, z)}{(n_2+1)!} \right] \\ & + \frac{d}{dx(z)} \left[\frac{F_{h,n-1}(z, \dots, z)}{(n-1)!} \right] \left(-\frac{d \log y(z)}{dx(z)} - \frac{1}{2y(z)} \sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot) \right) \\ & + \frac{d^2}{dx(u_1) dx(u_2)} \left[\frac{F_{h-1,n+1}(u_1, u_2, z, \dots, z)}{(n-1)!} - \frac{F_{h,n-1}(u_1, u_2, z, \dots, z)}{(n-3)!} \right]_{u_1=u_2=z} \\ & = -\sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \frac{d_{z'} F_{h,n}(z', z, \dots, z)}{(n-1)!}. \end{aligned} \quad (\text{A.19})$$

To simplify this expression, let us compute

$$\begin{aligned} -\sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot) & = \text{Res}_{z' \rightarrow z, \sigma(z)} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot) \\ & = 2 \text{Res}_{z' \rightarrow z} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot), \end{aligned} \quad (\text{A.20})$$

where the first equality follows from the absence of boundary term and the second one from the invariance of the integrand under $z' \rightarrow \sigma(z')$.

Let us now remind that

$$\int_{\sigma(z)}^z \omega_{0,2}(z', \cdot) = 2d_{z'} F_{0,2}(z', z) + \frac{dx(z')}{x(z')-x(z)} \quad (\text{A.21})$$

where $d_{z'} F_{0,2}(z', z)$ is holomorphic at $z' \rightarrow z$. Plugging this expression into eq. (A.20) gives

$$\begin{aligned} -\sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot) & = 2 \text{Res}_{z' \rightarrow z} \frac{y(z')}{x(z')-x(z)} \left[2d_{z'} F_{0,2}(z', z) + \frac{dx(z')}{x(z')-x(z)} \right] \\ & = 4 \frac{y(z)}{dx(z)} d_{z'} F_{0,2}(z', z) \Big|_{z'=z} + 2 \frac{dy(z)}{dx(z)} \end{aligned} \quad (\text{A.22})$$

and thus

$$-\frac{d \log y(z)}{dx(z)} - \frac{1}{2y(z)} \sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \int_{\sigma(z)}^z \omega_{0,2}(z', \cdot) = 2 \frac{dF_{0,2}(z', z)}{dx(z')} \Big|_{z'=z}. \quad (\text{A.23})$$

Plugging this into eq. (A.19), one gets

$$\begin{aligned}
& \left(\frac{d}{dx(z)} \right)^2 \left[\frac{F_{h,n-1}(z,z,z,\dots,z)}{(n-1)!} \right] + \sum_{\substack{h_1 + h_2 = h \\ n_1 + n_2 = n-1}} \frac{d}{dx(z)} \left[\frac{F_{h_1,n_1+1}(z,\dots,z)}{(n_1+1)!} \right] \frac{d}{dx(z)} \left[\frac{F_{h_2,n_2+1}(z,\dots,z)}{(n_2+1)!} \right] \\
& + \frac{d^2}{dx(u_1)dx(u_2)} \left[\frac{F_{h-1,n+1}(u_1,u_2,z,\dots,z)}{(n-1)!} - \frac{F_{h,n-1}(u_1,u_2,z,\dots,z)}{(n-3)!} \right]_{u_1=u_2=z} \\
& = - \sum_{p \in \mathcal{P}} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!}.
\end{aligned} \tag{A.24}$$

Summing over h and n such that $2h - 2 + n = m \geq 2$, one gets

$$\frac{\partial^2 S_{m-1}^{+\text{pert}}(x)}{\partial x^2} + \sum_{m_1+m_2=m-1} \frac{\partial S_{m_1}^{+\text{pert}}(x)}{\partial x} \frac{\partial S_{m_2}^{+\text{pert}}(x)}{\partial x} = - \sum_{2h-2+n=mp \in \mathcal{P}} \sum_{z' \rightarrow p} \text{Res}_{z' \rightarrow p} \frac{y(z')}{x(z')-x(z)} \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!}. \tag{A.25}$$

Let us now interpret the right hand side in terms of the variational formulas. To do so, one shall compute the residues as $z \rightarrow p$ whose expressions in terms of local coordinates depend on whether $x(p) = \infty$ or not.

— For $p = b_\nu^\pm$, a local coordinate is $x(z') - X_\nu$. Thus,

$$\text{Res}_{z' \rightarrow b_\nu^\pm} \frac{y(z')}{x(z')-x(z)} \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!} = - \sum_{k=0}^{\infty} (x(z)-X_\nu)^{-k-1} \text{Res}_{z' \rightarrow b_\nu^\pm} y(z') \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!} (x(z')-X_\nu)^k. \tag{A.26}$$

Since

$$y(z') = \pm \sum_{l=1}^{r_\nu} T_{\nu,l} (x(z') - X_\nu)^{-l} + O(1) \tag{A.27}$$

and

$$\forall K \geq 2, \quad \text{Res}_{z' \rightarrow b_\nu^\pm} d_{z'} F_{h,n}(z',z,\dots,z) (x(z') - X_\nu)^{-K+1} = \pm (K-1) \frac{\partial F_{h,n-1}(z,\dots,z)}{\partial T_{\nu,K}}, \tag{A.28}$$

one has

$$\text{Res}_{z' \rightarrow b_\nu^\pm} \frac{y(z')}{x(z')-x(z)} \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!} = - \sum_{K=2}^{r_\nu+1} U_{\nu,K}(x(z)) \frac{\partial F_{h,n-1}(z,\dots,z)}{\partial T_{\nu,K}} \tag{A.29}$$

where

$$U_{\nu,K}(x) = (K-1) \sum_{l=K-1}^{r_\nu} T_{\nu,l} (x - X_\nu)^{-l+K-2} \tag{A.30}$$

is a rational function of x .

— For $p = b_\infty^\pm$ when $n_\infty = 0$, a local coordinate is $x(z')^{-1}$. Let us write

$$\text{Res}_{z' \rightarrow b_\infty^\pm} \frac{y(z')}{x(z')-x(z)} \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!} = \sum_{k=0}^{\infty} x(z)^k \text{Res}_{z' \rightarrow b_\infty^\pm} y(z') \frac{d_{z'} F_{h,n}(z',z,\dots,z)}{(n-1)!} x(z')^{-k-1}. \tag{A.31}$$

Reminding that

$$y(z) = \mp \sum_{k=1}^{r_\infty} T_{\infty,k} x(z)^{k-2} + O(x(z)^{-2}) \tag{A.32}$$

and

$$\forall K \geq 2, \quad \text{Res}_{z' \rightarrow b_\infty^\pm} d_{z'} F_{h,n}(z',z,\dots,z) x(z')^{K-1} = \mp (K-1) \frac{\partial F_{h,n-1}(z,\dots,z)}{\partial T_{\infty,K}}, \tag{A.33}$$

one has

$$\operatorname{Res}_{z' \rightarrow b_\infty^\pm} \frac{y(z')}{x(z') - x(z)} \frac{d_{z'} F_{h,n}(z', z, \dots, z)}{(n-1)!} = - \sum_{K=2}^{r_\infty-2} U_{\infty,K}(x(z)) \frac{\partial F_{h,n-1}(z, \dots, z)}{\partial T_{\infty,K}} \quad (\text{A.34})$$

where

$$U_{\infty,K}(x) = (K-1) \sum_{l=K+2}^{r_\infty} T_{\infty,l} x^{l-K-2}. \quad (\text{A.35})$$

— For $p = b_\infty$ when $n_\infty = 1$, a local coordinate is $x(z)^{-\frac{1}{2}}$. Reminding that

$$y(z) = - \sum_{k=1}^{r_\infty} \frac{T_{b_\infty,k}}{2} x(z)^{k-\frac{5}{2}} + O(x(z)^{-\frac{5}{2}}) \quad (\text{A.36})$$

and

$$\forall K \geq 2, \quad \operatorname{Res}_{z' \rightarrow b_\infty} d_{z'} F_{h,n}(z', z, \dots, z) x(z')^{K-\frac{3}{2}} = (2K-3) \frac{\partial F_{h,n-1}(z, \dots, z)}{\partial T_{b_\infty,K}}, \quad (\text{A.37})$$

one has

$$\operatorname{Res}_{z' \rightarrow b_\infty} \frac{y(z')}{x(z') - x(z)} \frac{d_{z'} F_{h,n}(z', z, \dots, z)}{(n-1)!} = - \sum_{K=2}^{r_\infty-2} U_{\infty,K}(x(z)) \frac{\partial F_{h,n-1}(z, \dots, z)}{\partial T_{b_\infty,K}} \quad (\text{A.38})$$

where

$$U_{\infty,K}(x) = \left(K - \frac{3}{2} \right) \sum_{l=K+2}^{r_\infty} T_{b_\infty,l} x^{l-K-2}. \quad (\text{A.39})$$

Plugging this into eq. (A.25) proves for $m \geq 2$:

$$\begin{aligned} 0 &= \frac{\partial^2 S_m^+ \text{pert}(x)}{\partial x^2} + \sum_{m_1+m_2=m-1} \frac{\partial S_{m_1}^+ \text{pert}(x)}{\partial x} \frac{\partial S_{m_2}^+ \text{pert}(x)}{\partial x} \\ &- \sum_{K=2}^{r_\infty-2} U_{\infty,K}(x(z)) \frac{\partial S_{m-1}^+ \text{pert}(x)}{\partial T_{\infty,K}} - \sum_{\nu} \sum_{K=2}^{r_{b_\nu}+1} U_{b_\nu,K}(x(z)) \frac{\partial S_{m-1}^+ \text{pert}(x)}{\partial T_{b_\nu,K}} \\ &- \sum_{k=0}^{\infty} \delta_{m+1,2k} \left[\sum_{K=2}^{r_\infty-2} U_{\infty,K}(x(z)) \frac{\partial F_{k,0}}{\partial T_{\infty,K}} \sum_{\nu} \sum_{K=2}^{r_{b_\nu}+1} U_{b_\nu,K}(x(z)) \frac{\partial F_{k,0}}{\partial T_{b_\nu,K}} \right]. \end{aligned} \quad (\text{A.40})$$

A.2.2 Case $2h - 2 + n = 1$

Let us now proceed in the same way for $(h, n) = (0, 3)$. Applying eq. (A.20) to the holomorphic differential eq. (A.13), one gets

$$\begin{aligned} &d_{z_1} F_{0,3}(z_1, z_2, z_3) + \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot)}{4\omega_{0,1}(z_1)} \\ &- \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_2, \cdot)}{4\omega_{0,1}(z_2)} - \frac{\int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_1, \cdot) \int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_3, \cdot)}{4\omega_{0,1}(z_3)} \\ &= - \frac{dx(z_1)}{2y(z_1)} \sum_{p \in \mathcal{P}} \operatorname{Res}_{z' \rightarrow p} \frac{y(z')}{x(z') - x(z_1)} \left[d_{z'} F_{0,3}(z', z_2, z_3) \right. \\ &- \left. \frac{\int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z', \cdot) \int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z_2, \cdot)}{4\omega_{0,1}(z_2)} - \frac{\int_{\sigma(z_3)}^{z_3} \omega_{0,2}(z', \cdot) \int_{\sigma(z_2)}^{z_2} \omega_{0,2}(z_3, \cdot)}{4\omega_{0,1}(z_3)} \right]. \end{aligned} \quad (\text{A.41})$$

the sum of eq. (A.49) and eq. (A.50) reads

$$2 \frac{dS_{-1}^{+ \text{pert}}(x)}{dx} \frac{dS_1^{+ \text{pert}}(x)}{dx} + \left(\frac{dS_0^{+ \text{pert}}(x)}{dx} \right)^2 + \frac{d^2 S_0^{+ \text{pert}}(x)}{dx^2} = \sum_{K=2}^{r_\infty-2} U_{\infty, K}(x) \frac{\partial F_{1,0}}{\partial T_{b_\infty, K}} + \sum_{\nu=1}^n \sum_{K=2}^{r_\nu+1} U_{\nu, K}(x) \frac{\partial F_{1,0}}{\partial T_{b_\nu, K}} + \sum_{K=2}^{r_\infty-2} U_{\infty, K}(x) \frac{\partial S_0^{+ \text{pert}}(x)}{\partial T_{b_\infty, K}} + \sum_{\nu=1}^n \sum_{K=2}^{r_\nu+1} U_{\nu, K}(x) \frac{\partial S_0^{+ \text{pert}}(x)}{\partial T_{b_\nu, K}}. \quad (\text{A.51})$$

A.2.3 Cases $2h - 2 + n \leq 0$

For $2h - 2 + n = -1$, one has, by definition of the spectral curve,

$$\left(\frac{\partial S_{-1}^{+ \text{pert}}(x)}{\partial x} \right)^2 = \frac{\phi(x)}{dx^2}. \quad (\text{A.52})$$

For $2h - 2 + n = 0$, thanks to eq. (A.23) and the expression of the residues at poles in terms of variational formulas, one can write

$$2y(z) \frac{dF_{0,2}(z', z)}{dx(z')} \Big|_{z'=z} + \frac{dy(z)}{dx(z)} = \sum_{K=2}^{r_\infty-2} U_{\infty, K}(x(z)) \frac{\partial S_{-1}^{+ \text{pert}}(x(z))}{\partial T_{\infty, K}} + \sum_{\nu=1}^n \sum_{K=2}^{r_\nu+1} U_{\nu, K}(x(z)) \frac{\partial S_{-1}^{+ \text{pert}}(x(z))}{\partial T_{\nu, K}}, \quad (\text{A.53})$$

i.e.

$$2 \frac{dS_{-1}^{+ \text{pert}}(x)}{dx} \frac{dS_0^{+ \text{pert}}(x)}{dx} + \frac{d^2 S_{-1}^{+ \text{pert}}(x)}{dx^2} = \sum_{K=2}^{r_\infty-2} U_{\infty, K}(x) \frac{\partial S_{-1}^{+ \text{pert}}(x)}{\partial T_{\infty, K}} + \sum_{\nu=1}^n \sum_{K=2}^{r_\nu+1} U_{\nu, K}(x) \frac{\partial S_{-1}^{+ \text{pert}}(x)}{\partial T_{b_\nu, K}}. \quad (\text{A.54})$$

A.2.4 Conclusion of the proof

Summing the contributions coming from eq. (A.52), eq. (A.54), eq. (A.51) and eq. (A.40) for $m \geq 2$ with the appropriate \hbar factors, one gets the ODE satisfied by the non-perturbative wave function as stated in Theorem 4.1.

B System of ODE for the non-perturbative wave functions

B.1 Proof of Lemma 5.2

Let us compute

$$\frac{\partial W(x)}{\partial x} = \hbar \left(\frac{\partial^2 \Psi_+}{\partial x^2} \Psi_- - \Psi_+ \frac{\partial^2 \Psi_-}{\partial x^2} \right). \quad (\text{B.1})$$

Since Ψ_+ and Ψ_- are both solutions to eq. (5-22),

$$\hbar \frac{\partial^2 \Psi_\pm}{\partial x^2} = \hbar \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p, k}(x) \frac{\partial \Psi_\pm}{\partial T_{p, k}} + \hbar^{-1} \mathcal{H}(x) \Psi_\pm. \quad (\text{B.2})$$

plugging this back into the expression above gives

$$\frac{\partial W(x)}{\partial x} = \sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p, k}(x) W_{T_{p, k}}(x) \quad (\text{B.3})$$

and the lemma follows.

B.2 Proof of Lemma 5.3

We shall prove this fundamental result by following the footsteps of [26]. Let us first write down the compatibility of the system eq. (5-23). For any $(p, k) \in \mathcal{P} \times K_p$, writing down the equality of $\frac{\partial^2}{\partial x^2} \left[\frac{\partial \Psi}{\partial T_{p,k}} \right] = \frac{\partial}{\partial T_{p,k}} \left[\frac{\partial^2 \Psi}{\partial x^2} \right]$ and matching the coefficients of Ψ et $\frac{\partial \Psi}{\partial x}$, one gets

$$0 = 2 \frac{\partial Q_{p,k}}{\partial x} - \hbar \frac{\partial R}{\partial T_{p,k}} + \hbar R_{p,k} \frac{\partial R}{\partial x} + \hbar R \frac{\partial R_{p,k}}{\partial x} + \hbar \frac{\partial^2 R_{p,k}}{\partial x^2} \quad (\text{B.4})$$

and

$$0 = 2(\hbar Q + \mathcal{H}) \frac{\partial R_{p,k}}{\partial x} + \hbar R_{p,k} \frac{\partial Q}{\partial x} - \hbar R \frac{\partial Q_{p,k}}{\partial x} + R_{p,k} \frac{\partial \mathcal{H}}{\partial x} - \frac{\partial}{\partial T_{p,k}} (\hbar Q + \mathcal{H}) + \hbar \frac{\partial^2 Q_{p,k}}{\partial x^2}. \quad (\text{B.5})$$

Let us now write the expansion in \hbar of these equality order by order. For this purpose, let us remark that both equations can be put under the form of a trans-series as functions of \hbar

$$\sum_{m, \mathbf{k}} \alpha_{m, \mathbf{k}} \hbar^m \exp \left(\hbar^{-1} \sum_{j=1}^g k_j v_j \right) = 0 \quad (\text{B.6})$$

meaning that the coefficients of the trans-monomials vanish. Summing over vectors \mathbf{k} , this implies that the coefficient of \hbar^m is vanishing for any m . In order to compute it, let us remarks that derivation with respect to x preserves this \hbar grading while differentiation with respect to the times decreases the degree by 1.

To leading order, this reads

$$0 = 2 \frac{\partial Q_{p,k}^{(0)}}{\partial x} - \frac{\partial \phi}{\partial T_{p,k}} \frac{\partial \hat{R}^{(0)}(x, \mathbf{T}, v)}{\partial v} \Bigg|_{v=\frac{\phi+\rho}{\hbar}} \quad (\text{B.7})$$

and

$$0 = 2\mathcal{H}^{(0)} \frac{\partial R_{p,k}^{(0)}}{\partial x} + R_{p,k}^{(0)} \frac{\partial \mathcal{H}^{(0)}}{\partial x} - \frac{\partial \phi}{\partial T_{p,k}} \frac{\partial \hat{Q}^{(0)}(x, \mathbf{T}, v)}{\partial v} \Bigg|_{v=\frac{\phi+\rho}{\hbar}} - \frac{\partial \mathcal{H}^{(0)}}{\partial T_{p,k}}. \quad (\text{B.8})$$

Let us now assume that the functions $Q_{p,k}^{(0)}$ and $R_{p,k}^{(0)}$ have a pole at a critical value u_i such that they behave as

$$\begin{cases} \hat{Q}_{p,k}^{(0)}(x, \mathbf{T}, v) = \frac{q_{p,k}^{(0)}}{(x-u_i)^{d_{p,k}}} + O((x-u_i)^{-d_{p,k}+1}) \\ \hat{R}_{p,k}^{(0)}(x, \mathbf{T}, v) = \frac{r_{p,k}^{(0)}}{(x-u_i)^{d'_{p,k}}} + O((x-u_i)^{-d'_{p,k}+1}) \end{cases} \quad (\text{B.9})$$

for some positive degrees $(d_{p,k}, d'_{p,k})$ and write in the same way

$$\begin{cases} \hat{Q}^{(0)}(x, \mathbf{T}, v) = \frac{q^{(0)}}{(x-u_i)^d} + O((x-u_i)^{-d+1}), \\ \hat{R}^{(0)}(x, \mathbf{T}, v) = \frac{r^{(0)}}{(x-u_i)^{d'}} + O((x-u_i)^{-d'+1}). \end{cases} \quad (\text{B.10})$$

The leading order in $x \rightarrow u_i$ of eq. (B.7) reads

$$-2d_{p,k} \frac{q_{p,k}^{(0)}}{(x-u_i)^{d_{p,k}+1}} = \frac{\partial \phi}{\partial T_{p,k}} \frac{\partial r^{(0)}}{\partial v} \Bigg|_{v=\frac{\phi+\rho}{\hbar}} \frac{1}{(x-u_i)^{d'}} \quad (\text{B.11})$$

meaning that $d_{p,k}$ is independent of (p, k) and $d_{p,k} = d' - 1$ for any pair (p, k) . From the definition of Q , it behaves as

$$\hat{Q}^{(0)}(x, \mathbf{T}, v) = \left[\sum_{p \in \mathcal{P}} \sum_{k \in K_p} U_{p,k}(u_i) q_{p,k}^{(0)} \right] \frac{1}{(x-u_i)^{d'-1}} + O((x-u_i)^{-d'+2}). \quad (\text{B.12})$$

The leading order of eq. (B.8) reads

$$(-2d_{p,k} + 1) \left[\frac{\partial \mathcal{H}^{(0)}}{\partial x} \right] \Big|_{x=u_i} \frac{r_{p,k}^{(0)}}{(x-u_i)^{d'_{p,k}+1}} = \frac{\partial \phi}{\partial T_{p,k}} \frac{\partial q^{(0)}}{\partial v} \Big|_{v=\frac{\phi+p}{\hbar}} \frac{1}{(x-u_i)^d} \quad (\text{B.13})$$

which implies that $d'_{p,k} = d - 1 = d' - 2$ for any pair (p, k) which contradicts the definition of R .

Thus $R^{(0)}(x)$ and $Q_{p,K}^{(0)}(x)$ are holomorphic at the critical values. This implies that $Q^{(0)}(x)$ is holomorphic as well at these points. Finally, eq. (B.8) together with the fact that $Q^{(0)}(x)$ is holomorphic implies that $R_{p,K}^{(0)}(x)$ is holomorphic as well at these points.

We shall now proceed by induction on h for proving that the coefficients $R_{p,K}^{(h)}$ and $Q_{p,K}^{(h)}$ are holomorphic at the critical values. For this purpose, one can write the h 'th order of the compatibility conditions as

$$2 \frac{\partial Q_{p,K}^{(h)}}{\partial x} - \hbar \frac{\partial \phi}{\partial T_{p,K}} \frac{\partial \tilde{R}^{(h)}(x, \mathbf{T}, v)}{\partial v} \Big|_{v=\frac{\phi}{\hbar}} = \text{lower order terms} \quad (\text{B.14})$$

and

$$2\mathcal{H}^{(0)} \frac{\partial R_{p,K}^{(h)}}{\partial x} + R_{p,K}^{(h)} \frac{\partial \mathcal{H}^{(0)}}{\partial x} - \frac{\partial \phi}{\partial T_{p,K}} \frac{\partial \tilde{Q}^{(h)}(x, \mathbf{T}, v)}{\partial v} \Big|_{v=\frac{\phi}{\hbar}} - \frac{\partial \mathcal{H}^{(h)}}{\partial T_{p,K}} = \text{lower order terms} \quad (\text{B.15})$$

where the right hand sides are lower order terms in the \hbar expansion which are holomorphic at the critical values by induction. Using the same argument as for the leading order, one can conclude that any $R_{p,K}^{(h)}$ and $Q_{p,K}^{(h)}$ are holomorphic at these points.

C Expansions of S^\pm and Wronskians around X_ν and ∞

In this appendix, we recall the expansions around the poles of certain useful quantities.

— **Around** $x = X_\nu$, **for** $\nu \in \llbracket 1, n \rrbracket$.

The logarithm of the wave functions read

$$S_\pm(x) = \mp \hbar^{-1} \sum_{k=2}^{r_{b_\nu}} \frac{T_{\nu,k}}{k-1} \frac{1}{(x-X_\nu)^{k-1}} \pm \hbar^{-1} T_{\nu,1} \log(x-X_\nu) + \sum_{k=0}^{\infty} A_{\nu,k}^\pm (x-X_\nu)^k. \quad (\text{C.1})$$

The Wronskian in x behaves as

$$W(x) = \frac{2T_{\nu,r_\nu}}{(x-X_\nu)^{r_\nu}} \exp(A_{\nu,0}^+ + A_{\nu,0}^-) + O((x-X_\nu)^{-r_\nu+1}). \quad (\text{C.2})$$

The Wronskians in the spectral times at finite pole read

$$\begin{aligned} & \forall \nu' \in \llbracket 1, n \rrbracket, \forall k \geq 2 : W_{T_{\nu',k}}(x) \\ &= \left[-\frac{2\delta_{\nu,\nu'}}{(k-1)(x-X_\nu)^{k-1}} + \hbar \frac{\partial (A_{\nu,0}^+ - A_{\nu,0}^-)}{\partial T_{\nu',k}} + O((x-X_\nu)) \right] \Psi_+(x) \Psi_-(x), \end{aligned} \quad (\text{C.3})$$

i.e

$$\forall k \geq 2, W_{T_{\nu,k}}(x) = -\frac{2}{k-1} \frac{\exp(A_{\nu,0}^+ + A_{\nu,0}^-)}{(x-X_\nu)^{k-1}} + O((x-X_\nu)^{-k+2}) \quad (\text{C.4})$$

and

$$\forall \nu' \neq \nu, \forall k \geq 2 : W_{T_{\nu',k}}(x) = \hbar \frac{\partial (A_{\nu,0}^+ - A_{\nu,0}^-)}{\partial T_{\nu',k}} \exp(A_{\nu,0}^+ + A_{\nu,0}^-) + O((x-X_\nu)). \quad (\text{C.5})$$

The Wronskians in the spectral times at ∞ read

$$\forall k \geq 2 : W_{T_{\infty,k}}(x) = \hbar \frac{\partial (A_{\nu,0}^+ - A_{\nu,0}^-)}{\partial T_{\infty,k}} \exp(A_{\nu,0}^+ + A_{\nu,0}^-) + O((x-X_\nu)). \quad (\text{C.6})$$

— **Around** $x = \infty$ **if** $n_\infty = 0$. The logarithms of the wave functions read

$$S_\pm(x) = \mp \hbar^{-1} \sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{k-1} x^{k-1} \mp \hbar^{-1} T_{\infty,1} \log(x) - \frac{\log x}{2} + \sum_{k=0}^{\infty} A_{\infty,k}^\pm x^{-k}. \quad (\text{C.7})$$

The Wronskian in x behaves as

$$W(x) = -2T_{\infty,r_\infty} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) x^{r_\infty-3} + O(x^{r_\infty-4}). \quad (\text{C.8})$$

The Wronskians in the spectral times at finite poles read

$$\forall \nu \in \llbracket 1, n \rrbracket, \forall k \geq 2 : W_{T_{\nu,k}}(x) = O(x^{-1}). \quad (\text{C.9})$$

The Wronskians in the spectral times at ∞ read

$$\forall k \geq 2, W_{T_{\infty,k}}(x) = -\frac{2}{k-1} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) x^{k-2}. \quad (\text{C.10})$$

— **Around** $x = \infty$ **if** $n_\infty = 1$.

$$S_\pm(x) = \mp \hbar^{-1} \sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{2k-3} x^{\frac{2k-3}{2}} \mp \hbar^{-1} T_{\infty,1} \log(x) - \frac{\log x}{4} + \sum_{k=1}^{\infty} A_{\infty,k}^\pm x^{-\frac{k}{2}}. \quad (\text{C.11})$$

The Wronskian in x behaves as

$$W(x) = -T_{\infty,r_\infty} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) x^{r_\infty-3} + O(x^{r_\infty-3-\frac{1}{2}}). \quad (\text{C.12})$$

The Wronskians in the spectral times at finite poles read

$$\forall \nu \in \llbracket 1, n \rrbracket, \forall k \geq 2 : W_{T_{\nu,k}}(x) = O(x^{-\frac{1}{2}}). \quad (\text{C.13})$$

The Wronskians in the spectral times at ∞ read

$$\forall k \geq 2 : W_{T_{\infty,k}}(x) = -\frac{2}{2k-3} \exp(A_{\infty,0}^+ + A_{\infty,0}^-) x^{k-2}. \quad (\text{C.14})$$

Note that one may obtain the next orders in the expansions of the Wronskians from those of S_\pm with the formula:

$$\begin{aligned} W(x) &= \hbar \left(\frac{\partial S_+(x)}{\partial x} - \frac{\partial S_-(x)}{\partial x} \right) \exp(S_+(x) + S_-(x)) \\ W_{T_{\nu,k}}(x) &= \hbar \left(\frac{\partial S_+(x)}{\partial T_{\nu,k}} - \frac{\partial S_-(x)}{\partial T_{\nu,k}} \right) \exp(S_+(x) + S_-(x)) \end{aligned} \quad (\text{C.15})$$