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Supporting the reinvention of the slope of a curve in a point: A smooth slope to slide is a smooth slide to slope

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This paper discusses a task, designing a slide, to support the reinvention of the notion of slope of a curve in a point. The field test show that the lesson scenario allows teachers to utilize students' informal models to introduce slope more formally.

Keywords: Mathematics education, reinvention, slope.

Introduction

Introducing slope of a curve in a point and the derivative to students is a didactical challenge for teachers. It is tempting to choose for an instrumental approach swiftly progressing to differentiation techniques, both easier to teach and to learn. Without conceptual understanding this entails merely meaningless manipulation of symbols and execution of recipes. The meaning of limit, difference quotient and other concepts related to the slope of a curve at a point form serious obstacles for students (e.g. Tall, 2013; Zandieh, 2000).

This paper discusses an attempt to deal with these obstacles inspired by design heuristics from the theory of Realistic Mathematics Education (RME) suggesting that a way for students to learn mathematics in a meaningful way is to involve them in a process of reinvention (Freudenthal, 1991). A common approach in RME is to introduce a new concept through a task of which the context provides opportunities for students to focus on reasoning and representations related to the notion aimed at. As Gravemeijer and Doorman (1999) state it

The students should first experience a qualitative, global, introduction of a mathematical concept. This qualitative introduction then should create the need for a more formal description of the concepts involved. (p. 113)

This study reports initial experiences of an attempt to support students in (re)inventing the notion of slope of a curve in a point and related representations (e.g. Zandieh, 2000). Students, collaborating in groups of three, are asked to design a slide consisting of a bended and a straight part joining *without bumps*. The desired outcomes are concrete equations describing a line and a curve that meet smoothly (imagine the purpose is to feed these equations into a 3D-printer to print the slide). The task aims for students to discuss what it means for the line and curve to meet in a not-bumpy, i.e. smooth, way. They should search for methods to design a slide and to decide to what extent smoothness is achieved. The hypothesis is that they will come up with ideas that are essential to the notion of slope of a curve in a point. Informal building blocks, meaningful to the student, that can be used by the teacher to afterwards introduce the slope of a curve in a point more formally.

The context of the task is a non-kinematic one. Previous research (e.g. Doorman, 2005) exploited kinematic contexts. Historically this approach makes sense, given the origin of the notion of derivative in mechanics. The presence of dimensions (like meters/second) helps to understand the meaning of slope and more generally to make a connection from derivative in mathematics to the physics curriculum. The slide-context offers opportunities that are not present in the kinematic approach. Students have a tactile and visual understanding of what it means for a surface (or better, a curve) to be *smooth*, as also pointed out by Tall (2013). He suggests students slide their hands along a curve to sense the changing slope. The slide task allows students to mathematize this tactile and visual experience. A main merit of the slide task is that it is formulated in a very *open* way: it allows for various approaches by students. We discern three options: the standard textbook approach through secants of the curve, the locally linear/linear approximation/"zooming in" approach (Tall, 2013), and a (more algebraic) approach based on local bounding lines. The most important advantage of the slide task is that it allows the teacher *connect the student's work to any or all of these approaches*, depending on what students produce, possibly providing *multiple views on the notion of slope*.

The design of the lesson plan (scenario) for the task is based on the Theory of Didactical Situations (TDS) (Brousseau, 2002). We investigated whether combining RME and TDS as frameworks for the design of an inquiry based mathematics lesson can lead to a successful scenario. This approach of combining ideas from RME and TDS to task design for inquiry based mathematics teaching (IBMT) is pioneered in the Erasmus+ project Meria (Winsløw, 2017).

The research question addressed in this paper is: How do RME-inspired task characteristics and a TDS-inspired teaching scenario support students' invention of the notion of slope of a curve in a point?

RME perspective on the slide task

The design of the task originates from the RME-principle of a *didactical phenomenology* (Freudenthal, 1983) of the slide context:

What a didactical phenomenology can do is ... starting from those phenomena that beg to be organized, and from that starting point teaching the learner to manipulate these means of organising. (Didactical phenomenology of mathematical structures)

The challenge of realizing a smooth connection in the slide context is a phenomenon that begs to be organized by the tangent line. Visually at first, but then it begs for symbolic means to decide whether the candidate line and curve really join in a smooth way. Some questions that arise are: *how good is the slide at begging?* Does it lead students to mathematize in direction of the learning goal, or even reinvent the slope of a curve at a point? What can be expected from students can be explained using the *emergent models* design heuristic (Doorman & Gravemeijer, 2009). The context-problem is expected to invite students to develop their own situation-specific solution methods. These methods will have to be discussed and compared. A (situational) model expected to emerge from the slide task is that of the slope of a curve at a point as the slope of a tangent line. This is a model *of*, produced after one episode of mathematizing. At best, this is a starting point for

further mathematizing towards symbolic and computational aspects of the slope at a point and the derivative.

The slide task as a TDS-scenario

How to organize the classroom for a reinvention activity? Freudenthal (1991) claims that “guiding means striking a delicate balance between the force of teaching and the freedom of learning” (p. 55). TDS might provide a suitable framework to balance the two. Central to the theory is the difference between didactical and adidactical situations (Brousseau, 2002). In a didactical situation the teacher acts intentionally to share his knowledge. In an adidactical situation the teacher purposefully withdraws, leaving space for students to develop their own activities. Students need this space to invoke their own meaningful strategies to address the slide problem.

For Brousseau it served another purpose: to change the expectations the students and the teacher have of each other. He calls it *finding a new didactical contract*. During this adidactical situation the teacher cannot be expected to be involved in the mathematizing process. Instead (groups of) students interact with the *milieu*. For the slide task the milieu consists of the problem itself together with the artifacts that may be needed to tackle it, for example GeoGebra.

The slide task lesson plan is set up according to phases from TDS and starts with the teacher explaining the problem and introducing artifacts that can be used to work on it. Then the teacher symbolically hands over the milieu to the students and withdraws. What follows is an adidactical *action phase* of 20 to 30 minutes. In this phase students work on the problem in groups of three. They may apply any approach they think is useful. The teacher, even though not interacting with the students, is not inactive: she registers solution strategies from the students and identifies examples that might be used in the following *formulation phase*. The teacher makes sure that for groups that have different strategies one student explains their approach on the blackboard. Then follows a *validation phase*. The teacher asks: “are some solutions better than others? Is there a best solution? How do you know?”. The classroom discussion that these questions provoke form the input for the last phase: the *institutionalization phase*. In this phase the teacher is expected to be able to organize the ideas and strategies presented by students into solution models of the slide problem. The teacher makes a start with transforming the emerging *model of* the situation produced by students into a *model for* mathematical reasoning.

Method

Context. This task was conceived during a project meeting of the Erasmus+ project Meria on inquiry based mathematics teaching. A working group designed a detailed scenario describing teacher and student activities and possible solution strategies.

Procedure. The data for this pilot include self-reports, including students work, of pilot lessons by three Dutch teachers (from now on referred to as teacher A, B and C). Teacher B is involved with the whole Meria project as a teacher who performs pilots; the other two teachers signed up for the course after reading about it in a national newsletter. They were trained to give the lesson in a three hour course taught by the authors. In the course we had a try-out of the scenario and discussed the

scenario in detail. The potential of the didactical phase(s) was stressed as this is an uncommon position for teachers to be in. At some point in the following four weeks they taught the lesson. After the lesson teachers assembled the work of the students and wrote a report including their observations and experiences during the lesson. In a second meeting, after those four weeks, we discussed the experiences of the pilot and their reports with the teachers.

Other data is the self-report on a pilot conducted by the first author.

Analysis.

In an a priori analysis we discerned three hypothetical student approaches, based on three formal approaches to the tangent line. In the a posteriori analysis, we consider signs or traces of each approach as an indicator of a *situational student model*, which may later be *vertically mathematized* into a more formal mathematical model given by the definition.

1. (Bounding line approach) Students choose a free line and then move (translate and rotate) it until there seems to be just one intersection point in the area of focus. This informal approach relates to a formal definition of the tangent line as a unique local bounding line (this we will explain more precisely in the continuation of this research project).
2. (Secant lines approach) Students choose one point on the curve, the intended point of tangency. Then they choose another point on the curve, draw the line between the two points, and move the second point closer to the first (this works best in Geogebra or similar) to obtain a smoother fit. This informal approach relates, of course, to the standard school book approach of the tangent line and slope.
3. (Linear approximation approach) Students choose one point on the curve, draw a line and then try to adjust the slope so that it fits best against the curve (perhaps by wiggling their ruler). In GeoGebra or similar this can be done by moving the second point needed to draw the line. This informal approach relates to the more formal approach of the tangent line as the best linear approximation at a point of the curve (this will also be explained more precisely in future work).

The main distinction between the approaches is in the way the smoothness of the slide designs is evaluated or controlled by the students in the slider context: (1) by looking for intersection points, (2) by moving the point closer and closer and (3) by looking for intuitive smoothness. If students allow GeoGebra to compute the intersection point of the line and the curve, then approach 1 and 3 will resemble approach 2, since they will see the two intersection points approaching each other as they move closer to the desired line. In approach 1 students fix the slope and search for corresponding point, whereas in approach 2 they fix the point and find the corresponding slope. In approach 3 both options are present.

After students have drawn some lines and curves and written corresponding equations, they will have to find out whether their solution is ‘smooth’ enough. In a second a priori analysis, of what might be discussed by the students and/or the teacher in the validation phase, we expect three possible ways for students to evaluate their own design (in line with Zandieh, 2000):

- I. (Visual). Some will rely on their visual evaluation of the design: if it looks good, then it is good. Students may also choose to zoom in on the curve (work on a smaller scale), for example using GeoGebra or similar, to check whether their fit is smooth. Zooming in the curve should more or less coincide with line (see the Locally Straight Approach in Tall 2013).
- II. (Algebraic). The students may compute whether their system of equation has the intended intersection point as a solution. Moreover, students could try and compute that there is *precisely* one solution to the system of two equations (at least locally). For example, if the curve is described by $y = x^2$ and the line is translated, so $y = x + b$. Finding the value of b for which the discriminant of the equation $x^2 = x + b$ is zero, gives the desired line. For general algebraic curves one could study for which value of b the intersection point has multiplicity ≥ 2 .
- III. (Numerical). As a continuation of the second and third approach it would seem natural to choose the second point very close to the first and see how the slope of the obtained line compares to the suggested line. For example, if $y = x^2$ and $y = 2x - 1$, the line through $(x_0, y_0) = (1, 1)$ and $(x_1, y_1) = (1.001, 1.001^2)$ has a slope very close to 2. Students may even study a sequence of points (x_1, y_1) ever closer to $(1, 1)$, which would bring them very close to the definition of a derivative through the difference quotient.

We collected the work of the students. In some cases that was individual paper work, but in most cases groups handed in one A4 with their notes and solutions. Furthermore, the teachers reported about the lessons, what they observed during the action phase, to what extent they were able to follow the scenario and what they saw as main results of the lesson. We analyzed the data for occurrences (of traces) of approaches 1, 2 and 3 during the action phase. In addition, these resources were used to find occurrences of the evaluation methods I, II and III during the *action phase* and *validation phase* (possibly the teacher introducing them during a *didactical validation phase*). These traces were used to hypothesize what informal situational mathematical models emerged from the students' work.

Finally these observed approaches were analyzed for their connection to what the teachers reported to have *institutionalized* in the final phase of the lessons. In particular, we tried to decide whether the potential in the students' work was seen and used by the teacher.

Results

We discuss the work of some groups that represent various observed approaches. A group of grade 8 pre-university school students from teacher A used approach 1: in GeoGebra they translated the line with fixed slope until it seemed to be tangent to the hyperbola $y = \frac{10}{x}$ (see Figure 1). The suggested line has equation $y = -x + 6,4$. This looks acceptable (evaluation method I), but actually the two intersection points are approximately 2 apart. Clearly these students have neither

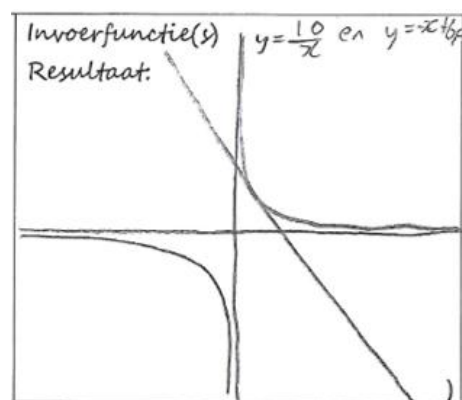


Figure 1. Sample student work

computed the intersection points (algebraic evaluation) nor zoomed in on the graphs (enhanced visual evaluation).

The teacher reports that students mention that they are looking for the line such that there is one intersection point. So these students' informal model of the situation relates to the bounding line definition of tangent line. The teacher writes in her report that she is disappointed that the students did not arrive at the notion of slope of a curve at a point. But, in this case, in the institutionalization phase the teacher could have used the students' explanation of their strategy and the fact that students mention that they are looking for one intersection point to discuss the tangent line as a bounding line. Next, with the students, she could discuss evaluation method II as a step towards the target knowledge. This makes clear that there is a difficulty for teachers to realize the potential in the informal solutions students present. The challenge is to relate students' models to the target knowledge.

A group of grade 10 vocational school students from teacher B draws a tangent line to $y = x^2$ at the point $(-2,4)$ (see Figure 2). First they seem to have drawn a secant line through $(-1,1)$ and $(-2,4)$, but this is then dismissed for the sequel. They somehow come up with a $\Delta y = 5.7$ for this candidate tangent line on the interval $[-3, -1]$. They compute the slope of the line. What follows is unfortunately nonsense, but nevertheless the teacher was very excited about their approach and let them present it at the blackboard during the formulation phase. Rightfully so, since these students' informal solution method is precisely the model one needs to introduce the slope of a curve at a point. The teacher writes in her self-report: "The notion of ... slope in a point became meaningful".

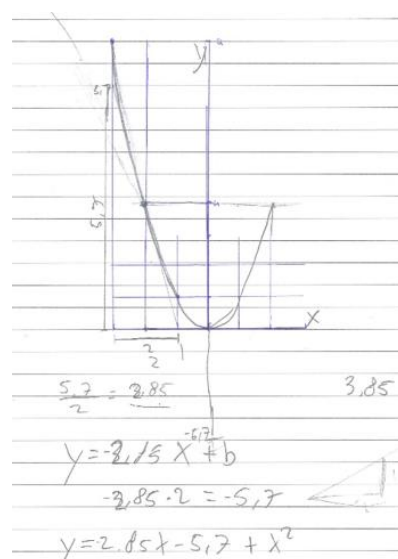


Figure 2: Sample student work

Another group from teacher B goes through a sequence of improved slide designs (see Figure 3). The first one has an asymptote; students write "does not have a straight bit". The second one is correct, but is dismissed, because the straight bit is horizontal instead of going up. The third is still with decimal coefficients, but the next four all are clearly attempts at exact solution (5 and 6 the same). Their approach can be classified as type 3. Teacher B told the class that zooming in was a good idea, so all students had a means of evaluating their solutions (method I). What happens in the last attempt is a "nice number wonder": by choosing nice numbers -2 and $\frac{1}{2}$ the students hit the bull's eye.

Teacher C did not formulate the task properly. She said: "Figure out in groups which functions could describe

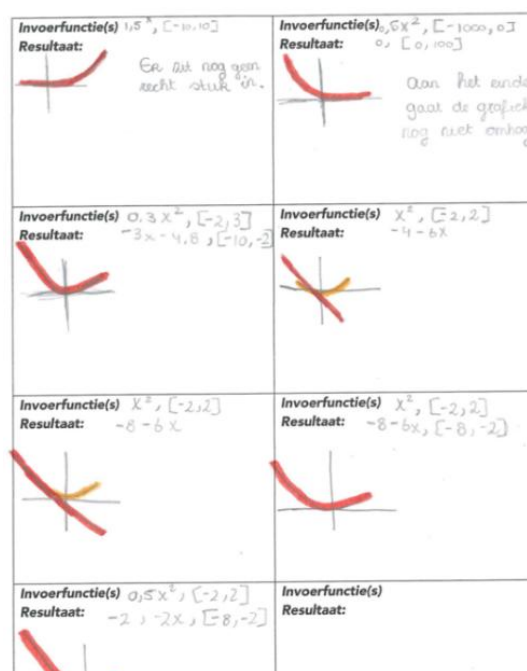


Figure 3: Sample student work

this curve". As a consequence, most students did not work with a line and a non-linear curve and the teacher got stuck trying to institutionalize the target knowledge as aimed for in the scenario.

The first author performed a pilot study with twelve grade 9 pre-university students equally divided in three groups. One group made drawings of circle segments and approximate tangent lines, using that the radius is perpendicular to the tangent. Within a few minutes they say: "the slope of the tangent line is important". Then they solve the problem by choosing a circle $x^2 + y^2 = 2$ which goes through the point $(-1, -1)$. For this point they infer that the slope of the tangent line is -1 (also visually obvious), which leads them to the equation $y = -x - 2$. This group's approach seems to be a case of vertical mathematization. Tangents to circles were already meaningful to them and for them solving the problem easily changes into symbolic work: doing some computations. After they showed the teacher their solution, he asked them to find a solution using a parabola. Their quick solution was a vertical line ($x = 0$) and a parabola ($x = y^2$). Then they asked: "How can you decide the right slope for a different point on the parabola?". This was a very welcome question, later referred to by the teacher in the institutionalization phase.

Another group quickly decided they wanted to use a tangent line with a parabola and a line design and surprisingly knew the algebraic way (II) to evaluate it: using the discriminant to decide whether the intersection point is unique. It turned out that the class made an exercise about exactly this a few weeks ago. Still the group took a long time to decide on a suitable line, finally settling for a correct one. According to the students this was obtained by guessing!

In the institutionalization phase the teacher used the students' solution strategies and remarks to stress the importance of the slope of the tangent line (both in the circle solution and parabola solution). He explained this was called the slope of the curve at a point. With the students they decided that for a circle they could compute the slope in every point, but for the parabola new techniques would be necessary. This would be the subject for later lessons.

Discussion

Did the task and scenario provide opportunities for students to "experience a qualitative, global, introduction" to the slope of a curve in a point, as was the goal of the task? More concretely: did students come up with ideas that are essential to the notion of slope of a curve in a point? Based on the observations above one could conclude the students did.

Two ideas that are essential to the notion of slope did not come up naturally for students in the pilot. Firstly, the secant lines approach (2) did not occur in any of the pilot lessons, neither did any numerical evaluation of the design (III). This indicates that for the students in the pilot this approach does not come naturally (given the current Dutch curriculum and the task). Many students do have a dynamical approach, in the sense that they try to improve their initial and following attempts by varying parameters of the curve or the line to end up with a (visually) seemingly smoother result. But using two points on the curve to construct such a sequence of better and better fitting lines seems not a natural idea to come up with. On the positive side: the dynamical approach of "improving on your previous line" is a first step toward understanding the tangent line as the result of limit *procedure*. Secondly, the idea of *zooming in* or *changing scale* as an evaluation method has not occurred naturally to students. Once told by teacher B, students immediately

grasped the idea and could use it, but they did not reinvent it. Students seem to prefer to stick to the scale at which they drew their first picture and work with reasonably nice numbers within that scale.

If students discover that some of their solutions have two intersection points, in the institutionalization phase the teacher could try and use this discovery to form a bridge to approach 2 and the difference quotient. As soon as you have two intersection points the idea of moving one of them to find a new line is a reasonable step.

A more radical conclusion in line with RME principles would be to go with the students' informal models all the way, instead of trying to facilitate a bridge towards the traditional way to introduce the slope. If the students' approach suggests the bounding line definition of a tangent line is the more meaningful for them, then the teaching sequence should be based on it. One can design instructional sequences following on approach 1 and 3 (in addition to existing sequences based on approach 2). These questions will be topic of study during the next steps in the context of this project.

Finally, in this project we combined principles from TDS and RME. This experience showed that the TDS emphasis on organizing didactic and adidactic phases by a teacher, in connection with the RME principle of reinvention guided by emergent models during these phases, supported the design of an inquiry-based scenario for students.

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