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Abstract We consider a portfolio optimization problem for financial markets described by exponential Lévy processes with jumps. For this problem we obtain and study the Hamilton-Jacobi-Bellman (HJB) equation which is an integral and partial differential equation of the second order. For this problem we show the corresponding verification theorem and construct the optimal consumption/investment strategies. For the power utility function we find the optimal strategies in the explicit form. Finally, we do the Monte Carlo simulations to illustrate numerically the obtained theoretical results.
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1 Introduction

1.1 Motivations

In this paper we consider an optimal investment/consumption problem during a fixed time interval $[0,T]$ for a financial market generated by exponential Lévy process. It should be noted that, the optimal investment/consumption problems are very popular in the stochastic financial markets theory. Beginning with the classical paper of Merton [30], where in the first time an optimal investment problem was studied for the Black-Scholes models, the interest in these problems constantly increases until today (see, for example, [31,22,25, 17] and the references therein). It should be emphasized that financial markets defined by the continuous stochastic processes similar to the geometric brownian motion are very limited for the practical applications and they do not allow us to describe situations of abrupt, impulsive changes in price processes observed during of the crises and instability in financial markets. It seems that for the first time an optimization portfolio problem for financial markets with jumps was studied in the paper [15], in which the authors using the stochastic Pontryagin maximum principle constructed optimal investment/consumption strategies. The main drawback of the maximum principle approach is that this method can't consider the objective function including the term depending of the terminal wealth value which is very important for the applications of this strategies. This means that for this case the optimization problem does not take into account the portfolio value, i.e. indeed, this is only optimal consumption problem which has not enough practical interest. Later, these problems were studied for more complex market models and in different settings, in [27] authors considered a maximization utility problem in general semimartingale settings and found the necessary and sufficient condition on a utility functions. In the papers [34,12,2,29] such problems were considered for stochastic volatility markets (as example, in [29] authors analyzed the Bellman equation but did not proved the verification theorem), in [13,21] for the market defined by the the affine processes, in the papers [23,24,9,14] the authors considered the portfolio optimization problems with constraints. The similar problem was considered in [20] with semimartingale prices framework without constraints on risks measures. Authors of [4,33] considered a problem of optimal consumption in the specific budget form. In particular, similar approach was developed in [10] under restricted information. In [28] considered pure consumption problem on Lévy markets with infinite time horizon under proportional transaction costs where they used geometric approach and viscosity solutions in similar spirit as it was done in [6] (without transacion costs). In the paper [7] studied optimal investment for finite time horizon and consumption problem for infinite horizon using stochastic duality for market with jumps. Closed solution was found in case of variance gamma class with HARA utility function. Authors of [5] characterized q-optimal equivalent martingale measures for Lévy processes and solved 2 examples of optimal investment and consumption problem for some special form of utility functions. In the papers [11,28] the authors studied a special geometric model for the optimization problem on the Lévy markets with integrated jumps. In the papers [1,3] the optimization problems were considered for the spread markets. Such problems are of prime interest for practical investors such as those in the electricity and gas markets, and also in other sectors like microstructure level within the aircraft construction, for the precious metals (see, for example, [8,32,16]). The three main drawbacks of the papers are consideration only pure consumption problem which is not enough for practical interest, as it was noted earlier, or merely improvement existence (and in some works uniqueness) of optimal strategy without computing these strategies, or

finding investment strategies which has practical interest but for some cases it is also not enough.

The main goal of our work is to study the classical investment and consumption problem for the general Lévy markets, i.e. in the case when the objective functions are constructed trough the both the integrated consumption term and the function of the terminal capital on the finite time interval. Moreover, we are interested to find the optimal solutions in the explicit form and to illustrate its behavior by the Monte - Carlo methods.

1.2 Main results

In this paper we consider a portfolio optimization problem for financial markets with jumps on the basis of the stochastic dynamic programming method. To this end we obtain and study the Hamilton–Jacobi–Bellman (HJB) equation. The challenge here is that we could not use directly the classical HJB analysis method from [22], which was due to the additional integral term corresponding to the jumps in the market model. Therefore, we need to develop a special analytical tool to analyze this equation and to construct optimal strategies. Similar to [23,24,12,2] we study this problem through the verification theorem method. So, in this paper, in the first time we show a special verification theorem for objective functions containing terminal functionals in non gaussian Lévy financial markets framework. Then, using this theorem we construct the optimal strategies, and, finally, for the power utility functions we provide the solutions for such optimisation problems in explicit form. Finally, we illustrate the obtained theoretical results by the numeric simulations.

1.3 Plan of the paper

The rest of the paper is organized as follows. In Section 2 we introduce the financial market and all main conditions. In Section 3 we obtain the HJB equation and we show the verification theorem. In Section 4 we construct the optimal investment/consumption strategies in the explicit form for the power utility functions. In Section 5 we give the Monte Carlo simulations for the obtained optimal strategies. Finally the proofs of the auxiliary results are given in Appendix A.

2 Problem

Let us consider the problem of constructing optimal investment strategies for financial markets described by Lévy processes with jumps that are widely used in financial mathematics (see, for example, [19]). Suppose the financial market with finite time horizon $T < \infty$ consists of a risk-free asset (bond) $(B_t)_{0\le t\le T}$ and risky assets (stocks) $(S_t)_{0\le t\le T}$, $S_t \in \mathbb{R}^n_+$, $i = 1, \dots, n$,

$$\begin{cases} \mathrm{d}B_t = r\, B_t\, \mathrm{d}t\,, \quad B_0 = 1\,, \\ \mathrm{d}S^i_t = \mu_i\, S^i_t\, \mathrm{d}t + S^i_{t-} \sum\limits_{j=1}^{n} \left(\sigma_{ij}\, \mathrm{d}W^j_t + \varsigma_{ij}\, \mathrm{d}L^j_t\right)\,, \end{cases} \tag{2.1}$$

where $r \ge 0$ is a risk-free interest rate, $\mu = (\mu_1, \dots, \mu_n)^{'} \in \mathbb{R}^n$ is a drift vector, the volatilities $\sigma = (\sigma_{i,j})_{1\le i,j\le n}$ and $\varsigma = (\varsigma_{i,j})_{1\le i,j\le n}$ are $n \times n$ matrices such that σ is positive

definite and, for any $i,\, j = 1,\ldots,n$, $0 \le \varsigma_{ij} \le 1$. The prime $'$ denotes the transposition. Moreover, $(W^1_t)_{0\le t\le T}\,,\ldots,(W^n_t)_{0\le t\le T}$ are independent standard brownian motions and $(L^1_t)_{0\le t\le T}\,,\ldots,(L^n_t)_{0\le t\le T}$ are independent pure jumps Lévy processes, i.e.

$$L^j_t = \int\limits_0^t \int\limits_{\mathbb{R}_*} y\,\Big(\nu^{\,j}(\omega;\,\mathrm{d}y\,,\mathrm{d}s) - \widetilde{\nu}^{\,j}(\mathrm{d}y\,,\mathrm{d}s)\Big)\,,$$

where $\nu^{\,j}(\omega;\,\mathrm{d}y\,,\mathrm{d}s)$ are random jump measures with compensators $\widetilde{\nu}^{\,j}(\mathrm{d}y\,,\mathrm{d}s) = \Pi^j(\mathrm{d}y)\,\mathrm{d}s$ and $\Pi^j(\cdot)$ are the corresponding Lévy measures on $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$. It should be noted that in this case $\Delta L^j_t\,\Delta L^i_t = 0$, for any $t \le T$ and $i \ne j$. We also assume the following conditions

$$\Pi^j\,(]-\infty,-1]) = 0 \quad \text{and} \quad \Pi^j(y^2) = \int\limits_{\mathbb{R}_*} y^2\,\Pi^j(\mathrm{d}y) < \infty\,, \tag{2.2}$$

which imply almost sure positivity of the processes $S^{\,i}_t$, for any $1 \le i \le n$ and $0 \le t \le T$. Everywhere bellow we use the filtration $(\mathcal{F}_t)_{0\le t\le T}$ with $\mathcal{F}_t = \sigma\{W_u, L_u\,:\,0 \le u \le t\}$. Similar to [23] in this paper we use the fractional strategies defined as

$$\theta^{\,i}_t = \frac{\alpha^i_t\,S^{\,i}_t}{X_t} \quad \text{and} \quad c_t = \frac{\zeta_t}{X_t}\,, \tag{2.3}$$

where α^i_t is number of units of i-th risky asset purchased by an investor at time moment t, X_t is her amount of capital, i.e. (we define $\check{\alpha}_t$ as number of risk-free asset)

$$X_t = \sum_{i=1}^{n} \alpha^i_t\,S^{\,i}_t + \check{\alpha}_t\,B_t\,, \tag{2.4}$$

and from the equation above we can deduce

$$1 - \sum_{i=1}^{n} \theta^{\,i}_t = \frac{\check{\alpha}_t\,B_t}{X_t}\,.$$

Moreover, the process ζ_t in (2.3) is the consumption intensity i.e. it is non negative integrated process for which the integral $\int_0^t \zeta_s\,\mathrm{d}s$ is the consumption on time interval $[0,t]$. Now using the self-financing-consumption principle for the wealth process (2.4) (see, for example, [22]) and the definition (2.1), we obtain that

$$\begin{cases} \mathrm{d}X_t = X_t\,(r + \theta^{\,'}_t\,\breve{\mu} - c_t)\,\mathrm{d}t + X_t\,\theta^{\,'}_t\,\sigma\,\mathrm{d}W_t + X_{t-}\,\theta^{\,'}_{t-}\,\varsigma\,\mathrm{d}L_t\,, \\ X_0 = x > 0\,, \end{cases} \tag{2.5}$$

where $\breve{\mu} = (\mu - r\,\mathbf{1}_n)$, $\mathbf{1}_n = (1,\ldots,1)^{\,'} \in \mathbb{R}^n$ and $\theta_t = (\theta^1_t,\ldots,\theta^n_t)^{\,'}$. It should be noted that to provide the positivity of the portfolio value X_t the jump sizes of the process

$$\breve{L}_t = \int\limits_0^t \theta^{\,'}_{s-}\,\varsigma\,\mathrm{d}L_s$$

have to be greater than -1, i.e. $\Delta \check{L}_s > -1$. To this end we assume that the financial strategy $\theta_t = (\theta_t^1, \dots, \theta_t^n)'$ is a càdlàg process with the values in the set $[0,1]^n$ such, that for any fixed $0 \le t \le T$ almost sure $\sum_{j=1}^n \theta_t^j \le 1$. In the sequel we denote by

$$\Theta = \left\{ x = (x_1, \dots, x_n)' \in [0,1]^n \, : \, \sum_{j=1}^n x_j \le 1 \right\}. \tag{2.6}$$

Now, using this set we introduce the admissible strategies for which we will consider the optimal consumption and investment problem.

Definition 2.1. *A stochastic process $\upsilon = (\theta_t, c_t)_{0 \le t \le T}$ is called admissible if the first component $(\theta_t)_{0 \le t \le T}$ is adapted, càdlàg process with the values in the set (2.6) and the process $(c_t)_{0 \le t \le T}$ is adapted, non negative integrated on the interval $[0,T]$ for which the equation (2.5) has unique strong strictly positive solution $(X_t)_{0 \le t \le T}$.*

We denote by $\mathcal{V}$ the set of admissible strategies. Now, for any $\upsilon \in \mathcal{V}$, we set the objective function as

$$J(x, \upsilon) := \mathbf{E}_x \left(\int_0^T U(X_s \, c_s) \, \mathrm{d}s + U(X_T) \right), \tag{2.7}$$

where $\mathbf{E}_x$ is the conditional expectation given $X_0 = x$ and the utility function $U(x) = x^\gamma$, $0 < \gamma < 1$. Our goal is to find a strategy $\upsilon^* \in \mathcal{V}$, such that

$$J(x, \upsilon^*) = \sup_{\upsilon \in \mathcal{V}} J(x, \upsilon) =: J^*(x). \tag{2.8}$$

According to the dynamic programming principle we define the value function on $[t,T]$ as

$$J^*(t,x) := \sup_{\upsilon \in \mathcal{V}} J(t,x,\upsilon), \tag{2.9}$$

where the objective function on the time interval $[t,T]$ is given as

$$J(t,x,\upsilon) := \mathbf{E}_{x,t} \left(\int_t^T U(X_s \, c_s) \, ds + U(X_T) \right).$$

Here $\mathbf{E}_{x,t}$ is the conditional expectation with respect to $X_t = x$. Note that in our case $J(t,x,\upsilon)$ can be equal to ∞ for some strategy υ.

Remark 2.2. *Note that the optimal consumption and investment problem without terminal fucntional for the model (2.1) is studied in [15] (the problem (4.2)) through the maximum Pontryagin principle.*

3 Verification theorem

Now we will generalize the controlled process (2.5) and the definition of admissible strategies. For such framework we formulate and prove the verification theorem. Let define duffusion jumps controlled process X_t whose takes values in an open convex set $\mathcal{X} \subseteq \mathbb{R}$ (we need convexity to apply Ito's Lemma for a process $z(t, X_t)$ and to guarantee the existance of derivative $z_x(t,x)$ on interior of $\mathcal{X}$ and openess to define a finite summand which corresponds to jumps in HJB's equation) and control process υ_t with values in closed set $\mathcal{K}$ (we need closure to be sure that if, for any $t \in [0,T]$, $\upsilon_t \in \mathcal{K}$, then $\upsilon_{t-} \in \mathcal{K}$), where $\mathcal{K} \subseteq \mathbb{R}^n$. The main difference between classical HJB's approach and our solution is to use arbitrary sets $\mathcal{X}$ and $\mathcal{K}$ which allows us do not define stopping time $\tau = \inf\{s \in [0,T] \,:\, X_s \notin \mathcal{X}\}$. In particular case, for financial markets (when $\mathcal{X} =]0, \infty[$), stopping time τ means ruin moment. Although our solution guarantees that the controlled process will take values in $\mathcal{X}$. But it does not mean that optimal portfolio will have low volatility over time interval $[0,T]$.

Let $X = (X_t)_{0 \le t \le T}$, $X_t \in \mathcal{X}$ be Ito's process in form

$$\begin{cases} \mathrm{d}X_t = a(X_t, \upsilon_t)\,\mathrm{d}t + b^{'}(X_{t-}, \upsilon_{t-})\,\mathrm{d}V_t\,, \\ \\ X_0 = x \in \mathcal{X}\,, \end{cases} \tag{3.1}$$

where $V_t = \sigma\, W_t + \varsigma\, L_t$. $(L_t)_{t \ge 0}$, $(W_t)_{t \ge 0}$ are independent n-dimentional standart brownian motion and pure jumps Lévy process. Also, the processes $(L_t^1)_{0 \le t \le T}, \ldots, (L_t^n)_{0 \le t \le T}$ are mutually independent,

$$L_t^j = \int\limits_0^t \int\limits_{\mathbb{R}_*} y \left(\nu^{\,j}(\omega;\, \mathrm{d}y\,, \mathrm{d}s) - \widetilde{\nu}^{\,j}(\mathrm{d}y\,, \mathrm{d}s) \right)\,,$$

here $\nu^{\,j}(\omega;\, \mathrm{d}y\,, \mathrm{d}s)$ is a random measure of j-th component of jumps with it's deterministic compensator

$$\widetilde{\nu}^{\,j}(\mathrm{d}y\,, \mathrm{d}s) = \Pi^j(\mathrm{d}y)\,\mathrm{d}s\,,$$

where $\Pi^j(\cdot)$ is Lévy measure on $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ with

$$\Pi^j(y^2) < \infty\,.$$

$\sigma \in \mathbb{R}^{n \times n}$ is positive definite matrix, and, for any $i,\, j = 1, \ldots, n$, $0 \le \varsigma_{ij} \le 1$. Functions

$$a : \mathcal{X} \times \mathcal{K} \to \mathbb{R} \quad \text{and} \quad b : \mathcal{X} \times \mathcal{K} \to \mathbb{R}^n$$

are not random, continuous and such that for any fixed non-random point $\kappa \in \mathcal{K}$ the equation (3.1) with $\upsilon_t \equiv \kappa$ has an unique strong solution $X_t \in \mathcal{X}$ on time interval $[0,T]$. For fixed x and κ, $b(x,\kappa)$ is a vector-column. Moreover, for given continuous utility functions $U_1(t,x,\kappa)$ and $U_2(x)$ with values in $\mathbb{R}$,

$$\mathbf{E}\left(\int\limits_0^T (U_1(s, X_s, \kappa))_-\,\mathrm{d}s + (U_2(X_T))_- \right) < \infty\,,$$

where, for any $x \in \mathbb{R}$, $(x)_- = \max(0, -x)$. The conditions on functions a and b guarantee existence of control processes (at least $\upsilon_t \equiv \kappa = const \in \mathcal{K}$).

Definition 3.1. *We will say, a stochastic process $\upsilon = (\upsilon_t)_{0\le t\le T}$ is called admissible if it is $(\mathcal{F}_t)_{0\le t\le T}$ -adapted, has càdlàg trajectories, takes values in the set $\mathcal{K}$, and such that the equation (3.1) on time interval $[0,T]$ has an unique strong solution, such that, for any $t\in[0,T]$: $X_t\in\mathcal{X}$, $X_{t-}\in\mathcal{X}$,*

$$\int_0^T \Big(|a(X_t,\upsilon_t)|+|b(X_t,\upsilon_t)|^2\Big)\,\mathrm{d}t<\infty \qquad \mathbf{P}\text{-a.s.} \tag{3.2}$$

and

$$\mathbf{E}\left(\int_0^T (U_1(s,X_s,\upsilon_s))_-\,\mathrm{d}s+(U_2(X_T))_-\right)<\infty\,. \tag{3.3}$$

We define $\mathcal{V}$ as a set of all admissible strategies (note that $\mathcal{K}\subseteq\mathcal{V}$) and the value function on the interval $[t,T]$ as

$$J^*(t,x):=\sup_{\upsilon\in\mathcal{V}} J(t,x,\upsilon)\,, \tag{3.4}$$

where the objective function defined

$$J(t,x,\upsilon):=\mathbf{E}_{t,x}\left(\int_t^T U_1(s,X_s,\upsilon_s)\,\mathrm{d}s+U_2(X_T)\right).$$

The goal will be to find an admissible strategy $\upsilon^*\in\mathcal{V}$, such that

$$J(0,x,\upsilon^*)=J^*(0,x)=\sup_{\upsilon\in\mathcal{V}} J(0,x,\upsilon)\,. \tag{3.5}$$

Remark 3.2. *Note the process (2.5) can be obtained as a special case*

$$a(x,\kappa)=x\,(r+\kappa_1^{'}\breve{\mu}-\kappa_2)\quad \textit{and}\quad b(x,\kappa)=x\,\kappa_1\,,$$

with $\kappa=(\kappa_1,\kappa_2)\in\mathcal{K}:=\Theta\times\mathbb{R}_+$, $\kappa_1=(\kappa_1^1,\dots,\kappa_1^n)$ is a vector-column.

To apply the dynamic programming method, we will need to introduce Hamilton function. For any twice continuously differentiable function $g(t,x)$, $g:[0,T]\times\mathcal{X}\to\mathbb{R}$ with respect to x and continuously differentiated with respect to t, such that

$$\sup_{x\in\mathcal{X}}\frac{|g(t,x)|}{1+|x|}<\infty\,,$$

we define

$$H(t,x,g):=\sup_{\kappa\in\mathcal{K}} H_0(t,x,g,\kappa)\,, \tag{3.6}$$

where

$$\begin{aligned} H_0(t,x,g,\kappa)=a(x,\kappa)\,g_x(t,x)+\frac{1}{2}\,\mathrm{tr}\left\{\sigma^{'}\,b(x,\kappa)\,b^{'}(x,\kappa)\,\sigma\right\}\,g_{xx}(t,x)\\ +U_1(t,x,\kappa)+\overline{g}(t,x,\kappa)\,, \end{aligned} \tag{3.7}$$

here notation tr A means trace of matrix $A\in\mathbb{R}^{n\times n}$ and

$$\overline{g}(t,x,\kappa)=\sum_{i=1}^{n}\int_{\mathbb{R}_*}\Upsilon(t,x,b^{'}(x,\kappa)\,\varsigma_i\,y)\,\Pi^i(\mathrm{d}y)\,, \tag{3.8}$$

where $\varsigma_i = (\varsigma_{1i}, \dots, \varsigma_{ni})'$ is i-th column of the matrix ς,

$$\Upsilon(t,x,v) = [g(t,x+v) - g(t,x) - g_x(t,x)\,v]\,\mathbf{1}_{\{x+v\in\mathcal{X}\}}\,, \tag{3.9}$$

and $v \in \mathbb{R}_*$. Here $\Pi^i(\mathrm{d}y)$ is Levy measure of the process L_t^i. We need an indicator $\mathbf{1}_{\{x+v\in\mathcal{X}\}}$ since there are no assumptions on measure $\Pi(\cdot)$ to guarantee $x+v$ belongs to $\mathcal{X}$, for any $v \in \mathbb{R}_*$. Notations g_t, g_x, g_{xx} mean corresponding derivatives of function $g(t,x)$.

It is proved in Appendix A.1 that $\overline{g}$ is finite and it will be used as jumps compensator of the process $g(t,X_t)$. Note that this function is a sum of integrals since L_t consists of mutually independent Lévy processes, see Remark A.2 in Appendix.

To solve the problem (3.5), we need to study the following Hamilton–Jacobi–Bellman equation with respect to the function of two variables $z(t,x)$,

$$\begin{cases} z_t(t,x) + H(t,x,z) = 0\,, & t\in[0,T]\,, \\ \\ z(T,x) = U_2(x) & x\in\mathcal{X}\,. \end{cases} \tag{3.10}$$

Next, we need the following conditions.

$\mathbf{H}_1$) *There exists solution* $z \in \mathcal{C}^{1,2}([0,T]\times\mathcal{X},\mathbb{R})$ *of the equation* (3.10)*, such that*

$$z_* = \inf_{0\le s\le T}\inf_{x\in\mathcal{X}} z(s,x) > -\infty\,, \tag{3.11}$$

and, for any fixed $t\in[0,T]$,

$$\sup_{x\in\mathcal{X}}\frac{|z(t,x)|}{1+|x|} < \infty\,. \tag{3.12}$$

$\mathbf{H}_2$) *There exists* $[0,T]\times\mathcal{X}\to\mathcal{K}$ *measurable function* υ^*, *such that*

$$H(t,x,z) = H_0(t,x,z,\upsilon^*(t,x))\,,$$

for any $0\le t\le T$ *and* $x\in\mathcal{X}$.

$\mathbf{H}_3$) *For any* $x\in\mathcal{X}$, *there exists an unique almost surely solution* $X^* = (X_t^*)_{0\le t\le T}$ *with values in the set* $\mathcal{X}$ *such that* $X_{t-}^*\in\mathcal{X}$ *of equation*

$$\mathrm{d}X_t^* = a^*(t,X_t^*)\,\mathrm{d}t + (b^*(t-,X_{t-}^*))'\,\mathrm{d}V_t\,,\quad X_0^* = x\,, \tag{3.13}$$

where $V_t = \sigma\,W_t + \varsigma\,L_t$,

$$a^*(t,x) = a(x,\upsilon^*(t,x)) \quad \textit{and} \quad b^*(t,x) = b(x,\upsilon^*(t,x))\,.$$

Moreover, the process $\upsilon^* = (\upsilon_t^*)_{0\le t\le T}$, $\upsilon_t^* = \upsilon^*(t,X_t^*)$ *is admissible, i.e. belongs to* $\mathcal{V}$.

$\mathbf{H}_4$) *For any* $0\le t\le T$ *and* $x\in\mathcal{X}$,

$$\mathbf{E}_{t,x}\sup_{t\le s\le T}|z(s,X_s^*)| < \infty\,. \tag{3.14}$$

Remark 3.3. *Here* $\mathbf{H}_1$) *and* $\mathbf{H}_4$) *are technical conditions. In particular,* z_* *will be used to apply Fatou's lemma for limit transition under conditional expectation. The condition* (3.12) *guarantees finiteness of the jumps compensator (function* $\overline{z}$*) of the process* $z(X_t,t)$, *where* X_t *is Lévy process and* $\mathbf{H}_4$) *will be used to apply Lebesgue convergence theorem to prove optimality of the process* $\upsilon^*(t,X_t^*)$.

Remark 3.4. *We don't need uniqueness of a solution $z(t,x)$ of (3.10). But if conditions of theorem 3.6 are satisfied then $z(t,x)$ will be unique and $J(t,x,\upsilon^*) < \infty$, for any $t \in [0,T]$ and $x \in \mathcal{X}$.*

Remark 3.5. *Indeed, we use the same type of the conditions as it is done in paper [23] to show the verification theorem.*

Using the approach proposed in [23], we show the following verification theorem.

Theorem 3.6. *Suppose conditions* $\mathbf{H}_1$) – $\mathbf{H}_4$) *are hold. Then, for any* $0 \le t \le T$ *and* $x \in \mathcal{X}$,

$$z(t,x) = J^*(t,x) = J(t,x,\upsilon^*)\,,$$

where the optimal strategy $\upsilon^* = (\upsilon_s^*)_{t\le s\le T}$, $\upsilon_s^* = \upsilon^*(s, X_s^*)$ *is determined in terms of* $\mathbf{H}_2$) – $\mathbf{H}_4$).

Proof. We will consider an arbitrary fixed time moment $t \in [0,T[$. The proof is conducted in two parts: the first step is shown that, for an arbitrary control $\upsilon \in \mathcal{V}$ and any $x \in \mathcal{X}$, the following inequality hold:

$$J(t,x,\upsilon) \le z(t,x)\,.$$

Which implies, for $\forall x \in \mathcal{X}$, $t \in [0,T[$:

$$\sup_{\upsilon\in\mathcal{V}} J(t,x,\upsilon) \le z(t,x)\,.$$

The second part is shown that, for $\upsilon^* = (\upsilon_s^*)_{t\le s\le T}$, the following equality achieved

$$J(t,x,\upsilon^*) = z(t,x)\,,$$

where $\upsilon_s^* = \upsilon^*(s, X_s^*)$.

1 part. Let $\upsilon \in \mathcal{V}$, $\upsilon = (\upsilon_s)_{t\le s\le T}$ be an arbitrary admissible control (we will consider the control on time interval $[t,T]$, while the control itself can be defined over the whole interval). And consider the stochastic differential equation on the interval $[t,T]$. By the definition of an admissible control process, the equation

$$\mathrm{d}X_s = a(X_s,\upsilon_s)\,\mathrm{d}s + b^{'}(X_{s-},\upsilon_{s-})\,\mathrm{d}V_s\,,\quad X_t = x \in \mathcal{X}\,,\quad s \in [t,T]\,,$$

with $V_s = \sigma W_s + \varsigma L_s$ must have an unique ($\mathbf{P}$-a.s.) strong solution $X = (X_s)_{t\le s\le T}$ (since it has an unique strong solution on the whole interval $[0,T]$).

By condition $\mathbf{H}_1$), function $z : [0,T]\times\mathcal{X} \to \mathbb{R}$ is the solution of HJB's equation with continuous $z_t(t,x)$, $z_x(t,x)$ and $z_{xx}(t,x)$. Applying Ito's formula we have, for arbitrary $t < u \le T$ and $X_t = x$,

$$z(u,X_u) = z(t,x) + \int_t^u z_t(s,X_s)\,\mathrm{d}s + \int_t^u z_x(s,X_{s-})\,\mathrm{d}X_s$$
$$+ \frac{1}{2}\int_t^u z_{xx}(s,X_{s-})\,\mathrm{d}\langle X^c\rangle_s + \sum_{t\le s\le u}\left(\Delta z(s,X_s) - z_x(s,X_{s-})\,\Delta X_s\right),$$

where $\langle X^c\rangle$ is variation of continuous part of semimartingale X (predictable quadratic variation) and

$$\Delta z(s,X_s) = z(s,X_s) - z(s,X_{s-})\,,\quad \Delta X_s = X_s - X_{s-}\,.$$

It is easy to compute (note, we have n-dimensional Wiener process in form of $\sigma\, W_s$), that

$$\mathrm{d}\langle X^c\rangle_s = \mathrm{tr}\left\{\sigma^{'}\, b(X_s,\upsilon_s)\, b^{'}(X_s,\upsilon_s)\,\sigma\right\}\,\mathrm{d}s\,.$$

Therefore Ito's formula has the following form

$$\begin{aligned} z(u,X_u) = z(t,x) + \int_t^u \Big[& z_t(s,X_s) + z_x(s,X_s)\, a(X_s,\upsilon_s) \\ & + \frac{1}{2}\, z_{xx}(s,X_s)\,\mathrm{tr}\left\{\sigma^{'}\, b(X_s,\upsilon_s)\, b^{'}(X_s,\upsilon_s)\,\sigma\right\}\Big]\,\mathrm{d}s \\ & + \int_t^u z_x(s,X_{s-})\, b^{'}(X_{s-},\upsilon_{s-})\,\mathrm{d}V_s \\ & + \sum_{t\le s\le u} \left(\Delta z(s,X_s) - z_x(s,X_{s-})\,\Delta X_s\right). \end{aligned}$$

The key idea to prove the 1st step will be the following, we need to compensate the jump term, add to and subtract from Ito's formula the integral $\int_t^u U_1(s,X_s,\upsilon_s)\,\mathrm{d}s$ and take a conditional expectation given $X_t = x$. But we cannot apply the last operation directly since there are no conditions on finitness of expectations (in particular, for stochastic integrals). Thus we introduce a sequence of stopping times $(\tau_m)_{m>0}$ (see Lemma A.5 in Appendix why they are stopping times)

$$\tau_m := \inf\left\{u\ge t : \int_t^u \left(|\beta(s)|^2 + |\overline{z}(s,X_{s-},\upsilon_{s-})|\right)\mathrm{d}s \ge m\right\}\wedge T\,, \tag{3.15}$$

where

$$\beta(s) = z_x(s,X_s)\, b^{'}(X_s,\upsilon_s)$$

and $\overline{z}$ is given as in (3.8) ($g\equiv z$),

$$\overline{z}(t,x,\kappa) = \sum_{i=1}^n \int_{\mathbb{R}_*} \Upsilon(t,x,b^{'}(x,\kappa)\,\varsigma_i\, y)\, \Pi^i(\mathrm{d}y)\,, \tag{3.16}$$

where $\varsigma_i = (\varsigma_{1i},\ldots,\varsigma_{ni})^{'}$ is i-th column of the matrix ς ,

$$\Upsilon(t,x,v) = [z(t,x+v) - z(t,x) - z_x(t,x)\, v]\,\mathbf{1}_{\{x+v\in\mathcal{X}\}}\,,$$

$\Pi^i(\cdot)$ is Lévy measure of L_t^i . Note that the integrals in (3.16) are finite (see Lemma A.1 in Appendix). Here we used obvious equalities

$$\Delta z(s,X_s) = z(s,X_{s-}+\Delta X_s) - z(s,X_{s-})$$

and

$$\Delta X_s = b^{'}(X_{s-},\upsilon_{s-})\,\varsigma\,\Delta L_s = \sum_{i=1}^n b^{'}(X_{s-},\upsilon_{s-})\,\varsigma_i\,\Delta L_s^i\,,$$

to find a jumps compensator (see Remark A.2 in Appendix how we can define it)

$$\mathbf{E}\sum_{t\le s\le\tau_m}\left(\Delta z(s,X_s)-z_x(s,X_{s-})\,\Delta X_s\right)=\mathbf{E}\int_t^{\tau_m}\overline{z}(s,X_{s-},\upsilon_{s-})\,\mathrm{d}s\le m\,. \tag{3.17}$$

Also note that in this case

$$\mathbf{E}_{t,x}\int_t^{\tau_m}z_x(s,X_{s-})\,b^{'}(X_{s-},\upsilon_{s-})\,\mathrm{d}V_s=0\,. \tag{3.18}$$

As the result, considering the definition (3.7) of function $H_0(t,x,z,\kappa)$, replace u to τ_m in Ito's formula, we have

$$\begin{aligned}\int_t^{\tau_m}U_1(s,X_s,\upsilon_s)\,\mathrm{d}s+z\left(\tau_m,X_{\tau_m}\right)&=z(t,x)+\int_t^{\tau_m}\left[z_t(s,X_s)+H_0(s,X_s,z,\upsilon_s)\right]\mathrm{d}s\\&+\int_t^{\tau_m}z_x(s,X_{s-})\,b^{'}(X_{s-},\upsilon_{s-})\,\mathrm{d}V_s\\&+\sum_{t\le s\le\tau_m}\left(\Delta z(s,X_s)-z_x(s,X_{s-})\,\Delta X_s\right)-\int_t^{\tau_m}\overline{z}(s,X_s,\upsilon_s)\,\mathrm{d}s\,.\end{aligned} \tag{3.19}$$

Note, that

$$\int_t^{\tau_m}\left[z_t(s,X_s)+H_0(s,X_s,z,\upsilon_s)\right]\mathrm{d}s\le 0\quad\mathbf{P}\text{-a.s.}\,,$$

since, by the condition $\mathbf{H}_1$), for any $s\in[t,T]$ and $x\in\mathcal{X}$, $z(s,x)$ is the solution of HJB's equation, i.e.

$$z_t(s,x)=-H(s,x,z)\,,$$

from (3.6), it follows that, for any $\kappa\in\mathcal{K}$,

$$H(s,x,z)\ge H_0(s,x,z,\kappa)\,.$$

Therefore, from (3.19), we can obtain the following inequality,

$$\begin{aligned}\int_t^{\tau_m}U_1(s,X_s,\upsilon_s)\,\mathrm{d}s+z\left(\tau_m,X_{\tau_m}\right)&\le z(t,x)+\int_t^{\tau_m}z_x(s,X_{s-})\,b^{'}(X_{s-},\upsilon_{s-})\,\mathrm{d}V_s\\&+\sum_{t\le s\le\tau_m}\left(\Delta z(s,X_s)-z_x(s,X_{s-})\,\Delta X_s\right)-\int_t^{\tau_m}\overline{z}(s,X_s,\upsilon_s)\,\mathrm{d}s\,.\end{aligned} \tag{3.20}$$

Now we can take the conditional expectation (given $X_t=x$) in inequality (3.20), considering (3.17) and (3.18) (note that V_t is square integrable martingale),

$$\mathbf{E}_{t,x}\left(\int_t^{\tau_m}U_1(s,X_s,\upsilon_s)\,\mathrm{d}s+z\left(\tau_m,X_{\tau_m}\right)\right)\le z(t,x)\,. \tag{3.21}$$

The next step will be to $m \to \infty$ (i.e. $\tau_m \to T$) and do limit transition under expectation $\mathbf{E}_{t,x}$ by using Fatou's lemma. To do it, for $m \in \mathbb{N}$, let's define random variable

$$\xi_m := \int_t^{\tau_m} U_1(s, X_s, \upsilon_s)\, \mathrm{d}s + z\left(\tau_m, X_{\tau_m}\right).$$

By hypothesis $\mathbf{H}_1$) of the theorem, we have $z_* = \inf_{0\le t\le T}\inf_{x\in\mathcal{X}} z(t,x) > -\infty$. Therefore, for any $m \in \mathbb{N}$,

$$\begin{aligned}\xi_m &= z\left(\tau_m, X_{\tau_m}\right) + \int_t^{\tau_m} U_1(s, X_s, \upsilon_s)\, \mathrm{d}s \\ &\ge z_* - \int_t^{\tau_m} [U_1(s, X_s, \upsilon_s)]_-\, \mathrm{d}s \ge z_* - \int_t^{T} [U_1(s, X_s, \upsilon_s)]_-\, \mathrm{d}s\,, \quad \mathbf{P}\text{-a.s.} \end{aligned} \tag{3.22}$$

Let's define

$$\eta := z_* - \int_t^{T} [U_1(s, X_s, \upsilon_s)]_-\, \mathrm{d}s\,,$$

and note that, by Definition 3.1 of admissible strategies, $\mathbf{E}_{t,x}|\eta| < +\infty$. By (3.22), we have, for any $m \in \mathbb{N}$,

$$(\xi_m)_- \ge \eta\,.$$

Therefore, by Fatou's lemma for conditional expectation, we have next inequality

$$\mathbf{E}_{t,x} \liminf_{m\to\infty} \xi_m \le \liminf_{m\to\infty} \mathbf{E}_{t,x}\, \xi_m\,.$$

Thus inequality (3.21) can be rewritten as following

$$\mathbf{E}_{t,x}\left(\liminf_{m\to\infty} \xi_m\right) \le z(t,x)\,. \tag{3.23}$$

Note that $\lim_{m\to\infty} \xi_m$ is well-defined, namely,

$$\lim_{m\to\infty} \xi_m = z(T, X_T) + \int_t^{T} U_1(s, X_s, \upsilon_s)\, \mathrm{d}s\,,$$

since Lebesgue integral is continuous (as function of upper limit), $\tau_m \to T$ as $m \to \infty$, $\mathbf{P}$-a.s., and X is Lévy process, in particular, at time moment T there is no jump (i.e. X is continuous at $t = T$), therefore

$$\begin{aligned}\liminf_{m\to\infty} \xi_m = \lim_{m\to\infty} \xi_m &= z(T, X_T) + \int_t^{T} U_1(s, X_s, \upsilon_s)\, \mathrm{d}s \\ &= U_2(X_T) + \int_t^{T} U_1(s, X_s, \upsilon_s)\, \mathrm{d}s\,,\end{aligned}$$

and inequality (3.23) can be rewritten in the following form

$$\mathbf{E}_{t,x} \liminf_{k\to\infty} \xi_k = \mathbf{E}_{t,x} \left(U_2(X_T) + \int_t^T U_1(s, X_s, \upsilon_s)\, \mathrm{d}s \right)$$

$$= J(t,x,\upsilon) \le z(t,x)\,. \tag{3.24}$$

Taking into account that this inequality holds, for any admissible strategy $\upsilon \in \mathcal{V}$, we obtain that

$$\sup_{\upsilon\in\mathcal{V}} J(t,x,\upsilon) =: J^*(t,x) \le z(t,x)\,. \tag{3.25}$$

2 part. By the hypotheses $\mathbf{H}_1$)–$\mathbf{H}_4$) of the theorem, there exist: a solution $z(t,x)$ of the Hamilton-Jacobi-Bellman equation, the measurable function $\upsilon^* : [0,T] \times \mathcal{X} \to \mathcal{K}$, such that, $\forall x \in \mathcal{X}$ and $\forall s \in [0,T]$:

$$H(s,x,z) = H_0(s,x,z,\upsilon^*(s,x))\,,$$

the stochastic process $X^* = (X^*_s)_{0\le s\le T}$ and admissible control process $\upsilon^* = (\upsilon^*_s)_{0\le s\le T}$, $\upsilon^*_s = \upsilon^*(s, X^*_s)$, such that

$$\mathrm{d}X^*_s = a^*(s,X^*_s)\,\mathrm{d}s + (b^*(s-,X^*_{s-}))'\,\mathrm{d}V_s\,, \quad X^*_0 = x \in \mathcal{X}\,, \quad s\in[0,T]\,.$$

And, for $\forall u \in [0,T]$,

$$\mathbf{E}_{u,x} \sup_{u\le s\le T} |z(s,X^*_s)| < +\infty\,,$$

where

$$a^*(s,x) = a(x,\upsilon^*(s,x))\,, \quad b^*(s,x) = b(x,\upsilon^*(s,x))\,.$$

In the first part we have considered an arbitrary admissible control process $\upsilon \in \mathcal{V}$, and noted that, for any $m > 0$, τ_m defined by (3.15) is stopping time. Then, for any $m > 0$, a random variable τ^*_m is also stopping time (in case of υ^*), where

$$\tau^*_m = \inf\left\{ u \ge t : \int_t^u \left(|\beta^*(s)|^2 + |\overline{z}^*(s, X_{s-}, \upsilon_{s-})| \right) \mathrm{d}s \ge m \right\} \wedge T\,, \tag{3.26}$$

here, for $s \in [t,T]$,

$$\beta^*(s) = z_x(s,X^*_s)\ \left(b^*(s,X^*_s)\right)'\,,$$

$$\overline{z}^*(t,x,\kappa) = \sum_{i=1}^n \int_{\mathbb{R}_*} \Upsilon(t,x,\left(b^*(x,\kappa)\right)'\,\varsigma_i\, y)\, \Pi^i(\mathrm{d}y)\,,$$

where $\varsigma_i = (\varsigma_{1i}\,, \ldots, \varsigma_{ni})'$ is i-th column of the matrix ς and $\Upsilon(s,x,v)$ defined by (3.16).

With analogously to the previous arguments in the first part, using the Ito formula, we obtain the process $z(u, X^*_u)$, $u \in]t,T]$. Then, adding to and subtracting from Ito's formula the sum of Lebesgue integrals ($\upsilon^*_s = \upsilon^*(s, X^*_s)$)

$$\int_t^{\tau^*_m} \overline{z}^*(s, X^*_s, \upsilon^*_s)\,\mathrm{d}s + \int_t^{\tau^*_m} U_1(s, X^*_s, \upsilon^*_s)\,\mathrm{d}s\,,$$

we get (similarly to expression (3.19))

$$\int_t^{\tau_m^*} U_1(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s + z\left(\tau_m^*, X_{\tau_m^*}^*\right) = z(t,x) + \int_t^{\tau_m^*}\left[z_t(s,X_s^*) + H_0(s,X_s^*,z,\upsilon_s^*)\right]\mathrm{d}s$$
$$+ \int_t^{\tau_m^*} z_x(s,X_{s-}^*)\,\left(b^*(s-,X_{s-}^*)\right)'\,\mathrm{d}V_s$$
$$+ \sum_{t\le s\le\tau_m^*}\left(\Delta z(s,X_s^*) - z_x(s,X_{s-}^*)\,\Delta X_s^*\right) - \int_t^{\tau_m^*}\overline{z}^*(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s\,. \tag{3.27}$$

Further,

1) It follows from the conditions $\mathbf{H}_1$)–$\mathbf{H}_3$) that

$$\int_t^{\tau_m^*}\left[z_t(s,X_s^*) + H_0(s,X_s^*,z,\upsilon_s^*)\right]\mathrm{d}s = 0\,,\quad \mathbf{P}\text{-a.s.}$$

2) In the first part we have showed (for an arbitrary admissible control $\upsilon\in\mathcal{V}$ and corresponding stopping time τ_m) finiteness of the expectation (3.17) and equality (3.18), i.e.

$$\mathbf{E}_{t,x}\int_t^{\tau_m^*} z_x(s,X_{s-}^*)\,\left(b^*(s-,X_{s-}^*)\right)'\,\mathrm{d}V_s = 0\,.$$

3) By the condition $\mathbf{H}_4$), for any $t\in[0,T]$, $x\in\mathcal{X}$,

$$\mathbf{E}_{t,x}\sup_{t\le s\le T}|z(s,X_s^*)| < +\infty\,,$$

hence the sequence $\{z(\tau_m^*, X_{\tau_m^*})\}_{m>0}$ is integrated (w.r.t. $\mathbf{P}$) and all integrals are uniformly bounded, that is, for any $m\in\mathbb{N}$,

$$\mathbf{E}_{t,x}\left|z\left(\tau_m^*, X_{\tau_m^*}^*\right)\right| \le \mathbf{E}_{t,x}\sup_{t\le s\le T}\left|z(s,X_s^*)\right| < +\infty\,.$$

4) by the definition of an admissible control ($m\in\mathbb{N}$):

$$\mathbf{E}_{t,x}\int_t^{\tau_m^*}\left[U_1(s,X_s^*,\upsilon_s^*)\right]_-\,\mathrm{d}s \le \mathbf{E}_{t,x}\int_t^{T}\left[U_1(s,X_s^*,\upsilon_s^*)\right]_-\,\mathrm{d}s < +\infty\,.$$

On the one hand, based on the points 3) and 4) we have, for any $m>0$,

$$\mathbf{E}_{t,x}\int_t^{\tau_m^*}\left|U_1(s,X_s^*,\upsilon_s^*)\right|\mathrm{d}s + \mathbf{E}_{t,x}\left|z\left(\tau_m^*, X_{\tau_m^*}^*\right)\right| < +\infty\,.$$

On the other hand, it follows from points 1) and 2), considering (3.27), that

$$\mathbf{E}_{t,x}\int_t^{\tau_m^*} U_1(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s + \mathbf{E}_{t,x}\, z\left(\tau_m^*, X_{\tau_m^*}^*\right) = z(t,x)\,. \tag{3.28}$$

Now we show that there exists a limit

$$\lim_{m\to\infty} \mathbf{E}_{t,x}\, z\left(\tau_m^*, X_{\tau_m^*}^*\right)\,.$$

From the point 3) and since the function $z(t,x)$ is continuous, the process X^* is càdlàg (i.e. there is no jump at maturity time T) and $\mathbf{P}$–$\lim_{m\to\infty}\tau_m^* = T$, it follows that

$$\mathbf{P}\text{–}\lim_{m\to\infty} z\left(\tau_m^*, X_{\tau_m^*}^*\right) = z(T,X_T^*) = U_2(X_T^*)\,.$$

Therefore, by the Dominated convergence theorem, we have

$$\lim_{m\to\infty}\mathbf{E}_{t,x}\, z\left(\tau_m^*, X_{\tau_m^*}^*\right) = \mathbf{E}_{t,x}\lim_{m\to\infty} z\left(\tau_m^*, X_{\tau_m^*}^*\right) = \mathbf{E}_{t,x}\, z(T,X_T^*) = \mathbf{E}_{t,x}\, U_2(X_T^*)\,.$$

To prove the following limit transition

$$\lim_{m\to\infty}\mathbf{E}_{t,x}\int_t^{\tau_m^*} U_1(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s = \mathbf{E}_{t,x}\int_t^T U_1(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s$$

we use twice Levi's monotone convergence theorem for

$$\int_t^{\tau_m^*}\left[U_1(s,X_s^*,\upsilon_s^*)\right]_-\mathrm{d}s \quad\text{and}\quad \int_t^{\tau_m^*}\left[U_1(s,X_s^*,\upsilon_s^*)\right]_+\mathrm{d}s$$

separately.

Thus, we obtain that the limit transition is valid for (3.28) :

$$\begin{aligned}
&\lim_{m\to\infty}\left(\mathbf{E}_{t,x}\int_t^{\tau_m^*} U_1(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s + \mathbf{E}_{t,x}\, z\left(\tau_m^*, X_{\tau_m^*}^*\right)\right)\\
&= \mathbf{E}_{t,x}\int_t^T\left[U_1(s,X_s^*,\upsilon_s^*)\right]_+\mathrm{d}s - \mathbf{E}_{t,x}\int_t^T\left[U_1(s,X_s^*,\upsilon_s^*)\right]_-\mathrm{d}s + \mathbf{E}_{t,x}\, z(T,X_T^*)\\
&= \mathbf{E}_{t,x}\int_t^T U_1(s,X_s^*,\upsilon_s^*)\,\mathrm{d}s + \mathbf{E}_{t,x}\, U_2(X_T^*) = J(t,x,\upsilon^*) = z(t,x).
\end{aligned} \tag{3.29}$$

In the first part of the proof we have obtained the inequality

$$\sup_{\upsilon\in\mathcal{V}} J(t,x,\upsilon) =: J^*(t,x) \le z(t,x)\,,$$

therefore, since there is only one supremum on the set $\mathcal{V}$, it follows that

$$J(t,x,\upsilon^*) = J^*(t,x) = z(t,x)\,,$$

and $z(t,x)$ is the unique solution of HJB's equation (3.10). Hence we proved the theorem. □

4 Optimal Investment and Consumption

Now we apply results obtained in section 3 to the problem (2.8). In this case the controlled process driven by (2.5) with state space $\mathcal{X} =]0,\infty[$, admissible strategy $\upsilon = (\theta_t, c_t)_{0\le t\le T} \in \mathcal{V}$ with values in $\mathcal{K} = \Theta \times \mathbb{R}_+$, where the set Θ defined by (2.6). So, for given utility functions $U_1(x,\kappa) = (x\,\kappa_2)^\gamma$, $U_2(x) = x^\gamma$, $0 < \gamma < 1$, optimization problem consists to find an admissible strategy $\upsilon^* \in \mathcal{V}$, such that

$$\upsilon^* = \arg\sup_{\upsilon\in\mathcal{V}} \left(\mathbf{E}_x \int_0^T U_1(X_s, \upsilon_s)\,\mathrm{d}s + \mathbf{E}_x\, U_2(X_T) \right) , \tag{4.1}$$

where $\mathbf{E}_x$ is conditional expectation with respect to $X_0 = x$.

Note that according to the definition 2.1, for any $\upsilon \in \mathcal{V}$, there exists unique strictly positive solution $X = (X_t)_{0\le t\le T}$ of the equation (2.5) which can be derived explicitly

$$X_t = x\, e^{r\,t + \int_0^t \theta_s^{'}\,\breve{\mu}\,\mathrm{d}s - \int_0^t c_s\,\mathrm{d}s}\,\mathcal{E}_t(\check{V})\,, \tag{4.2}$$

where

$$\check{V}_t = \int_0^t \theta_s^{'}\,\sigma\,\mathrm{d}W_s + \int_0^t \theta_{s-}^{'}\,\varsigma\,\mathrm{d}L_s$$

and Doléan exponential

$$\mathcal{E}_t(\check{V}) = \exp\left(\check{V}_t - \frac{1}{2}\int_0^t \mathrm{tr}\left\{\sigma^{'}\,\theta_s\,\theta_s^{'}\,\sigma\right\}\mathrm{d}s \right) \prod_{0\le s\le t} \left(1 + \theta_{s-}^{'}\,\varsigma\,\Delta L_s\right) e^{-\theta_{s-}^{'}\,\varsigma\,\Delta L_s}\,.$$

Note that $\inf_{0\le t\le T}\mathcal{E}_t(V) > 0$, which implies that, for any $t \in [0,T]$, X_t, $X_{t-} \in \mathcal{X} =]0,\infty[$. As the result (since utility functions $U_1, U_2 \ge 0$), the definition 2.1 gives non-empty set of admissible strategies $\mathcal{V}$ and satisfies the definition 3.1.

The main result of the article can be formulated as theorem.

Theorem 4.1. *The solution $\upsilon^* = (\theta_t^*, c_t^*)_{0\le t\le T}$ of optimization problem (4.1) is given by*

$$\theta_t^* = \theta^* = \arg\max_{\theta\in\Theta} F(\theta) \tag{4.3}$$

and

$$c_t^* = \begin{cases} \left[-\dfrac{1-\gamma}{\varrho} + \left(1 + \dfrac{1-\gamma}{\varrho}\right)\exp\left(\dfrac{\varrho}{1-\gamma}\,(T-t)\right)\right]^{-1} & \text{if} \quad \varrho \ne 0\,, \\ (T-t+1)^{-1} & \text{if} \quad \varrho = 0\,, \end{cases} \tag{4.4}$$

where

$$F(\theta) = \gamma\,\theta^{'}\breve{\mu} + \frac{\gamma\,(\gamma-1)}{2}\,tr\left\{\sigma^{'}\theta\,\theta^{'}\sigma\right\} + \sum_{i=1}^{n}\int_{\mathbb{R}_*} \left[(1+\theta^{'}\varsigma_i\,y)^\gamma - 1 - \gamma\,\theta^{'}\varsigma_i\,y\right] \Pi^i(\mathrm{d}y)\,,$$

$\varsigma_i = (\varsigma_{1i}, \dots, \varsigma_{ni})^{'}$ *is* i*-th column of matrix* ς*, with*

$$\varrho = F(\theta^*) + r\,\gamma\,. \tag{4.5}$$

The corresponding optimal wealth processl $(X_t^*)_{0\le t\le T}$ *is given*

$$X_t^* = x\, e^{(r+(\theta^*)^{'}\,\breve{\mu})\,t - \int_0^t c_s^*\,\mathrm{d}s}\,\mathcal{E}_t(V)\,, \tag{4.6}$$

where $V_t = (\theta^*)^{'}\,\sigma\,W_t + (\theta^*)^{'}\,\varsigma\,L_t$ *and*

$$\mathcal{E}_t(V) = \exp\left(V_t - \frac{1}{2}\,tr\left\{\sigma^{'}\,\theta^*\,(\theta^*)^{'}\,\sigma\right\}t\right) \prod_{0\le s\le t}\left(1 + (\theta^*)^{'}\,\varsigma\,\Delta L_s\right)e^{-(\theta^*)^{'}\,\varsigma\,\Delta L_s}\,. \tag{4.7}$$

The proof is given in section 6.

5 Monte Carlo simulations

We define one dimentional market model with integrable jumps on $[0,T]$

$$\begin{cases} B_t = e^{r\,t}\,, \\ S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\,W_t + L_t\right]\prod\limits_{s\le t}(1+\Delta L_s)\,e^{-\Delta L_s}, \end{cases}$$

and constants $r = 0.02\,,\ \mu = 0.04\,,\ S_0 = 100\,,\ T = 10\,,$ Wiener process $W_t\,,$ Compound Poisson process L_t defined by $L_t = \sum_{j=1}^{N_t} Y_j$, where N_t is Poisson process with constant intensity $\lambda = 9.5$ and sequence $\{Y_j\}_{j\ge1}$ of i.i.d. real valued random variables with $\mathbf{E}\,Y_1 = 0$ and density function

$$p_Y(y) = \frac{\exp\left(-\frac{y^2}{2\,\eta^2}\right)}{\int_{-1}^{\infty}\exp\left(-\frac{z^2}{2\,\eta^2}\right)\mathrm{d}z}\,\mathbf{1}_{\{y>-1\}}\,,$$

where $\eta^2 = 0.005$ is called jumps variance.

For such case, under the given $U_1(x,\kappa) = (x\,\kappa_2)^\gamma\,,\ \kappa = (\kappa_1\,,\kappa_2) \in [0,1]\times\mathbb{R}_+\,,$ $U_2(x) = x^\gamma$ utility functions, optimal self-financing portfolio with consumption

$$X_t^* = x\,\exp\left(\left[r + (\mu - r)\,\theta^* - \frac{(\sigma\,\theta^*)^2}{2}\right]t - \int_0^t c_s^*\,\mathrm{d}s + \theta^*[\sigma\,W_t + L_t]\right) \prod_{0\le s\le t}\left(1+\theta^*\Delta L_s\right)e^{-\theta^*\Delta L_s}$$

determined by optimal strategy $(\theta^*, c_t^*)\,,$ where the fractional investment strategy θ^* defined as (see the equation (6.10))

$$F(\theta^*) = \max_{\theta\in[0,1]} F(\theta)\,,$$

$$F(\theta) = \gamma\,\theta\,(\mu - r) + \frac{\gamma\,(\gamma-1)}{2}\,\sigma^2\,\theta^2 + \int\limits_{-1}^{\infty} [(1+\theta\,y)^{\gamma} - 1 - \gamma\,\theta\,y]\,\Pi(\mathrm{d}y)\,,$$

with the Lévy measure

$$\Pi(\mathrm{d}y) = \lambda\,p_Y(y)\,\mathrm{d}y\,.$$

and optimal consumption rate c_t^* given by (4.4).

Note that in our case the solution of HJB's equation $z(t,x) = A(t)\,x^{\gamma}$ can be defined as supremum of conditional expectation over all admissible strategies

$$z(t,x) = \sup_{\upsilon\in\mathcal{V}} J(t,x,\upsilon)\,,$$

and

$$J(t,x,\upsilon) = \mathbf{E}_{t,x}\left(\int\limits_t^T U_1(X_s,\upsilon_s)\,\mathrm{d}s + U_2(X_T)\right)\,,$$

where υ_t is an admissible strategy and X_t is corresponding controlled process.

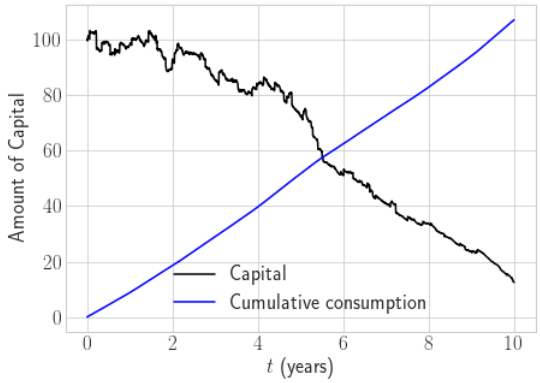

Fig. 1 The Optimal capital

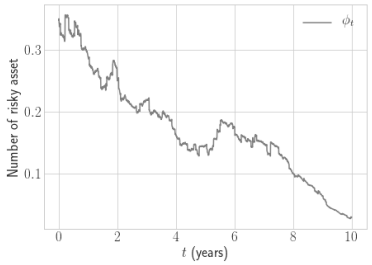

Fig. 2 Number of risky asset

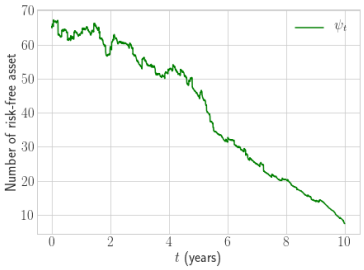

Fig. 3 Number of risk-free asset

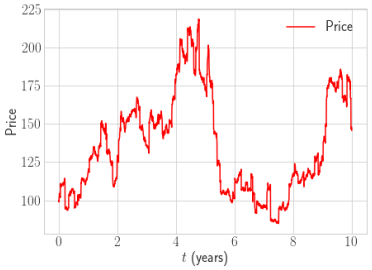

Fig. 4 Price

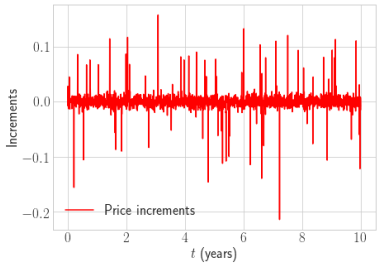

Fig. 5 Price increments

Remark 5.1. *Note that the numerical simulations confirm the natural behaviour for the optimal strategies, i.e. the more consumption, the less capital. Therefore, as it is seen in Figure 1, there is a point of equilibrium between consumption and wealth. The graphs in Figures 4 and 5 show the dynamics of the Lévy markets.*

6 Proof of Theorem 4.1

Proof. First we solve HJB's equation, from which will be found strategy υ^* and can be defined corresponding capital X^*. Then we prove that the conditions of Verification Theorem 3.6 hold which implies that processes X^* and υ^* are optimal. To this end, first we show, that the function $z(t,x)$ given by

$$z(t,x) = A(t)\, x^{\gamma} \tag{6.1}$$

is the solution of corresponding Hamilton–Jacobi–Bellman equation (3.10), where

$$A(t) = \begin{cases} \left[-\dfrac{1-\gamma}{\varrho} + \left(1 + \dfrac{1-\gamma}{\varrho}\right) \exp\left(\dfrac{\varrho}{1-\gamma}\, (T-t) \right) \right]^{1-\gamma} & \text{if} \quad \varrho \neq 0\,, \\ (T-t+1)^{1-\gamma} & \text{if} \quad \varrho = 0\,, \end{cases} \tag{6.2}$$

where ϱ defined in (4.5).
Indeed, note that the Hamilton–Jacobi–Bellman equation (3.10) with respect to the function of two variables $z(t,x)$ in this case has the form

$$\begin{cases} z_t(t,x) + \sup\limits_{\kappa \in \Theta \times \mathbb{R}_*} H_0(t,x,z,\kappa) = 0\,, & t \in [0,T]\,, \\ z(T,x) = x^{\gamma} & x > 0\,, \end{cases} \tag{6.3}$$

where $\kappa = (\theta, c)$,

$$H_0(t,x,z,\kappa) = x\,(r + \theta^{'}\,\check{\mu} - c)\, z_x(t,x) + \frac{x^2\, z_{xx}(t,x)}{2}\, \mathrm{tr}\left\{ \sigma^{'}\,\theta\,\theta^{'}\,\sigma \right\} + (c\,x)^{\gamma} + \overline{z}(t,x,\theta)\,, \tag{6.4}$$

$$\overline{z}(t,x,\theta)=\sum_{i=1}^{n}\int_{\mathbb{R}_*}\left[z(t,x+x\,\theta^{'}\varsigma_i\,y)-z(t,x)-z_x(t,x)\,x\,\theta^{'}\varsigma_i\,y\right]\Pi^i(\mathrm{d}y)\,,$$

$\varsigma_i=(\varsigma_{1i},\ldots,\varsigma_{ni})^{'}$ is i-th column of the matrix ς; and it is to compute, $\theta^{'}\varsigma_i\in[0,1]$.

Note that $\overline{z}$ is finite (see Lemma A.1 in Appendix) and depends only on θ, but in the verification theorem 3.6 we gave the proof in case of dependence on $\kappa=(\theta,c)\in\mathcal{K}$. The supremum in (6.3) can be rewritten as

$$\begin{aligned}\sup_{\kappa\in\Theta\times\mathbb{R}_*}H_0(t,x,z,\kappa)&=r\,x\,z_x(t,x)\\&+\max_{\theta\in\Theta}\left[x\,z_x(t,x)\,\theta^{'}\breve{\mu}+\frac{x^2\,z_{xx}(t,x)}{2}\,\mathrm{tr}\left\{\sigma^{'}\,\theta\,\theta^{'}\,\sigma\right\}+\overline{z}(t,x,\theta)\right]\\&+\max_{c\in\mathbb{R}_+}\left[(x\,c)^{\gamma}-x\,z_x(t,x)\,c\right],\end{aligned}$$

where the second maximum occurs (for fixed t and x) at the point

$$c^*=c^*(t,x)=\left(\frac{\gamma}{z_x(t,x)}\right)^{\frac{1}{1-\gamma}}\frac{1}{x}\,,\tag{6.5}$$

and, as a result, HJB's equation (6.3) can be presented as following

$$\begin{cases}z_t(t,x)+r\,x\,z_x(t,x)+\max_{\theta\in\Theta}\breve{F}(\theta)+(1-\gamma)\left(\frac{z_x(t,x)}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}=0\,,\\[2ex] z(T,x)=x^{\gamma}\end{cases}\tag{6.6}$$

where

$$\breve{F}(\theta)=x\,z_x(t,x)\,\theta^{'}\breve{\mu}+\frac{x^2\,z_{xx}(t,x)}{2}\,\mathrm{tr}\left\{\sigma^{'}\theta\,\theta^{'}\sigma\right\}+\overline{z}(t,x,\theta)\,.$$

Assume that the function $z(t,x)$ can be derived as

$$z(t,x)=A(t)\,x^{\gamma}\,,\tag{6.7}$$

where $A(t)$ is a continuously differentiable function. From (6.6) it follows that boundary condition on $A(t)$ should be $A(T)=1$. And, by definition, $\overline{z}$ can be computed as

$$\overline{z}(t,x,\theta)=A(t)\,x^{\gamma}\sum_{i=1}^{n}\int_{\mathbb{R}_*}\left[(1+\theta^{'}\varsigma_i\,y)^{\gamma}-1-\gamma\,\theta^{'}\varsigma_i\,y\right]\Pi^i(\mathrm{d}y)\,.\tag{6.8}$$

It is proved in Lemma A.3 in Appendix that the integral is finite.

Note that under the assumption (6.7) on $z(t,x)$, and given $c^*(t,x)$ in (6.5) we can rewrite equation (6.6) as following

$$\begin{cases}\dot{A}(t)\,x^{\gamma}+r\,\gamma\,A(t)\,x^{\gamma}+A(t)\,x^{\gamma}\max_{\theta\in\Theta}F(\theta)+(1-\gamma)A^{1+\frac{1}{\gamma-1}}(t)\,x^{\gamma}=0\,,\\[2ex] A(T)=1\end{cases}\tag{6.9}$$

where $\dot{A}(t) = \frac{dA(t)}{dt}$ and

$$F(\theta) = \gamma\,\theta^{'}\,\breve{\mu} + \frac{\gamma\,(\gamma-1)}{2}\,\mathrm{tr}\left\{\sigma^{'}\,\theta\,\theta^{'}\,\sigma\right\} + \sum_{i=1}^{n}\int\limits_{\mathbb{R}_*}\left[(1+\theta^{'}\,\varsigma_i\,y)^{\gamma} - 1 - \gamma\,\theta^{'}\,\varsigma_i\,y\right]\Pi^{i}(\mathrm{d}y)$$

is continuous function with finite integral term, for any $\theta \in \Theta$, and Θ is bounded set, thus there exists θ^*, such that

$$F(\theta^*) = \max_{\theta\in\Theta} F(\theta)\,. \tag{6.10}$$

Note that the optimal c^* defined in (6.5) under assumption (6.7) (at fixed time moment t) is given

$$c^* = c^*_t = A^{-\frac{1}{1-\gamma}}(t)\,. \tag{6.11}$$

Therefore HJB equation (6.9) can be rewritten as

$$\begin{cases} \dot{A}(t) + \varrho\,A(t) + (1-\gamma)\,A^{1+\frac{1}{\gamma-1}}(t) = 0\,, \quad t\in[0,T]\,, \\ A(T) = 1\,, \quad x>0\,, \end{cases}$$

where the constant ϱ is given

$$\varrho = F(\theta^*) + r\,\gamma\,. \tag{6.12}$$

Now we have got the equation which can be easily solved w.r.t. unknown function $A(t)$ with boundary condition $A(T)=1$, hence

$$A(t) = \begin{cases} \left[-\frac{1-\gamma}{\varrho} + \left(1+\frac{1-\gamma}{\varrho}\right)\exp\left(\frac{\varrho}{1-\gamma}\,(T-t)\right)\right]^{1-\gamma} & \text{if} \quad \varrho\neq 0\,, \\ (T-t+1)^{1-\gamma} & \text{if} \quad \varrho = 0\,, \end{cases} \tag{6.13}$$

Let's check the conditions $\mathbf{H}_1$) – $\mathbf{H}_4$).

$\mathbf{H}_1$), $\mathbf{H}_2$) are fulfilled, since we found the function $z(t,x)$ such that

$$\inf_{0\le s\le t}\inf_{x>0} z(s,x) = 0\,,$$

for any fixed $t\in[0,T]$,

$$\sup_{x\in]0,\infty[}\frac{z(t,x)}{1+x} = A(t)\left(\frac{\gamma}{1-\gamma}\right)^{\gamma}(1-\gamma) < \infty\,,$$

and for $\upsilon^*(t,x) = (\theta^*, c^*_t)$ (see (6.10) and (6.11)) we have the equality

$$H(t,x,z) = H_0(t,x,z,\upsilon^*(t,x)) = \varrho\,z(t,x) + (1-\gamma)\,A^{\frac{1}{\gamma-1}}(t)\,z(t,x)\,,$$

where $\varrho = F(\theta^*) + r\,\gamma$.

To prove $\mathbf{H}_3$) we need to solve the equation (2.5) in case of optimal control $\upsilon^*_t = \upsilon^*(t, X^*_t)$, i.e.

$$\mathrm{d}X^*_t = X^*_t\,(r + (\theta^*)^{'}\,\breve{\mu} - c^*_t)\,\mathrm{d}t + X^*_t\,(\theta^*)^{'}\,\sigma\,\mathrm{d}W_t + X^*_{t-}\,(\theta^*)^{'}\,\varsigma\,\mathrm{d}L_t\,,$$

$X_0 = x > 0$. According to Doléans exponential theorem, the solution of equation will be

$$X_t^* = x\, e^{(r+(\theta^*)'\,\tilde{\mu})\,t - \int_0^t c_s^*\,\mathrm{d}s}\,\mathcal{E}_t(V)\,, \tag{6.14}$$

where $V_t = (\theta^*)'\,\sigma\,W_t + (\theta^*)'\,\varsigma\,L_t$, and

$$\mathcal{E}_t(V) = \exp\left(V_t - \frac{1}{2}\,\mathrm{tr}\{\sigma'\,\theta^*\,(\theta^*)'\,\sigma\}\,t\right) \prod_{0\le s\le t}\left(1 + (\theta^*)'\,\varsigma\,\Delta L_s\right) e^{-(\theta^*)'\,\varsigma\,\Delta L_s}\,. \tag{6.15}$$

Note that $X_t^* > 0$ (i.e. for any $x > 0\,, t \in [0,T]\,, X_t^* \in \mathcal{X}$), since $(\theta^*)'\,\varsigma\,\Delta L_t > -1$ where $\theta^* \in [0,1]^n$ such that $\sum_{i=1}^n \theta_i^* \le 1$, and, for any $i, j\,,\ 0 \le \varsigma_{ij} \le 1$, by the definitions of L_t and constant θ^*. Also, for any $t \in [0,T]\,, X_t^*\,, X_{t-}^* \in \mathcal{X}$, since $\inf_{0\le t\le T} X_t^* > 0$. I.e. the strategy $(\theta^*, c_t^*)_{0\le t\le T}$ belongs to $\mathcal{V}$.

The last condition $\mathbf{H}_4$) consists to verify, for any $x \in \mathcal{X}\,, t \in [0,T]\,,$

$$\mathbf{E}_{t,x} \sup_{t\le s\le T} |z(s, X_s^*)| < \infty\,, \tag{6.16}$$

for

$$z(t, X_t^*) = A(t)\,(X_t^*)^\gamma\,,$$

where $0 < \gamma < 1$.

Note that, for any increasing continuous function $f(x)$, it is true

$$\sup_{x\in\mathcal{D}} f(x) = f(\sup_{x\in\mathcal{D}} x)\,, \tag{6.17}$$

where $\mathcal{D}$ is an arbitrary measurable set.

Obviously, that

$$z(t, X_t^*) \le C\,\left(\mathcal{E}_t(V)\right)^\gamma\,,$$

where

$$C = x \max_{0\le s\le T} A(s)\, e^{\gamma\,(r+(\theta^*)'\,\tilde{\mu})\,t - \gamma\,\int_0^t c_s^*\,\mathrm{d}s} < \infty\,, \tag{6.18}$$

and $\mathcal{E}_t(V)$ is Doléan exponential given by (4.7). Thus to prove (6.16) it is sufficient to show

$$\mathbf{E}_{t,x} \sup_{t\le s\le T} \left(\mathcal{E}_t(V)\right)^\gamma = \mathbf{E}_{t,x}\left(\sup_{t\le s\le T} \mathcal{E}_t(V)\right)^\gamma < \infty\,.$$

We will use Hölder's inequality,

$$\mathbf{E}_{t,x}\left(\sup_{t\le s\le T} \mathcal{E}_t(V)\right)^\gamma \le \left[\mathbf{E}_{t,x}\left(\sup_{t\le s\le T} \mathcal{E}_t(V)\right)^2\right]^{\frac{\gamma}{2}} = \left[\mathbf{E}_{t,x} \sup_{t\le s\le T} \left(\mathcal{E}_t(V)\right)^2\right]^{\frac{\gamma}{2}}\,.$$

Note that $\mathcal{E}_t(V)$ is square integrable martingale (see Lemma A.4 in Appendix) therefore we can apply martingale inequality (A.11) of Lemma A.6 in Appendix,

$$\mathbf{E}_{t,x}\sup_{t\le s\le T}\left(\mathcal{E}_t(V)\right)^2\le 2\,\mathbf{E}_{t,x}\left(\mathcal{E}_T(V)\right)^2<\infty\,.$$

Therefore we finally have proved the following chains inequalities

$$\begin{aligned}\mathbf{E}_{t,x}\sup_{t\le s\le T}|z(s,X^*_s)|&\le C\,\mathbf{E}_{t,x}\left(\sup_{t\le s\le T}\mathcal{E}_t(V)\right)^{\gamma}\\&\le 2^{\gamma/2}\,C\left[\mathbf{E}_{t,x}\left(\mathcal{E}_T(V)\right)^2\right]^{\frac{\gamma}{2}}<\infty\,,\end{aligned}$$

with the constant C given by (6.18).

Thus all conditions of Verification theorem 3.6 are satisfied and optimal fractional strategy $\upsilon^*_t=(\theta^*,c^*_t)$ is given by (4.3) and (4.4). □

Acknowledgments

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A Appendix

Lemma A.1. *Let $\mathcal{X}\subset\mathbb{R}$ be an open convex set, $T<\infty$. For any function $g\in\mathcal{C}^{1,2}([0,T]\times\mathcal{X},\mathbb{R})$, such that, for any fixed $t\in[0,T]$,*

$$\sup_{x\in\mathcal{X}}\frac{|g(t,x)|}{1+|x|}<\infty\,,\tag{A.1}$$

for any Lévy measure $\Pi(\cdot)$ on $\mathbb{R}_*=\mathbb{R}\setminus\{0\}$, with

$$\Pi\left(y^2\right)<\infty\,,\tag{A.2}$$

and, for any constant $\varpi\in\mathbb{R}$, we have, for fixed $t\in[0,T]$, $x\in\mathcal{X}$,

$$\int\limits_{\mathbb{R}_*}|\Upsilon(t,x,\varpi\,y)|\,\Pi(\mathrm{d}y)<\infty\,,\tag{A.3}$$

where

$$\Upsilon(t,x,v)=\left[g(t,x+v)-g(t,x)-g_x(t,x)\,v\right]\mathbf{1}_{\{x+v\in\mathcal{X}\}}$$

and $v\in\mathbb{R}_*$.

Proof. We consider two cases, $\varpi = 0$ and $\varpi \neq 0$. In the first case, obviously $\Upsilon(t,x,0) = 0$ and (A.3) is true.

Let $\varpi \neq 0$. Then by definition of open convex set, if $x \in \mathcal{X}$, then $\exists \varepsilon_0 > 0$ such that $\forall |v| < \varepsilon_0$, $x + v \in \mathcal{X}$. We set $\varepsilon = \frac{\varepsilon_0}{|\varpi|}$. If $|y| < \varepsilon$:

$$|\varpi\, y| = |\varpi|\, |y| < \varepsilon_0\,,$$

then, by definition of open convex set, $x + \varpi\, y \in \mathcal{X}$, for any $|y| < \varepsilon$. Thus, for given ϖ, x, we fixed $\varepsilon = \varepsilon(x,\varpi) > 0$ and present

$$\int\limits_{\mathbb{R}_*} |\Upsilon(t,x,\varpi\, y)|\, \Pi(\mathrm{d}y) = \int\limits_{\{|y|<\varepsilon\}} |\Upsilon(t,x,\varpi\, y)|\, \Pi(\mathrm{d}y) + \int\limits_{\{|y|\geq\varepsilon\}} |\Upsilon(t,x,\varpi\, y)|\, \Pi(\mathrm{d}y)\,.$$

Note that for $|y| < \varepsilon$

$$\begin{aligned} |\Upsilon(t,x,\varpi\, y)| &= \left| \int\limits_x^{x+\varpi\, y} \left[\int\limits_x^z g_{xx}(t,u)\, \mathrm{d}u \right] \mathrm{d}z \right| \\ &\leq \varpi^2\, \frac{y^2}{2} \max_{x-\varepsilon_0 \leq u \leq x+\varepsilon_0} |g_{xx}(t,u)| < \infty\,. \end{aligned}$$

Then, for some constant $0 < C_1 < \infty$,

$$\int\limits_{\{|y|<\varepsilon\}} |\Upsilon(t,x,\varpi\, y)|\, \Pi(\mathrm{d}y) \leq C_1 \int\limits_{\{|y|<\varepsilon\}} y^2\, \Pi(\mathrm{d}y)\,.$$

For the second integral we will use the fact (A.1), means for some constant $0 < C_2 < \infty$,

$$\begin{aligned} \int\limits_{\{|y|\geq\varepsilon\}} |\Upsilon(t,x,\varpi\, y)|\, \Pi(\mathrm{d}y) \leq C_2 &\int\limits_{\{|y|\geq\varepsilon\}} (1+|y|)\, \Pi(\mathrm{d}y) + |g(t,x)|\, \Pi(\{|y| \geq \varepsilon\}) \\ &+ |g_x(t,x)\, \varpi| \int\limits_{\{|y|\geq\varepsilon\}} |y|\, \Pi(\mathrm{d}y) < \infty\,, \end{aligned}$$

since, by Cauchy–Schwarz inequality,

$$\int\limits_{\{|y|\geq\varepsilon\}} |y|\, \Pi(dy) \leq \sqrt{\int\limits_{\{|y|\geq\varepsilon\}} y^2\, \Pi(\mathrm{d}y)}\, \sqrt{\Pi(\{|y| \geq \varepsilon\})} < \infty\,.$$

Thus for any t, x, ϖ (A.3) is hold. □

Remark A.2. *If n-dimensional Lévy process $L_t = (L_t^1, \ldots, L_t^n)$ consists of mutually independent processes then, for any vector $Y \in \mathbb{R}^n$, $x \in \mathbb{R}$ and continuously differentiable*

function $g\in\mathcal{C}^{1,2}([0,T]\times\mathcal{X},\mathbb{R})$, *we have the following reduction,*

$$\begin{aligned}
&\mathbf{E}\sum_{0\le t\le T}\Big[g(t,x+Y^{'}\Delta L_t)-g(t,x)-g_x(t,x)\,Y^{'}\Delta L_t\Big]\,\mathbf{1}_{\{x+Y^{'}\Delta L_t\in\mathcal{X}\}}\\
&=\mathbf{E}\sum_{0\le t\le T}\Big[g(t,x+Y^{'}\Delta L_t)-g(t,x)-g_x(t,x)\,Y^{'}\Delta L_t\Big]\,\mathbf{1}_{\{|\Delta L_t|\neq0\}}\,\mathbf{1}_{\{x+Y^{'}\Delta L_t\in\mathcal{X}\}}\\
&=\mathbf{E}\sum_{i=1}^{n}\sum_{0\le t\le T}\Big[g(t,x+Y_i\,\Delta L_t^i)-g(t,x)-g_x(t,x)\,Y_i\,\Delta L_t^i\Big]\,\mathbf{1}_{\{|\Delta L_t^i|\neq0\}}\,\mathbf{1}_{\{x+Y^{'}\Delta L_t\in\mathcal{X}\}}\\
&=\sum_{i=1}^{n}\int\limits_{[0,T]}\int\limits_{\mathbb{R}_*}[g(t,x+Y_i\,y)-g(t,x)-g_x(t,x)\,Y_i\,y]\,\mathbf{1}_{\{x+Y^{'}\Delta L_t\in\mathcal{X}\}}\,\Pi^i(\mathrm{d}y)\,\mathrm{d}t\,,
\end{aligned}$$

where $\Pi^i(\cdot)$ *is Lévy measure of* $(L_t^i)_{0\le t\le T}$ *on* $\mathbb{R}_*=\mathbb{R}\setminus\{0\}$, $i=1,...,n$, $y\in\mathbb{R}$ *and* $|\cdot|$ *is Euclidean norm. Since elements of the vector* L_t *are independent whence follows, for any* $i\neq j$,

$$\mathbf{P}\{\Delta L_t^i\,\Delta L_t^j=0\,,\,t\in[0,T]\}=1\,,$$

$$\mathbf{1}_{\{|\Delta L_t|\neq0\}}=\sum_{i=1}^{n}\mathbf{1}_{\{|\Delta L_t^i|\neq0\}}\,,$$

where $\mathbf{1}_{\{\Gamma\}}$ *is an indicator function of the event* Γ.

Lemma A.3. *Let Lévy measure* Π *be such that*

$$\Pi(y^2)<\infty\,,\quad \Pi(]-\infty,-1])=0\,. \tag{A.4}$$

Then, for any fixed $\theta\in[0,1]$ *and* $\gamma\in]0,1[$,

$$\int\limits_{\mathbb{R}_*}((1+\theta\,y)^{\gamma}-1-\gamma\,\theta\,y)\,\Pi(\mathrm{d}y)<\infty\,,$$

where $\mathbb{R}_*=\mathbb{R}\setminus\{0\}$.

Proof. Let's define

$$\check{F}(y)=(1+\theta\,y)^{\gamma}-1-\gamma\,\theta\,y\,.$$

We have, for any fixed $\delta\in]0,1[$,

$$\begin{aligned}
\int\limits_{\mathbb{R}_*}\check{F}(y)\,\Pi(\mathrm{d}y)=&\int\limits_{[-1,-1+\delta[}\check{F}(y)\,\Pi(\mathrm{d}y)+\int\limits_{[-1+\delta,1+\delta]\setminus\{0\}}\check{F}(y)\,\Pi(\mathrm{d}y)\\
&+\int\limits_{[1+\delta,+\infty[}\check{F}(y)\,\Pi(\mathrm{d}y)\,,
\end{aligned}$$

Let's estimate the first integral

$$\begin{aligned}
\left|\int\limits_{[-1,-1+\delta[}\check{F}(y)\,\Pi(\mathrm{d}y)\right|&\le\int\limits_{[-1,-1+\delta[}\big|1-(1+\theta\,y)^{\gamma}-\gamma\,\theta\,y\big|\,\Pi(\mathrm{d}y)\\
&\le\Pi([-1,-1+\delta[)+\gamma\,\theta\int\limits_{[-1,-1+\delta[}|y|\,\Pi(\mathrm{d}y)<\infty\,.
\end{aligned}$$

To estimate the second integral, we will use Taylor series to expand $\check{F}$ in a neighborhood of zero, $y \in [-1+\delta, 1+\delta]$,

$$\check{F}(y) = 1 + \check{F}^{'}(0)\, y + \frac{\check{F}^{''}(\xi)}{2}\, y^2\,,$$

where $\xi \in [0, y]$ is some fixed point,

$$\check{F}^{'}(y) = \frac{\gamma\,\theta}{(1+\theta\,y)^{1-\gamma}} - \gamma\,\theta\,, \quad \check{F}^{''}(y) = \frac{\gamma\,(\gamma-1)\,\theta^2}{(1+\theta\,y)^{2-\gamma}}\,.$$

Note that (since $\delta \in]0,1[$)

$$\max_{y\in[-1+\delta,1+\delta]} |\check{F}^{''}(y)| = \frac{\gamma\,(\gamma-1)\,\theta^2}{(1+\theta\,(-1+\delta))^{2-\gamma}} < \infty\,.$$

Thus we can present (given lemma's condition (A.4)), for some constant C,

$$\left| \int\limits_{[-1+\delta,1+\delta]\setminus\{0\}} \check{F}(y)\, \Pi(\mathrm{d}y) \right| \le \Pi\left([-1+\delta,1+\delta]\setminus\{0\}\right) + C\,\Pi(y^2) < \infty\,.$$

Estimation of the third integral will be given, for any $y \in]1+\delta, \infty[$, by the fact

$$(1+y)^{\gamma} \le 1 + y^{\gamma}\,, \quad y^{\gamma} \le y^2\,.$$

Therefore (given lemma's condition (A.4))

$$\left| \int\limits_{[1+\delta,\infty[} \check{F}(y)\, \Pi(\mathrm{d}y) \right| \le \int\limits_{[1+\delta,\infty]} y^{\gamma}\, \Pi(\mathrm{d}y) \le \int\limits_{[1+\delta,\infty]} y^2\, \Pi(\mathrm{d}y) < \infty\,.$$

□

Lemma A.4. *Let W_t and $L_t = (L_t^1, \ldots, L_t^n)$ be independent n-dimensional Wiener and pure jumps Lévy processes, where $(L_t^1)_{0\le t\le T}, \ldots, (L_t^n)_{0\le t\le T}$, are mutually independent and*

$$L_t^i = \int\limits_0^t \int\limits_{\mathbb{R}_*} y\, \left(\nu^i - \tilde{\nu}^{\,i}\right)(\omega;\, \mathrm{d}y\,, \mathrm{d}s)\,,$$

$\tilde{\nu}^{\,i}(\mathrm{d}y\,, \mathrm{d}s) = \Pi^i(\mathrm{d}y)\,\mathrm{d}s$, $i = 1, \ldots, n$, and such that the corresponding Lévy measures Π^i (defined on $\mathbb{R}_* = \mathbb{R}\setminus\{0\}$) have properties

$$\Pi^i(y^2) < \infty\,, \quad \Pi^i(]-\infty, -1]) = 0\,. \tag{A.5}$$

Then, for any vector $\theta \in [0,1]^n$ such that $\sum_{i=1}^n \theta_i \le 1$ and the process $V_t = \sigma\, W_t + \varsigma\, L_t$, where matrices $\sigma \in \mathbb{R}^{n\times n}$ is positive definite and $\varsigma \in [0,1]^{n\times n}$, Doléan exponential

$$\mathcal{E}_t(V) = \exp\left(\theta^{'}\, V_t - \frac{1}{2}\, tr\{\sigma^{'}\,\theta\,\theta^{'}\,\sigma\}\, t\right) \prod_{0\le s\le t} \left(1 + \theta^{'}\,\varsigma\,\Delta L_s\right) e^{-\theta^{'}\,\varsigma\,\Delta L_s}\,, \tag{A.6}$$

is square interable martingale, i.e. $\mathbf{E}\,[\mathcal{E}_t(V)]^2 < \infty$.

Proof. first notice that V_t is square integrable martingale and applying the Ito formula to Doléan exponential we have

$$\mathrm{d}\mathcal{E}_t(V) = \mathcal{E}_t(V)\,\theta^{'}\,\mathrm{d}V_t\,. \tag{A.7}$$

Therefore, to prove the lemma, it is sufficient to show that $\mathbf{E}\,[\mathcal{E}_t(V)]^2 < \infty$.
Because of W and L are independent, the fact that

$$\exp\left(\theta^{'}\,\sigma\,W_t - \frac{1}{2}\,\mathrm{tr}\{\sigma^{'}\,\theta\,\theta^{'}\,\sigma\}\,t\right)$$

is square integrable martingale and processes $L_t^1,\ldots,L_t^n$ don't have jumps simultaneously, it follows that we need to show only a process

$$e^{\theta^{'}\varsigma\,L_t}\prod_{0\le s\le t}\left(1+\theta^{'}\,\varsigma\,\Delta L_s\right)e^{-\theta^{'}\,\varsigma\,\Delta L_s}$$
$$= \exp\left(\sum_{j=1}^{n} a_j\,L_t^j + \sum_{j=1}^{n}\sum_{0\le s\le t}\left[\ln(1+a_j\,\Delta L_s^j) - a_j\,\Delta L_s^j\right]\right)$$

is square integrated. Where $a_j = \sum_{i=1}^{n}\theta_i\,\varsigma_{ij}$. Note, that $0 \le a_j \le 1$. To do this it is enough to show, for any $j = 1,\ldots,n$,

$$\mathbf{E}\exp\left(2\left[a_j\,L_t^j + \sum_{0\le s\le t}\left[\ln(1+a_j\,\Delta L_s^j) - a_j\,\Delta L_s^j\right]\right]\right) < \infty\,.$$

Let's define a process Z_t^j (for simplicity, the index j will be omitted below)

$$Z_t^j = Z_t = a_j\,L_t^j + \sum_{0\le s\le t}\left[\ln(1+a_j\,\Delta L_s^j) - a_j\,\Delta L_s^j\right].$$

Considering the definition of L_t^j and equality

$$\sum_{0\le s\le t}\left[\ln(1+a_j\,\Delta L_s^j) - a_j\,\Delta L_s^j\right] = \int_0^t\int_{\mathbb{R}_*}\left[\ln(1+a_j\,y) - a_j\,y\right]\nu^j(\omega;\,\mathrm{d}y\,,\mathrm{d}t)\,,$$

we can see that a process Z_t can be derived as following

$$Z_t = t\,\Pi^j\left(h\left(\ln(1+a_j\,y)\right) - a_j\,y\right) + h\left(\ln(1+a_j\,y)\right) * (\nu^j - \tilde{\nu}^{\,j})_t + \overline{h}\left(\ln(1+a_j\,y)\right) * \nu_t^j\,,$$

where we used standart notations

$$h(x) = \mathbf{1}_{\{|x|\le 1\}}\,,\quad \overline{h}(x) = \mathbf{1}_{\{|x|>1\}}\,,$$

$$h\left(\ln(1+a_j\,y)\right) * (\nu^j - \tilde{\nu}^{\,j})_t = \int_0^t\int_{\mathbb{R}_*}\ln(1+a_j\,y)\,\mathbf{1}_{\left\{|\ln(1+a_j\,y)|\le 1\right\}}(\nu^j - \tilde{\nu}^{\,j})(\omega;\,\mathrm{d}y\,,\mathrm{d}t)\,,$$

$$\overline{h}\left(\ln(1+a_j\,y)\right)*(\nu^j-\tilde{\nu}^{\,j})_t=\int\limits_0^t\int\limits_{\mathbb{R}_*}\ln(1+a_j\,y)\,\mathbf{1}_{\{|\ln(1+a_j\,y)|>1\}}(\nu^j-\tilde{\nu}^{\,j})(\omega;\,\mathrm{d}y\,,\mathrm{d}t)\,.$$

Therefore Z_t is Lévy process and the cumulant-generating function (here the index j is also omitted) $H:q\to\ln\mathbf{E}\left(e^{qZ_1}\right)$ of the random variable Z_1 defined as

$$H(q)=q\,\Pi^j\left(h(\ln(1+a_j\,y))-a_j\,y\right)+\Pi^j\left(e^{q\ln(1+a_j y)}-1-q\,h\left(\ln(1+a_j\,y)\right)\right)\,,$$

i.e., for any $j\,,q\,,\mathbf{E}\,e^{q\,Z_t}=e^{t\,H(q)}$ (if $H(q)$ is well defined). Note that using Taylor series near zero and inequality $a_j\,y\le a_j\,y^2$, when $|y|>1$, one can be shown that

$$\Pi^j\left(h(\ln(1+a_j\,y))-a_j\,y\right)<\infty\,. \tag{A.8}$$

Then, for $q=2$ (to prove square integrability of Z_1 which implies that Z_t is square integrable process), if we define a function

$$G(y)=e^{2\ln(1+a_j y)}-1-2\,h(\ln(1+a_j\,y))\,,$$

then

$$\begin{aligned}\Pi^j(G(y))&=\Pi^j\left(G(y)\,\mathbf{1}_{\{|\ln(1+a_j\,y)|\le1\}}\right)\\&\quad+\Pi^j\left(G(y)\,\mathbf{1}_{\{|\ln(1+a_j\,y)|>1\}}\right)<\infty\,,\end{aligned} \tag{A.9}$$

since, applying Taylor series to $G(y)$, for some constant $C>0$,

$$\Pi^j\left(G(y)\,\mathbf{1}_{\{|\ln(1+a_j\,y)|\le1\}}\right)\le C\,\Pi^j(y^2)<\infty\,,$$

and

$$\begin{aligned}\Pi^j\left(G(y)\,\mathbf{1}_{\{|\ln(1+a_j\,y)|>1\}}\right)&=\Pi^j\left(e^{2\ln(1+a_j y)}\,\mathbf{1}_{\{y>\varepsilon_1\}}\right)+\Pi^j\left(e^{2\ln(1+a_j y)}\,\mathbf{1}_{\{y<-\varepsilon_2\}}\right)\\&\quad-\Pi^j\left(\{y\in\mathbb{R}_*:|\ln(1+a_j y)|>1\}\right)\,,\end{aligned}$$

where $\varepsilon_1>0$ and $\varepsilon_2>0$ are such that, for any $y\in]-\varepsilon_2,\varepsilon_1[:\,|\ln(1+a_j y)|>1$. Here, using inequality $(1+a_j\,y)^2\le2+2\,y^2$, we can estimate

$$\Pi^j\left(e^{2\ln(1+a_j y)}\,\mathbf{1}_{\{y>\varepsilon_1\}}\right)=\Pi^j\left((1+a_j\,y)^2\,\mathbf{1}_{\{y>\varepsilon_1\}}\right)<\infty\,,$$

in case, when $\varepsilon_2<1$, considering lemma's condition $\Pi^j(\,]-\infty,-1])=0$, we can see, that $\ln(1+a_j\,y)<0$, for $-1<y<-\varepsilon_2$. Which implies $0<e^{2\ln(1+a_j\,y)}<1$, therefore

$$\Pi^j\left(e^{2\ln(1+a_j y)}\,\mathbf{1}_{\{y<-\varepsilon_2\}}\right)\le\Pi^j\left(\{-1<y<-\varepsilon_2\}\right)<\infty$$

and

$$\Pi^j\left(\{y\in\mathbb{R}_*:|\ln(1+a_j y)|>1\}\right)<\infty\,,$$

since L_t has a finite number of big jumps (property of Lévy process).

Therefore, from (A.8) and (A.9) follows that, for $q = 2$, the cumulant-generating function $H_j(2) = H(2) < \infty$, which implies (note, Z^j are independent processes)

$$\mathbf{E}\left[\mathcal{E}_t(V)\right]^2 = \mathbf{E}\,\exp\left(2\sum_{j=1}^{n} Z_t^j\right) = \prod_{j=1}^{n} e^{t\,H_j(2)} < \infty\,.$$

And considering stochastic integral representation (A.7) we get that $\mathcal{E}_t(V)$ is square interable martingale. □

Lemma A.5. *For an arbitrary twice continuously differentiable w.r.t x and differentiated w.r.t. t function $z(t,x)$, and Ito's process X*

$$\begin{cases} \mathrm{d}X_t = a(X_t,\upsilon_t)\,\mathrm{d}t + b^{'}(X_{t-},\upsilon_{t-})\,\mathrm{d}V_t\,, \\ X_0 = x > 0\,, \end{cases}$$

where non-random functions $a : \mathcal{X}\times\mathcal{K}\to\mathbb{R}$ and $b : \mathcal{X}\times\mathcal{K}\to\mathbb{R}^n$ are continuous and such that the process X is well-defined, $V_t = \sigma\,W_t + \varsigma\,L_t$ and L_t consists of independent pure jumps Lévy processes, a sequence of random variables

$$\tau_m(\omega) = \inf\left\{u \geq t : \int\limits_t^u \left(|\alpha(s,\omega)|^2 + |\beta(s,\omega)|\right)\mathrm{d}s \geq m\right\} \wedge T\,,$$

is a sequence of Markov moments, where

$$\alpha(s) = z_x(s,X_s)\,b^{'}(X_s,\upsilon_s)$$

and

$$\beta(s) = \sum_{i=1}^{n}\int\limits_{\mathbb{R}_*}\Big[z(s,X_{s-} + b^{'}(X_{s-},\upsilon_{s-})\,\varsigma_i\,y) - z(s,X_{s-}) - z_x(s,X_{s-})\,b^{'}(X_{s-},\upsilon_{s-})\,\varsigma_i\,y\Big]\Pi^i(\mathrm{d}y)\,.$$

Proof. Let's fix an arbitrary $m > 0$. By the definition of Markov moment, it is necessary to show that

$$\forall s\in[0,T]: \quad \{\omega\in\Omega : \tau_m(\omega)\leq s\}\in\mathcal{F}_s\,.$$

An arbitrary time moment $t\in[0,T[$ is fixed. We carry out the proof in 3 steps:

a) Let $s\in[0,t]$. By the definition, $\tau_m : \Omega\to[t,T]$, then

$$\{\omega : \tau_m(\omega)\leq s\} = \emptyset\,.$$

b) Let $s\in]t,T[$. Note that, for any $m\in\mathbb{N}$,

$$\{\omega\in\Omega : \tau_m(\omega)\leq s\} = \left\{\omega\in\Omega : \int\limits_t^s (|\alpha(v)|^2 + |\beta(v)|)\,\mathrm{d}v \geq m\right\}\,.$$

Denote by

$$A := \{\omega \in \Omega : \tau_m(\omega) \le s\} \quad \text{and} \quad B := \{\omega \in \Omega : F(s,\omega) \ge m\},$$

where

$$F(s,\omega) = \int_t^s \left(|\alpha(v,\omega)|^2 + |\beta(v,\omega)|\right) \mathrm{d}v .$$

We show the inclusion: $A \subseteq B$.
Let $\omega_0 \in A$, then $\tau_m(\omega_0) \le s$. By the definition of $\tau_m(\omega_0)$ and considering $s \in]t,T[$, we obtain that $F(s,\omega_0) \ge m$. Thus $A \subseteq B$.
Show the inclusion: $A \supseteq B$.
Let $\omega_0 \in B$, then

$$\int_t^s \left(|\alpha(v,\omega_0)|^2 + |\beta(v,\omega_0)|\right) \mathrm{d}v \ge m \;\Rightarrow\; s \in \{u \in [t,T] : F(u,\omega_0) \ge m\},$$

so the set $\{u \in [t,T] : F(u,\omega_0) \ge m\}$ is not empty. Denote it by

$$\Gamma_m(\omega_0) := \{u \in [t,T] : F(u,\omega_0) \ge m\}.$$

Since $\Gamma_m(\omega_0)$ is bounded below by the constant t, then there exists $\inf\{\Gamma_m(\omega_0)\} \ge t$. By the definition

$$\tau_m(\omega_0) = \inf\{\Gamma_m(\omega_0)\},$$

but $s \in \Gamma_m(\omega_0)$ then $\tau_m(\omega_0) \le s \;\Rightarrow\; \omega_0 \in A$. So, $A \supseteq B$.
Proceeding from the fact that $A \subseteq B$ and $A \supseteq B$ are fulfilled simultaneously, it follows that $A = B$.
Note, the process $F = (F(s,\omega))_{0 \le s \le T}$ is adapted, since the processes $\alpha(s)$ and $\beta(s)$ are càdlàg processes.
Thus $\{\omega \in \Omega : F(s,\omega) \ge m\} \in \mathcal{F}_s$ and hence

$$\{\omega \in \Omega : \tau_m(\omega) \le s\} \in \mathcal{F}_s, \quad \text{for any} \quad s \in (t,T).$$

c) Let $s = T$. As we noted earlier, by the definition, $\tau_m : \Omega \to [t,T]$. Then

$$\{\omega : \tau_m(\omega) \le T\} = \Omega .$$

Note that $\mathbf{P}(\tau_m(\omega) \le T) = 1$ by the construction, therefore $\tau_m(\omega)$ is stopping time,

$$\lim_{m\to\infty} \tau_m = T, \quad \mathbf{P}\text{-a.s.}.$$

□

Lemma A.6 (Martingales inequalities). *If the process X_t is a martingale and the condition $\mathbf{E}(|X_t|^p) < \infty$ is satisfied, then*

$$\mathbf{P}\left[\sup_{t\in[0,s]} |X_t| > \lambda\right] \le \frac{\mathbf{E}(|X_s|^p)}{\lambda^p} \quad (p \ge 1), \tag{A.10}$$

$$\mathbf{E}\sup_{t\in[0,s]} |X_t|^p \le \left(\frac{p}{p-1}\right)^p \mathbf{E}(|X_s|^p) \qquad (p > 1), \tag{A.11}$$

for all $s > 0$. Inequality (A.11) holds in the case when X is a nonnegative submartingale with $\mathbf{E}(|X_s|^p) < \infty$.

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