



**HAL**  
open science

# The concept of function in secondary school textbooks over time: An analysis made with the Theory of Conceptual Fields

Patricia Sureda, Laura Rossi

► **To cite this version:**

Patricia Sureda, Laura Rossi. The concept of function in secondary school textbooks over time: An analysis made with the Theory of Conceptual Fields. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02423540

**HAL Id: hal-02423540**

**<https://hal.science/hal-02423540>**

Submitted on 24 Dec 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# **The concept of function in secondary school textbooks over time: An analysis made with the Theory of Conceptual Fields**

Patricia Sureda<sup>1</sup> and Laura Rossi<sup>2</sup>

<sup>1</sup> CONICET - UNICEN, Tandil- Bs. As., Argentina; [psureda@exa.unicen.edu.ar](mailto:psureda@exa.unicen.edu.ar)

<sup>2</sup> UNICEN, Córdoba, Argentina; [lauraro737@gmail.com](mailto:lauraro737@gmail.com)

*The purpose of this paper is to analyse how functions have been taught throughout 22 secondary school textbooks over the last 50 years. We have adopted a temporal perspective, as the books examined were published from 1953 to 2010. Firstly, the definition of concept (situation, meaning, system of representation) was the one proposed in the theory of conceptual fields, in order to analyse the concept of "function" taught in each book. Secondly, the books were categorized according to a focus on Sets, Analytical - Graphic or Hybrid. Finally, the meaning of function emphasized in each approach was analysed according to the representational system. The analysis showed that the meaning changed over time, as well as the difference in each representational system.*

*Keywords: Mathematics teacher resources, textbooks, high school teachers, Theory of Conceptual Fields.*

## **Introduction**

The conceptual field of functions is taught from the first up to the sixth grade of high school. Its importance lies in its increasingly broad field of application.

In high school, the knowledge that the teacher selects for his teaching practice is the knowledge that is in his textbooks. Thus, the book is an unavoidable framework for both, the teacher and the student, and generates much of the mathematical discourse manifested in the teacher's practice.

The analyses of the 23 books were divided in two parts. In the first part, the teaching of functions was classified in three categories, proposed by Sureda and Otero (2007): Approach of Sets, Analytical Graph and Hybrid. Then, we analysed the proposal through the term of "Concept" taking into account the Theory of Conceptual Fields (Vergnaud, 1990). Accordingly, the situations used in the books to propose the functions were analysed along with the tasks and their resolutions, inferring from them the set of Operational Invariants presented in each representational system. The Operational Invariants can be inferred as the textbook proposed the tasks simultaneously with the correct resolutions, its definitions, and explanations.

## **Literature Review**

Due to the variety of investigations related to functions, we only mention those analysed in books, or those focused on in the analysis of representational systems. The latter can be found in the work of Lavaque, Méndez, and Villarroel (2006), who analysed the instruction of the notion of function through a coherent articulation of Duval's representational registers.

With regards to the research related to the secondary level textbooks, the work of Cordero, Cen Che and Téllez (2010) is worth mentioning, who proposed the use of graphic representations presented

in school books, in order to use them to understand the institutionalization of the concept of function.

This work mainly focuses on the meaning of the concept of function proposed in each book, as well the existence (or not) of a variation of such concept, throughout fifty years. This work is part of a larger study in which we propose to design and implement a set of situations to teach functions in secondary school. The Theory of Conceptual Fields allows us to analyse the gap between the idea of the concept of function, formalized in textbooks, with the construction of Operational Invariants that students work from the situation the textbook presents. The dialectic between concepts-in-action and theorems-in-action are central to achieve this.

### **The concept in the Theory of Conceptual Fields**

The Theory of Conceptual Fields (Vergnaud, 2013) assumes that knowledge is organized in Conceptual Fields, which develops over an extended period of time, through experience, maturity and learning. A Conceptual Field, as the one belonging to functions, is an informal and heterogeneous set of problems, situations, concepts, relationships, structures, contents and operations of thoughts, interconnected and probably linked during the acquisitional process. In this context, the concept is defined as a triplet of three sets:  $C(S, I, \Gamma)$ .

**The reference [S]:** It is the set of situations that give meaning to the concept. **The meaning [I]:** It is the set of Operational Invariants ((1) concepts-in-action, conformed by an object, a predicate or a category of thoughts considered pertinent, and (2) theorems-in-action, formed by propositions, where the real can be either true or false), and upon which rests the operation of the schemes. The schemes are the cognitive structure of the subject. **The signifier [ $\Gamma$ ]:** It is the set of linguistic and non-linguistic forms that allow to represent symbolically the concept, its properties, situations, and procedures.

Regarding this definition, it is important to emphasize that signifiers should not be confused with meanings: the words used recover different meanings according to the situation. Besides, the meaning constructed by the subject can be partially corresponding to the conventional meaning of words and sentences, or to the one given by the teacher, since there is no direct homomorphism but partial, between the reality and the language, including the scientific language.

Finally, the concept of Operational Invariants (Vergnaud, 2007), provides the elements to firstly determine the knowledges formalized in the textbooks to analyse the distance from the Operational Invariants that students will build in the classroom.

### **Material and Methods**

We analysed 23 books of mathematics teaching in high school published between 1953 and 2010. The first selection prioritized the textbooks which were more popular among secondary level teachers of mathematics. Then, the books in which functions were taught were chosen. In order to analyse different designs and styles, books from different publishers were carefully chosen. Moreover, in each book we analysed the Operative Invariants in the Representation Systems (RS) as well as the tasks. The RS are those proposed by Sureda and Otero (2013). At the same time, in each

book we analysed whether the concept of function corresponded to a more static (set) or more dynamic (analytical - graphic or hybrid) view of the concept (Sureda & Otero, 2007). **RS Numeric:** Tables and numerical calculations. **RS First Order Algebraic:** Algebraic procedures in which parameters are initialized ( $y = 2x^2 + 4$ ). **RS Second Order Algebraic:** Algebraic procedure in which parameters are not initialized ( $y = ax + b$ ). **RS Analytical-graphic:** Graph on Cartesian Axes. **RS Verbal Written:** written linguistic forms. **RS Pictorial:** Refers to the construction of schematic drawings such as the Venn Diagrams.

This analysis allowed us to classify them in the three mentioned approaches (Sureda & Otero, 2007). **Approach of Sets:** It is a teaching of the concept of function centred on the Theory of Sets. The study of the functions includes the teaching of ordered pair, Cartesian product, relations, equivalences, functions, set of departure and arrival, domain, image, etc. In this way, this approach is more precise and does not present ambiguities in the development of the concept of function. **In the Analytical-Graphic Approach,** the concept of function is determined through the graphical representation in Cartesian axes of a problem or concrete situation, in which a functional relation between variables is established. The analysis of the function and its properties is performed from the graph: domain, image, notable points, asymptotes, increasing or decreasing, etc. **The Hybrid Approach** has characteristics of the two already mentioned organizations. It establishes a relationship between variables, and also uses a set concepts such as set of departure and arrival, and allocation rules. However, it is problematic when giving meaning to the concepts it borrows from set theory.

These three approaches allowed textbooks to be classified into three disjoint and exhaustive categories, since those textbooks which did not correspond to the approach of sets, nor to the analytical-graphic, were a combination of both.

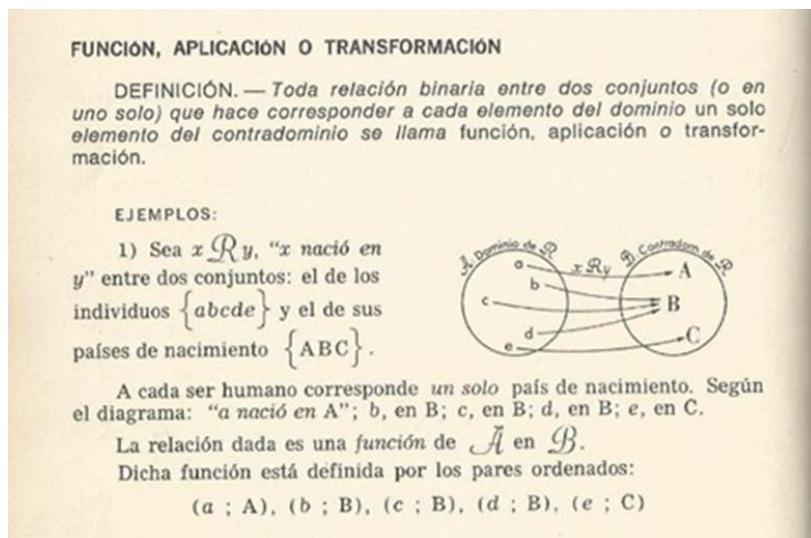
## Data Analysis and Results

**Approach of Sets:** The books analysed which fit in the Approach of Sets are: **1)** *Arithmetic and Algebra 3*, Publisher Estrada (1953); **2)** *Mathematics 4 Commercial Course*, Eds. Estrada (1961); **3)** *Elements of Calculus Difference and Integral*, Eds. Alsina (1970); **4)** *Mathematics 3*, Eds. Estrada (1980); **5)** *Mathematics 3*, Eds. Plus Ultra (1987); **6)** *Mathematics II For the Aula Taller*, Eds. Plus Ultra (1988); **7)** *Modern Mathematics 2*, Eds. Stella (1970); **8)** *Directed study of mathematics 3*, Eds. A-Z (1989). These books were published between 1950 and 1989. The analysis is similar to the one made to the book *Mathematical 4 Commercial Course* published in 1961 (from Eds. Estrada), presented below.

**The reference [S]:** The textbook introduces the concept of function (figure 1) through the definition: "*Any binary relation between two sets – or just in one set- that corresponds each element of the domain to a single element of the counter domain is called function, application, or transformation*".

Then, various Venn diagrams are presented, in which three examples of relationships are indicated. In the example 1) the relation "x was born in y" is proposed (figure 1), indicated as a function. In example 2) the relation "a x corresponds to the desk y" is proposed, indicated as a function. In

example 3) the relation "x is a multiple of y" is proposed, indicated as a non-function. Each example is accompanied by a Venn diagram in which the output sets  $A$  domain of  $R$  the arrival set  $D$  codomain of  $R$ ; and the relationship by the arrow  $x R y$ . Finally, the ordered pairs of the function are indicated.



**Figure 1: Definition of Function - Mathematics 4 Commercial Course, Publisher Estrada 1961**

**The signifier [Γ]:** The textbook uses the RS Verbal Written for the definition of Function; the Second Order Algebraic RS to define the sets and relationships of the example, and the Pictorial RP for the representation in Venn diagrams.

In short, the textbooks with this approach present the concept of function from the notions of ordered pair, cartesian product, and relationships. The sequence that these textbooks follow to accomplish it, is: theory - example - exercise. The Representation Systems used are: Verbal Written, Algebraic of Second Order and Pictorial. Thus, we can determine that another characteristic of the set theory is the relevance given to mathematical notation.

**The meaning [I]:** The Theorems and Concepts in Action that emerge from the concept of Function in this approach are the following. **RS Verbal Written:** *A Function is a binary relation between two sets that corresponds to each element of the domain in a single element of the codomain - If a Relationship complies with the uniqueness and existence, it is a Function - The Domain of a Function is included in the Reach set - The Image is included in the Range of a Function.* **RS Second Order Algebraic:** *The expression of a function is  $y = f(x)$  - Given the sets  $A = \{a, b, c, d, e\}$  and  $B = \{A, B, C\}$ , the relationship formed by the pairs  $(a; A) (b; B) (c; B) (d; B) (e; C)$  is a function of A in B - The Domain of a Function is included in the Scope set - The Image is included in the Range of a Function.* **RS Pictorial:** *Functions can be represented by Venn diagrams - From each element of the set called Domain, a single arrow appears towards an element of the set called Codomain.*

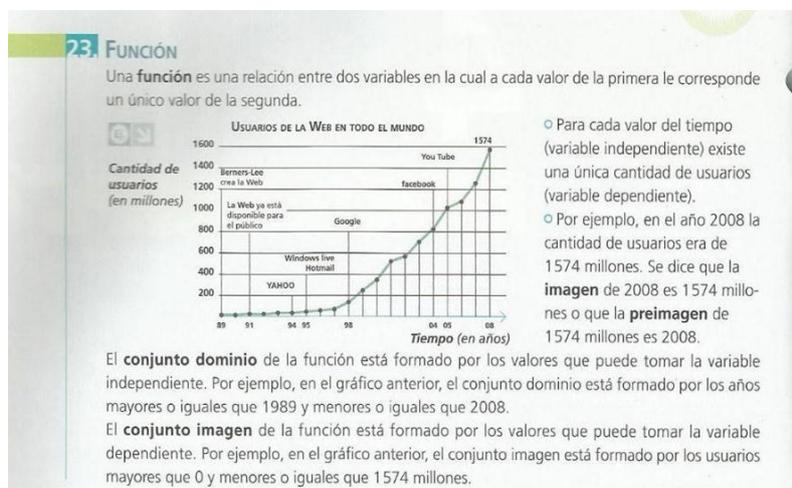
These Theorems-in-Action propitiate a static idea of the concept of Function. This can be seen in the examples and activities proposed. For example, the textbook presents the sets A and B by extension, and then defines the relationship between both sets. That is, the function can be defined by extension and there is no rule for assigning the variables.

**Analytical - Graphical Approach:** The textbooks analysed included in the Analytical - Graphical Approach are the following six: **1)** *Mathematics Bachillerato 2*, Eds. Anaya (1987); **2)** *Mathematics II*, Eds. Anaya (1989); **3)** *Mathematics 9*, Eds. A-Z (1997); **4)** *Mathematics 1*, Eds. Kapeluz (2004); **5)** *Mathematics 9*, Eds. McGraw Hill (1998); **6)** *Mathematics 3*, Eds. Puerto de Palos (2009). These textbooks were published between 1987 and 1998, yet there is one exception of 2009. The analysis is similar to the one made in the textbook *Mathematics 3* published in 2009 by Puerto de Palos, presented below.

**The reference [S]:** The textbook introduces the concept of function (figure 2) by means of the definition: "A function is a relation between two variables in which each value of the first corresponds to a single value of the second".

Then, it presents a graph that represents the users of the Web worldwide, and interprets it. Thus, the textbook indicates that for each value of time (independent variable) there is a unique number of users (dependent variable). That is, 1574 million users are the image of the year 2008, and 2008 is the preimage of 1574. Then, the textbook defines the domain as all the values that can be taken by the independent variable, in this case all the years greater or equal than 1989 and less than or equal to 2008; and all the values that the independent variable can take as the image set, formed by all users greater than 0 or less than or equal to 1574 million.

**The signifier [F]:** The textbook uses the RS Written Verbal for the definition of Function and domain and image set; and the RS Analytical - Graph to represent the function.



**Figure 2: Function Definition Book "Mathematics 3" - 2009, Edited by Puerto de Palos**

**The meaning [I]:** The Theorems and Concepts in Act that we reconstruct from the concept in this approach are the following. **RS Verbal Written:** A *Function is a Relationship between two variables in which each value of the independent variable corresponds to a single value of the dependent variable - The independent variable is x, which is represented on the abscissa axis - The dependent variable is y, which is represented on the ordinate axis - To find the characteristics of a Function, the problematic situation it represents must be taken into account.* **RS Analytical – Graph:** A *Function is represented through a graph in the Cartesian Axes – It is a Function if a single element of the vertical axis corresponds to each point of the horizontal axis - It is not a*

Function if the drawing of a vertical line from any point of the horizontal axis cuts the graph in more than one point.

These Theorems-in-Action give a dynamic idea of the concept of Function. In short, the books of this approach propose some situations of everyday life through a graph in Cartesian axes, describing the relationship between variables as a function, and then analyse its characteristics such as the analysis of variables, growth and decrease, maximum and minimum points, constant function, periodic function and zeros. In all cases, the definition is made in the RS Verbal Write and refers to a relation between variables that meet certain characteristics.

**Hybrid Approach:** The books analysed framed in the Hybrid Approach are the following: **1)** *Mathematics 2*, Eds. Santillana (1989); **2)** *Mathematics 3*, Eds. Santillana (1995); **3)** *Mathematics 8*, Eds. Santillana (2002); **4)** *Pythagoras 8*, Eds. SM (2004); **5)** *Mathematics, functions and Statistics*, Eds. A-Z (2005); **6)** *Functions 1*, Eds. Longseller (2002); **7)** *Mathematics I*, Eds. Estrada (2004), **8)** *Elementary Functions to construct mathematical models*, Eds. Inet (2010); **9)** *Mathematics 4*, Eds. Estrada (2010). These textbooks were published between 1995 and 2010, yet there is one exception in 1989. The analysis is similar to the one made in the book *Mathematics 3* published in 1995 by Santillana, presented below.

**The reference [S]:** The textbook introduces the concept of function by means of a real-life problem relative to the rate of a taxi according to the kilometres travelled, and then solves it. Based on the problem, it establishes the following (figure 3): "In this real-life problem, we observe the presence two sets; set D, formed by the distances that can be covered in a taxi, and another set P, which is the price corresponding to those distances. As seen above, each distance D corresponds to a single price of the set P. In mathematics, we call this correspondence function of D in P".

En este problema de la vida real observamos que aparecen dos conjuntos: un conjunto D, formado por las distancias que se pueden recorrer en un remise, y otro conjunto P, formado por los precios correspondientes a esas distancias. Según lo visto anteriormente, a cada distancia del conjunto D le corresponde un único precio del conjunto P. En matemática, a esta correspondencia la llamamos función de D en P.

**Concepto de función**

Dados dos conjuntos A y B, se llama función de A en B a la relación que a todo elemento de A le hace corresponder uno y sólo un elemento de B.

$f: A \rightarrow B$  Quiere decir que la función  $f$  está definida de A en B.  
 A es el conjunto de partida de la función  $x \in A$   
 B es el conjunto de llegada de la función  $y \in B$   
 Si a un elemento  $x$  le corresponde  $y$  a través de la función  $f$ , decimos que  $y$  es la imagen de  $x$ , y lo escribimos  $y = f(x)$ ; también podemos decir que  $x$  es la preimagen de  $y$ .

$f$  es función.

**Figure 3: Definition of Function, Book "Mathematics 3" - 1995, Publisher Santillana**

Then, the formal definition of function is written, as well as the image and preimage, using elements of the set theory.

**The signifier [Γ]:** The textbook uses the Written Verbal Representation Systems, Numeric, Analytical - Graphic and Pictorial to define and represent the function and concepts of image, preimage and domain.

**The meaning [I]:** The Theorems and Concepts in Act reconstructed from the concept in this approach, are the following. **RS Verbal Written:** *A Function is a specific relationship between two variables - The variable  $x$  is the independent variable - The variable  $y$  is the dependent - Each value of " $x$ " corresponds to a single value of " $y$ " - The Domain is the set of all the values that the independent variable " $x$ " can take - The Image is the set of all the values that the dependent variable " $y$ " can take - The Image is obtained by applying the Function to the elements of the Domain.* **RS Numeric:** *A function can be represented by a table - In the first column the values of the independent variable are located - In the second column the values of the dependent variable are located.* **RS Analytical – Graphic:** *A Function can be represented in Cartesian axes - The values of  $x$  are exemplified by the horizontal axis - The values of  $y$  are located on the vertical axis - The graph represents the points  $(x,y)$ , to draw the figure that represents the function - When drawing lines parallel to the  $y$  axis, they must cut the graph at a single point to be Function.* **RS Pictorial:** *Functions can be represented by Venn diagrams - From each element of the set called Domain, a single arrow appears towards an element of the set called Codomain.*

These Theorems-in-Action give a dynamic idea of the concept of Function, yet they preserve elements of the Sets approach. In synthesis, in these books, the proposed situations correspond to a situation directly connected to a certain context, with its formula or chart and analysis, using elements of the two previous approaches.

## Conclusion

The analyses of the concept of function showed that - except for some few exceptions- the approaches usually corresponded to a particular set of years, and that the changes in the textbooks coincided with the dates in which national laws on education were sanctioned in the country. The Approach of Sets (1953-1990) prevailed in the period of Law 1420 of Education. Then, the Graphic Analytical Approach (1993-2006) was consolidated during the Federal Education Law (24.195), passed in 1993. Finally, the Hybrid Approach corresponds to the current period, coinciding with the last National Education Law 26.206 enacted in 2006. At the same time, the analyses of the textbooks showed that the Operational Invariants, about function changes according to the approach.

The **Approach of Sets** emphasizes the idea of a binary relation between sets that complies with existence and uniqueness (for example " $x$  was born in  $y$ "), but relegates the dependence between variables. However, the positive aspect of this approach is the fact that it uses a lot of the RS Second Order Algebraic, which allows to have the symbolic notation and rigor characteristic of Mathematics. In the **Analytical - Graphical Approach**, the idea of the function is built on the system of graphic representation, in which the written verbal language explains that a Function is a Relationship between two variables, and that each value of the independent variable corresponds to a single value of the dependent variable. The independent variable is  $x$ , represented on the abscissa axis, and the dependent variable is  $y$ , represented on the ordinate axis. This approach manages to link the use of functions with everyday life but loses mathematics in the process. Finally, the **Hybrid Approach** presents the functions as a relation between two variables, but tries to recover elements from the focus of the approach of sets, such as Range, ordered pair, etc. Likewise, this

approach tries to make sense of the functions by linking them with everyday life, while at the same time there is an effort to recover the proper notations of mathematics. However, this effort is limited by the lack of elements typical of set theory, conditioned by its use.

Finally, the analysis of the reconstructed theorems-in-action for each RS showed that, as affirmed by Sureda and Otero (2013), the meaning of function changes according to the representational system. For it is not the same to think of functions as a relation that achieves "*from each element of the set called Domain a single arrow goes to an element of the set called Codomain*", to think of a Function as a relation in which "*each point of the axis horizontal corresponds to a single element of the vertical axis*", or that can be represented by a formula. Moreover, not only there is a change in meaning, but also a change in the operative invariants linked to them that are characteristic of the RS. For example, RS Graph has a number of operative invariants linked to the scale, and to the location of the points in the plan, which the Venn diagram lacks. The meaning of a concept is much more than the systems of representation; however, it is impossible to conceptualize complex concepts without any representation system. Particularly in secondary school, it is essential to foster situations that demand the use of the different representation systems which constitute the concept.

## References

- Cordero, F., Cen Che, C., & Suárez Téllez, C. (2010). Los funcionamientos y formas de las gráficas en los libros de texto: una práctica institucional en el bachillerato. *Revista Latinoamericana de Investigación en Matemática Educativa*, 2, 187-214.
- Lávaque Fuentes J., Méndez, N., & Villarroel, Y. (2006). Concepciones de los alumnos de la noción de Función. *Revista de Educación Matemática*. Número especial. Disponible en: <https://revistas.unc.edu.ar/index.php/REM/article/view/10467/11152>. Fecha de acceso: 10 May. 2019.
- Sureda, P., & Otero, R. (2011). Nociones fundamentales de la teoría de los campos conceptuales. *Revista Electrónica de Investigación en Educación en Ciencias (REIEC)*, 6(2), 124-138.
- Sureda, P., & Otero, M. R. (2013). Estudio sobre el proceso de conceptualización de la función exponencial. *Revista Educación Matemática*, 25(2), 89-118.
- Vergnaud, G. (2013). Conceptual development and learning. *Revista Currículum*, 26(3), 39-59.
- Vergnaud, G. (2007). ¿En qué sentido la teoría de los campos conceptuales puede ayudarnos para facilitar aprendizaje significativo? *Investigações em Ensino de Ciências*, 12(2), 285-302.
- Vergnaud (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior*, 17(2), 167-181.