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## Numerical Scheme Impacts on Time Domain Full Waveform Inversion

Pierre Jacquet, Andreas Atle, H el ene Barucq, Henri Calandra, Julien Diaz

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# Numerical Scheme Impacts on Time Domain Full Waveform Inversion

Mathias 2019

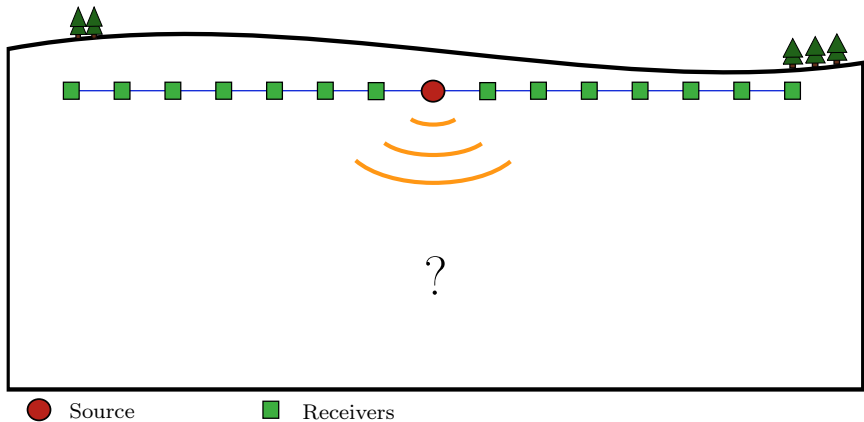
Pierre Jacquet

pierre.jacquet@inria.fr

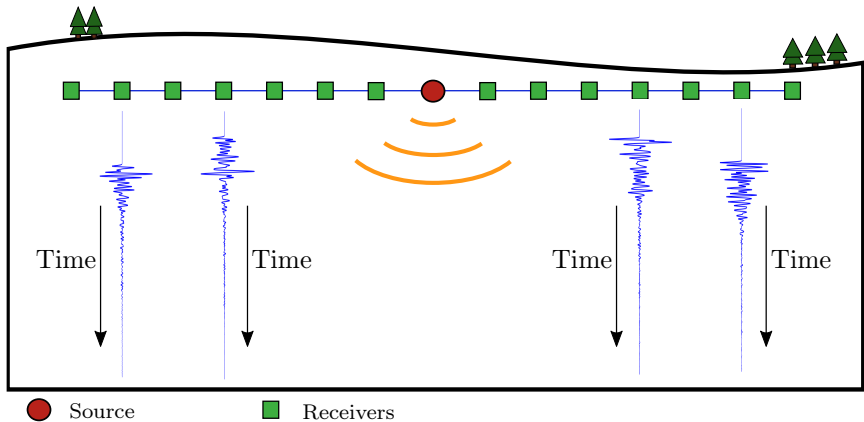
Atle Andreas, Barucq H el ene, Calandra Henri, Diaz Julien  
Second year PhD Student  
Inria - Magique 3D - DIP

Pau, FRANCE

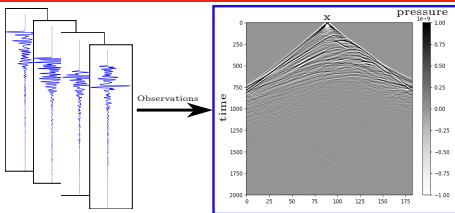




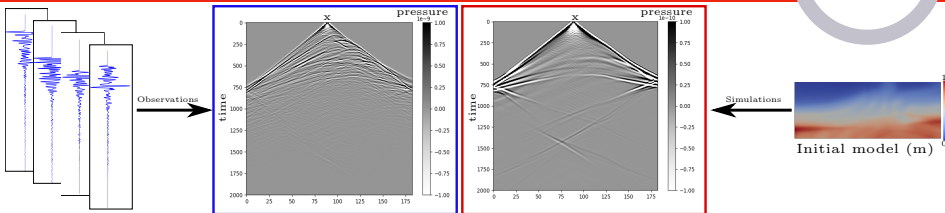
# Seismic Acquisition

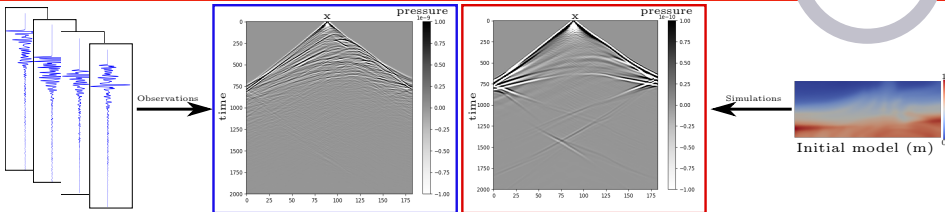


# FWI Workflow



# FWI Workflow





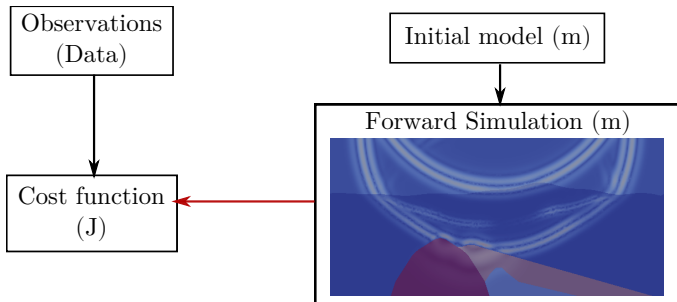
Cost function to minimize :

$$\mathcal{J}(\mathbf{m}) = \frac{1}{2} \|\mathbf{d}_{obs} - \mathcal{F}(\mathbf{m})\|^2$$

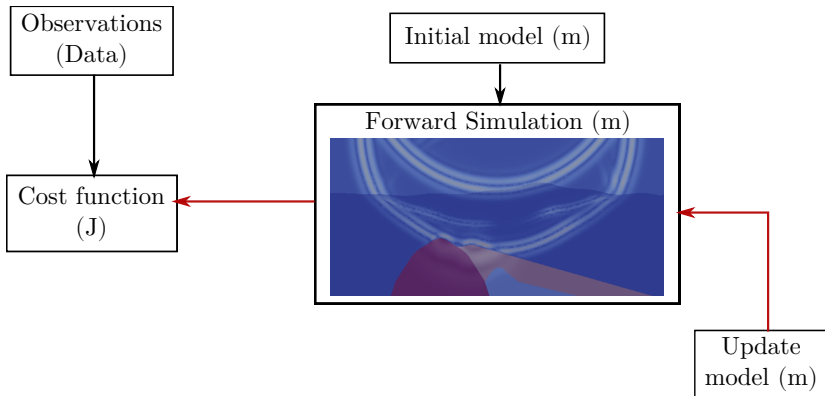
- ▶  $\mathcal{F}(\mathbf{m})$  is the restriction on the receivers of the simulated waves in the medium  $\mathbf{m}$ . (With  $\mathbf{m} = \mathbf{c}, \rho, \kappa \dots$ )
- ▶ FWI iterates until  $\mathcal{J}(\mathbf{m}) \rightarrow 0$

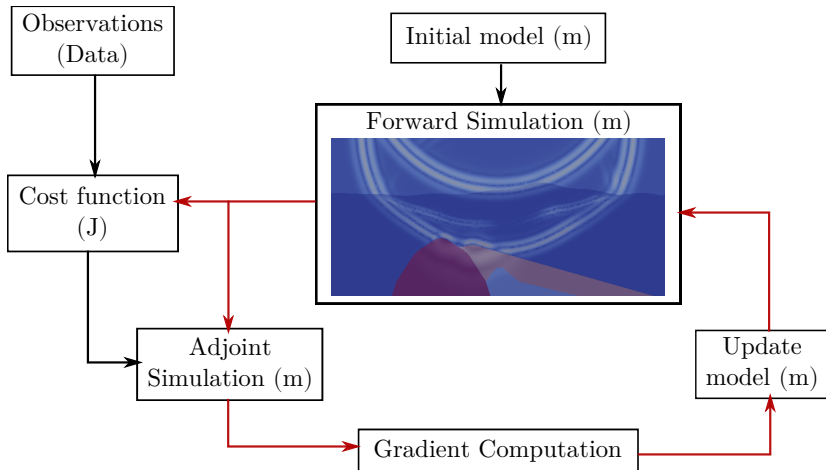
[1] Patrick Lailly  
The seismic inverse problem as a sequence of before stack migrations  
Conference on Inverse Scattering

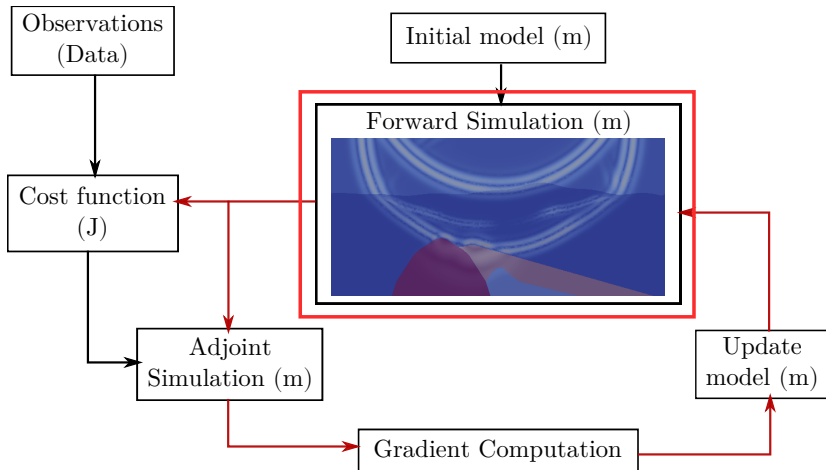
[2] Albert Tarantola  
Inversion of seismic reflection data in the acoustic approximation  
Geophysics, Vol. 49, 1984





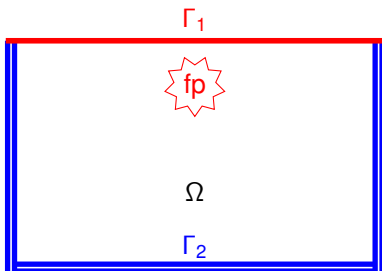






First order acoustic wave equation

$$\left\{ \begin{array}{l} \frac{1}{\rho c^2} \frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot \mathbf{v} = f_p \quad \text{on } \Omega \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{p} = 0 \quad \text{on } \Omega \\ \mathbf{p} = 0 \quad \text{on } \Gamma_1 \\ \frac{\partial \mathbf{p}}{\partial t} + \mathbf{c} \nabla \mathbf{p} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_2 \\ \mathbf{p}(0) = 0, \quad \mathbf{v}(0) = 0 \end{array} \right.$$



Domain with Absorbing Boundary  
Conditions



## Space Discretization : Discontinuous Galerkin Elements

- ▶ Nodal (Lagrangian / Jacobian)
- ▶ Modal (Bernstein-Bézier)

Space Discretization :  
Discontinuous Galerkin Elements

- ▶ Nodal (Lagrangian / Jacobian)
- ▶ Modal (Bernstein-Bézier)

Semi-discretized model :

$$\frac{\partial}{\partial t} \bar{\mathbf{U}}(t) = A \bar{\mathbf{U}}(t) + \bar{\mathbf{F}}(t)$$

with :

$$\bar{\mathbf{U}}(t) = \begin{pmatrix} \bar{\mathbf{P}}(t) \\ \bar{\mathbf{V}}(t) \end{pmatrix}$$

Space Discretization :  
Discontinuous Galerkin Elements

- ▶ Nodal (Lagrangian / Jacobian)
- ▶ Modal (Bernstein-Bézier)

Time schemes :

- ▶ Runge Kutta 2/4
- ▶ Adams Bashforth 3

Semi-discretized model :

$$\frac{\partial}{\partial t} \bar{\mathbf{U}}(t) = A\bar{\mathbf{U}}(t) + \bar{\mathbf{F}}(t)$$

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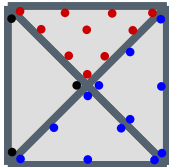


Assets of Discontinuous Galerkin Methods :

- ▶ Unstructured grid (enable to match the topography and medium irregularities)
- ▶ Robust to physical discontinuities
- ▶ hp-adaptivity
- ▶ Massively parallel performance properties



h-adaptivity



p-adaptivity with P1,  
P2, P3 elements



## Time Domain Full Waveform Inversion

Seismic Acquisition

FWI Workflow

Forward Discretization

## Some Results

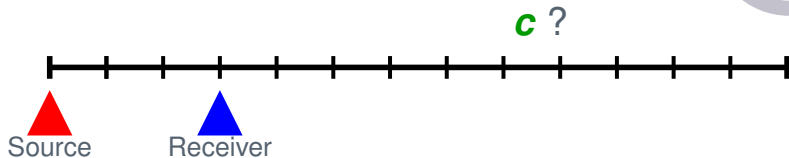
1D Results

2D Time Domain FWI Results

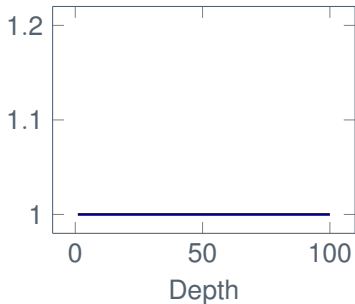
2D Multiscale Reconstruction

1D Preliminary tests

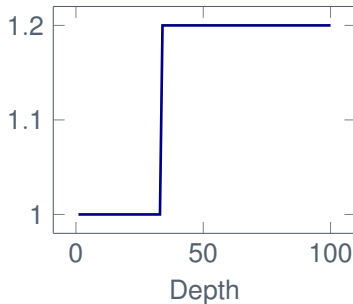
# One year ago...



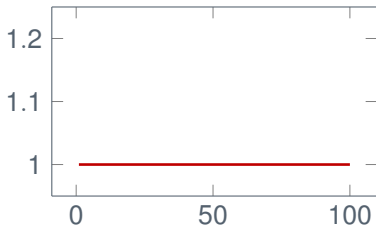
Initial  $c$  Model



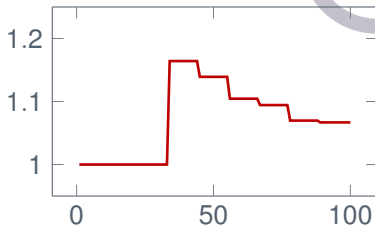
Target  $c$  Model



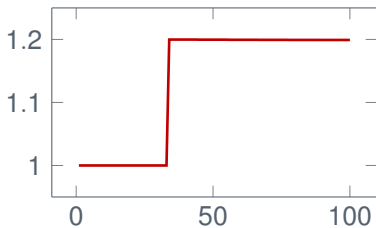
# FWI 1D Results



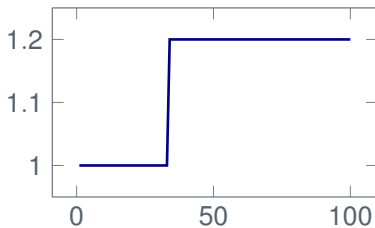
Initial  $c$  model



Intermediate  $c$  model (iter=20)



$c$  model Reconstructed (iter=50)



Target  $c$  model

2D FWI :

- ▶ Developed in Total environnement (DIP<sup>1</sup>)
- ▶ Nodal Space Operators (Lagrangian/Jacobian)
- ▶ Modal Space Operators (Bernstein-Bézier)
- ▶ Runge Kutta 2/4 and Adams Bashforth 3 time-schemes

Gradient expression :

$$\nabla_{\frac{1}{\kappa}} \mathcal{J} = \int_0^T \int_{\Omega} \frac{\partial \mathbf{p}}{\partial t} \lambda_1 d\Omega dt \quad \text{with : } \kappa = \rho \mathbf{c}^2$$

$\mathbf{c}$ ,  $\rho$  and  $\kappa$  Constant per elements

---

<sup>1</sup><http://dip.inria.fr/>

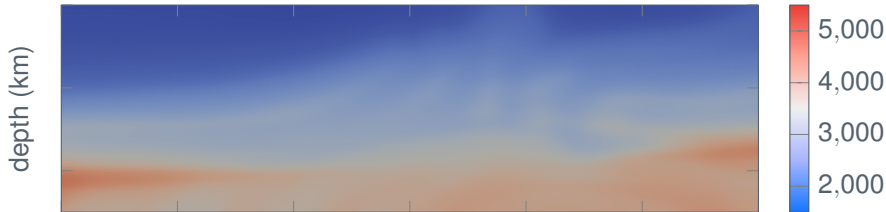
# 2D Time Domain FWI Reconstructions

Time-schemes comparison



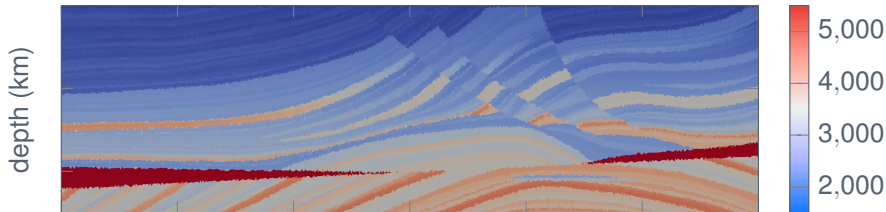
Initial **c** Model

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$



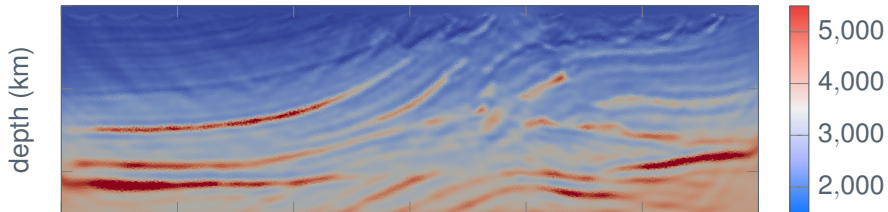
# 2D Time Domain FWI Reconstructions

Time-schemes comparison



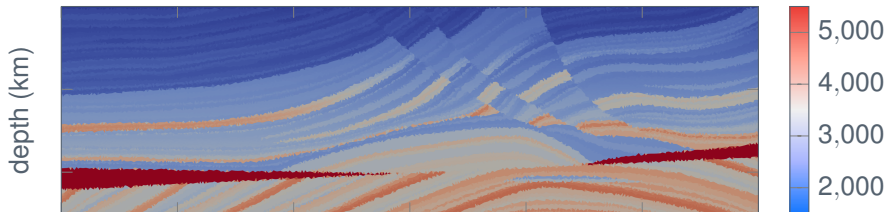
**RK2** Reconstructed **c** Model (30 iterations)

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$



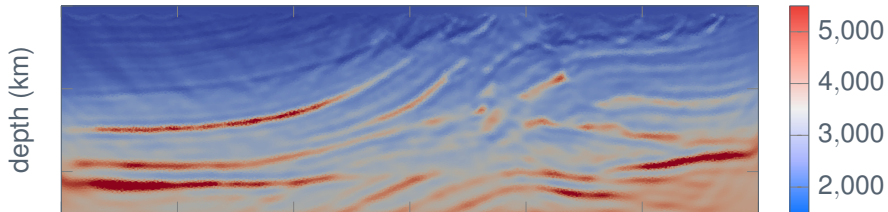
# 2D Time Domain FWI Reconstructions

Time-schemes comparison



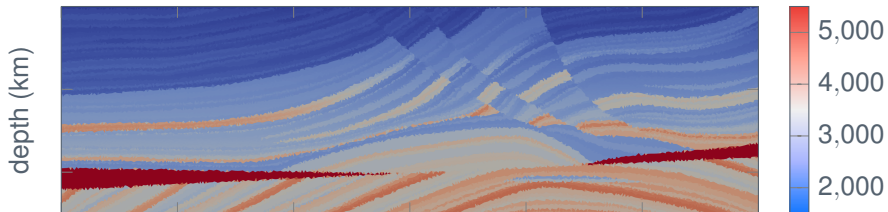
**RK4** Reconstructed **c** Model (30 iterations)

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$



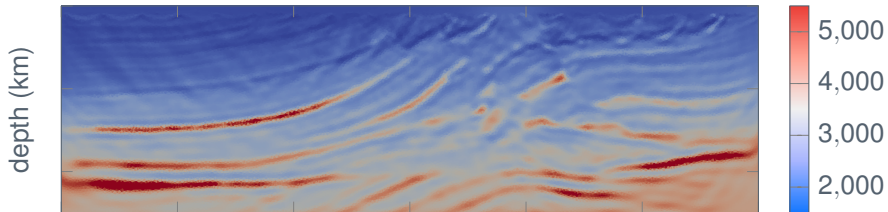
# 2D Time Domain FWI Reconstructions

Time-schemes comparison



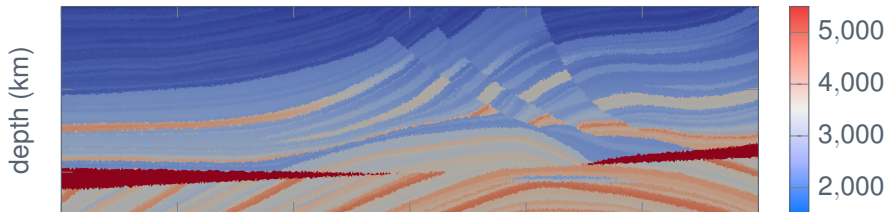
**AB3** Reconstructed **c** Model (30 iterations)

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$





# 2D Time Domain FWI Reconstructions

Time-Schemes Comparison

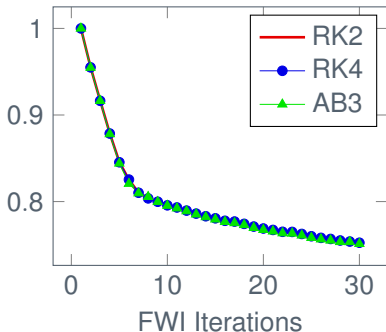


- ▶ 47k P1 elements
- ▶ Constant  $\rho$  model ( $\rho = 1$ )
- ▶ 19 sources / 181 Receivers
- ▶ Noise : SNR=10
- ▶ 30 iterations
- ▶ 120 cores
- ▶ Polynomial basis : Nodal

## Computational time :

- ▶ RK2 : **3h15**
- ▶ RK4 : **4h30**
- ▶ AB3 : **5h10**

Cost function evolution :



# 2D Time Domain FWI Reconstructions

Nodal/Modal Comparison

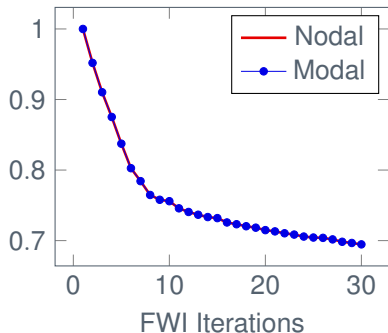


- ▶ 47k P1 elements
- ▶ Time Scheme : RK2
- ▶ Constant  $\rho$  model ( $\rho = 1$ )
- ▶ 19 sources / 181 Receivers
- ▶ Noise : SNR=10
- ▶ 30 iterations
- ▶ 120 cores

## Computational time :

- ▶ Nodal : **3h15**
- ▶ Modal: **4h30**<sup>[1]</sup>

Cost function evolution :



[1] Chan J. and Warburton T.  
GPU-Accelerated Bernstein Bézier Discontinuous Galerkin Methods for Wave Problems  
SIAM Journal on Scientific Computing 2017

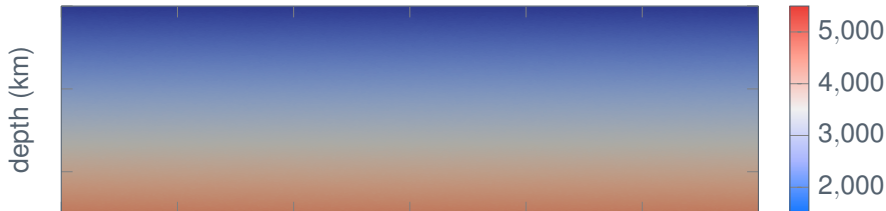
# 2D Multiscale Reconstructions

Reconstruction with an initial smooth model



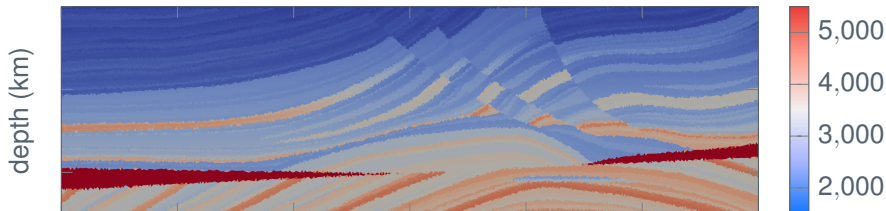
Initial **c** Model

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$



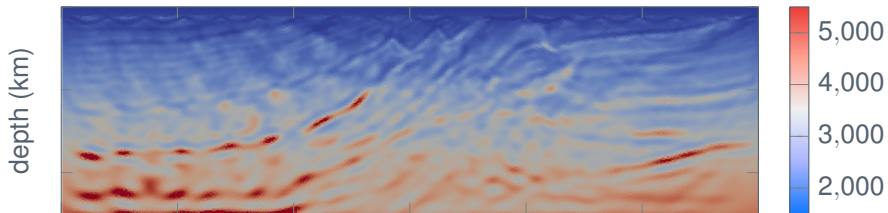
# 2D Multiscale Reconstructions

Reconstruction with an initial smooth model



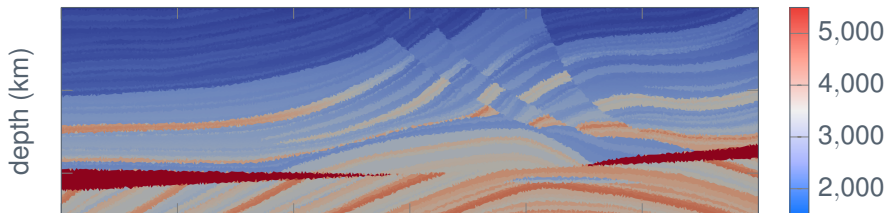
Reconstructed model **c** Model (30 iterations RK2)

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$



# 2D Multiscale Reconstructions

Multiscale Principle [1]

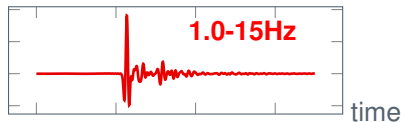
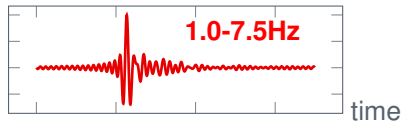
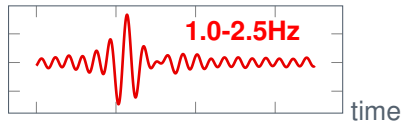


Low Frequencies  
↔  
Reconstruct **coarse** structures

High Frequencies  
↔  
Reconstruct **small** structures

Filtered Traces :

*p*



[1] C. Bunks, F. M. Saleck, S. Zaleski, and G. Chavent  
Multiscale seismic waveform inversion  
GEOPHYSICS, Vol. 60, No. 5, 1995

# 2D Multiscale Reconstructions

Multiscale Principle [1]



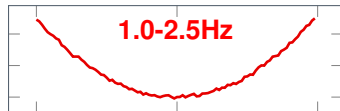
Heuristic Illustration :

$\mathcal{J}$

Low Frequencies

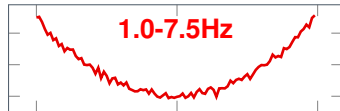


Reconstruct **coarse** structures



*m*

1.0-7.5Hz



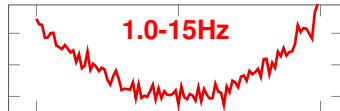
*m*

High Frequencies



Reconstruct **small** structures

1.0-15Hz



*m*

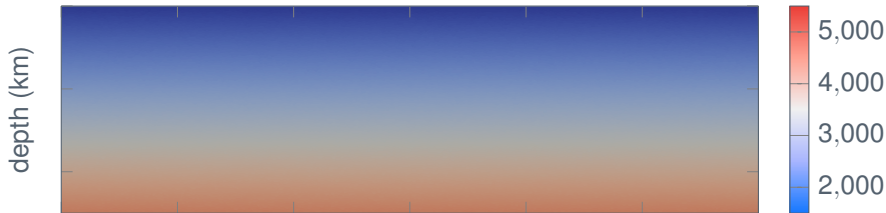
[1] C. Bunks, F. M. Saleck, S. Zaleski, and G. Chavent  
Multiscale seismic waveform inversion  
GEOPHYSICS, Vol. 60, No. 5, 1995

# 2D Multiscale Reconstructions

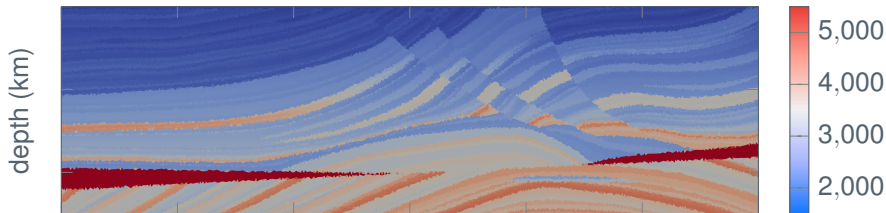
Reconstruction with an initial smooth model



Initial **c** Model



Target **c** Model



# 2D Multiscale Reconstructions

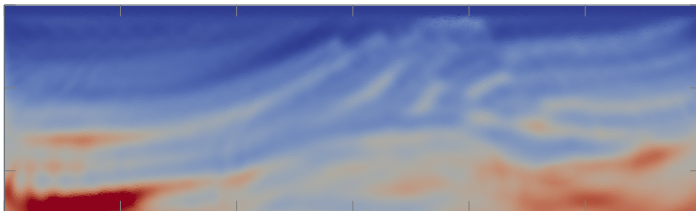
Reconstruction with an initial smooth model



Reconstructed **c** Model with 1.0-2.5Hz filter

$m \cdot s^{-1}$

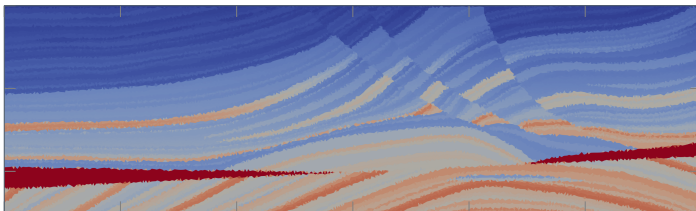
depth (km)



Target **c** Model

$m \cdot s^{-1}$

depth (km)





# 2D Multiscale Reconstructions

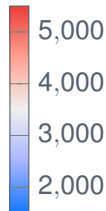
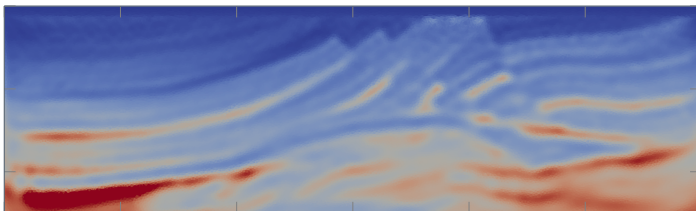
Reconstruction with an initial smooth model



Reconstructed **c** Model with 1.0-7.5Hz filter

$m \cdot s^{-1}$

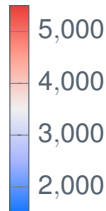
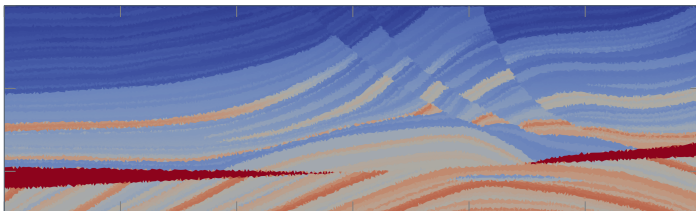
depth (km)



Target **c** Model

$m \cdot s^{-1}$

depth (km)



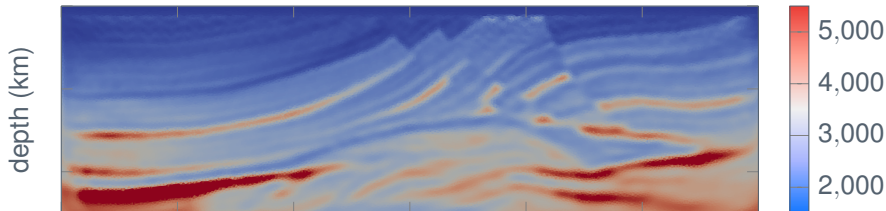
# 2D Multiscale Reconstructions

Reconstruction with an initial smooth model



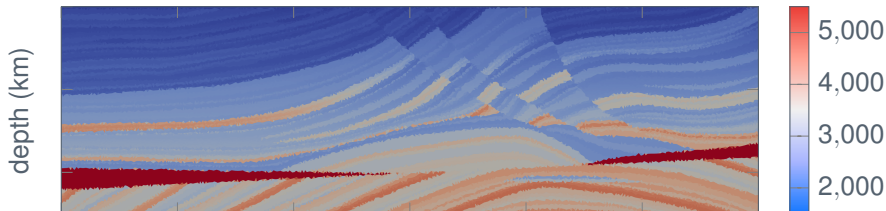
Reconstructed **c** Model with 1.0-10Hz filter

$m \cdot s^{-1}$



Target **c** Model

$m \cdot s^{-1}$



# 2D Multiscale Reconstructions

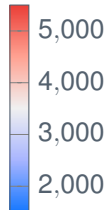
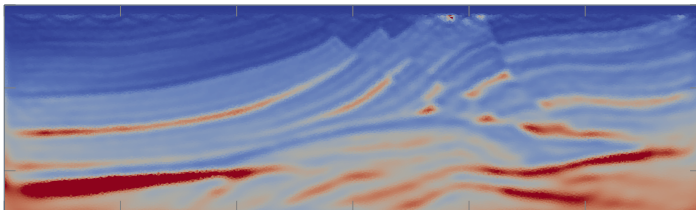
Reconstruction with an initial smooth model



Reconstructed **c** Model with 1.0-15Hz filter

$m \cdot s^{-1}$

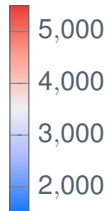
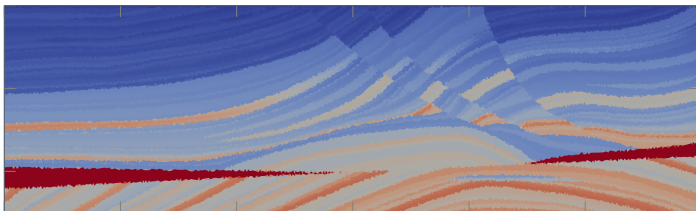
depth (km)



Target **c** Model

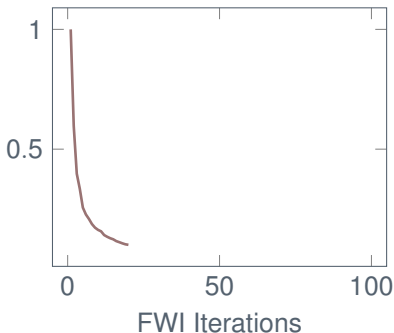
$m \cdot s^{-1}$

depth (km)



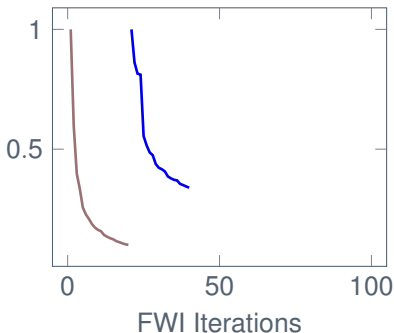
- ▶ 47k P1 elements
- ▶ Time Scheme : RK2
- ▶ Constant  $\rho$  model ( $\rho = 1$ )
- ▶ 19 sources / 181 Receivers
- ▶ Noise : SNR=10
- ▶ 120 cores
- ▶ 20 FWI iterations per filter
- ▶ Computation time : 10h
- ▶ Frequencies : 1-2.5Hz

Cost function evolution :



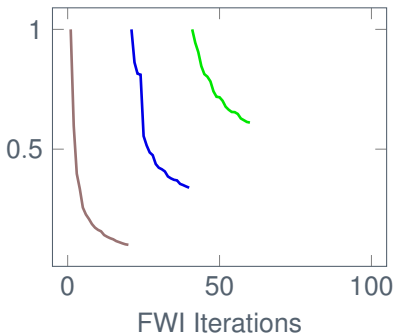
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- ▶ 120 cores
- ▶ 20 FWI iterations per filter
- ▶ Computation time : 10h
- ▶ Frequencies : 1-2.5Hz,  
1-5.0Hz

Cost function evolution :



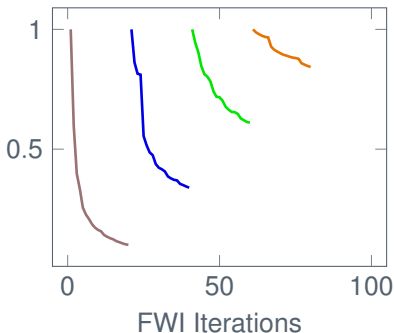
- ▶ 47k P1 elements
- ▶ Time Scheme : RK2
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- ▶ Computation time : 10h
- ▶ Frequencies : 1-2.5Hz,  
1-5.0Hz, 1-7.5Hz

Cost function evolution :



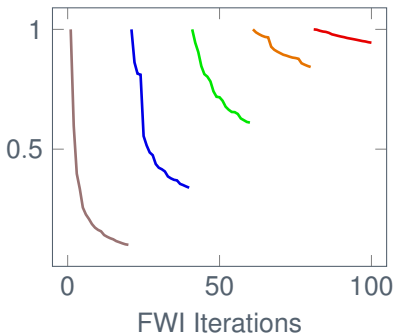
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- ▶ Computation time : 10h
- ▶ Frequencies : 1-2.5Hz,  
1-5.0Hz, 1-7.5Hz, 1-10Hz

Cost function evolution :



- ▶ 47k P1 elements
- ▶ Time Scheme : RK2
- ▶ Constant  $\rho$  model ( $\rho = 1$ )
- ▶ 19 sources / 181 Receivers
- ▶ Noise : SNR=10
- ▶ 120 cores
- ▶ 20 FWI iterations per filter
- ▶ Computation time : 10h
- ▶ Frequencies : 1-2.5Hz,  
1-5.0Hz, 1-7.5Hz, 1-10Hz,  
1-15Hz

Cost function evolution :





## Main Results :

- ▶ 2D Acoustic Reconstruction performed with different discretization
- ▶ Multiscale FWI implemented and working on Marmousi

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- ▶ Develop enhanced optimizers (NLCG, Limited BFGS)
- ▶ Adapt the code to use High order Model
- ▶ Extend the code to elastic and elasto-acoustic propagator
- ▶ Exploit coupled numerical method (SEM/DG) (Aurélien Citrain Thesis)

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**Thank you.**



Lagrangian functional [1] :

$$\mathcal{L}(\hat{\mathbf{u}}, \hat{\boldsymbol{\lambda}}, \mathbf{m}) = \frac{1}{2} \|d_{obs} - \mathcal{R}(\hat{\mathbf{u}})\|^2 + \langle \text{Forward}_{\mathbf{m}}(\hat{\mathbf{u}}) - f_p, \hat{\boldsymbol{\lambda}} \rangle$$

If  $\hat{\mathbf{u}} = \mathbf{u}$  Solution of the Direct Problem  $\iff (\text{Forward}_{\mathbf{m}}(\mathbf{u}) - f_p = 0)$  :

$$\mathcal{J}(\mathbf{m}) = \mathcal{L}(\mathbf{u}, \hat{\boldsymbol{\lambda}}, \mathbf{m})$$

[1] Plessix R-E

A review of the adjoint-state method for computing the gradient of a functional with geophysical applications  
Geophysical Journal International, Volume 167, Issue 2, 2006

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For  $Forward_{\mathbf{m}}(\mathbf{u}) - f_p = 0$  :

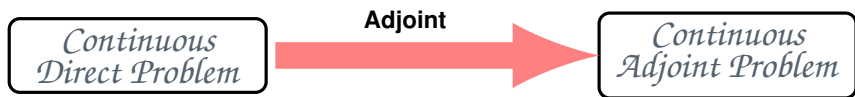
$$\partial_{\mathbf{m}_i} \mathcal{J}(\mathbf{m}) = \partial_{\mathbf{m}_i} \mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}, \mathbf{m}) = \partial_{\mathbf{m}_i} \langle Forward_{\mathbf{m}}(\mathbf{u}), \boldsymbol{\lambda} \rangle$$

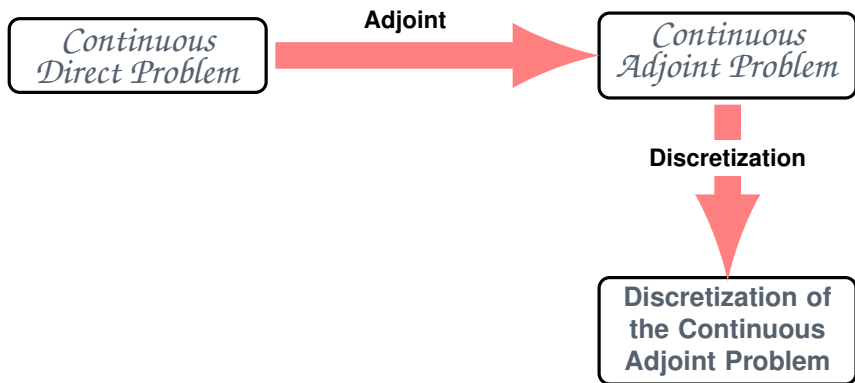
[1] Plessix R-E

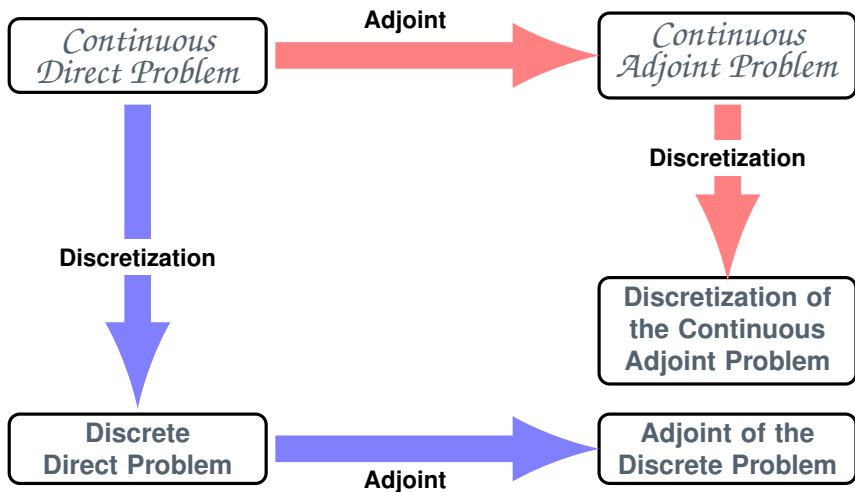
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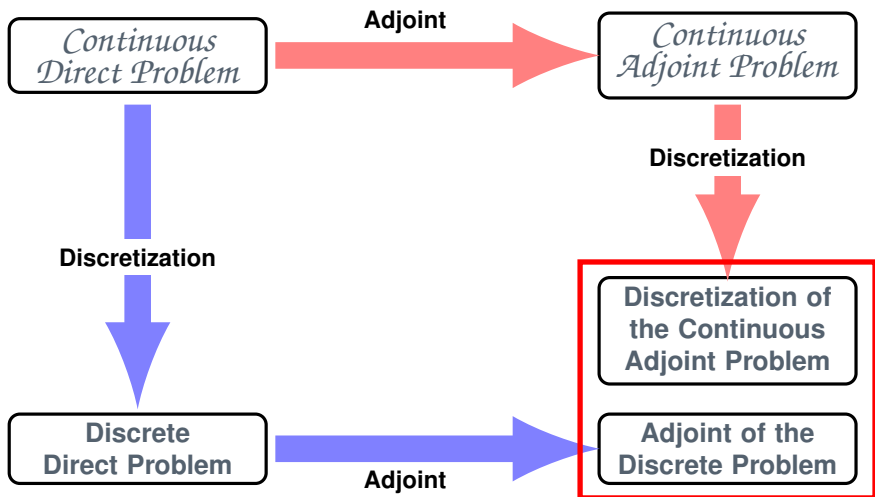
*Continuous  
Direct Problem*











$$\mathcal{J}(\mathbf{p}) = \frac{1}{2} \|d_{obs} - R\mathbf{p}\|^2$$

$$\left\{ \begin{array}{l} \frac{1}{\rho c^2} \frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot \mathbf{v} = f_p \quad \text{on } \Omega \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{p} = 0 \quad \text{on } \Omega \\ \mathbf{p} = 0 \quad \text{on } \Gamma_1 \\ \frac{\partial \mathbf{p}}{\partial t} + \mathbf{c} \nabla \mathbf{p} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_2 \\ \mathbf{p}(0) = 0, \quad \mathbf{v}(0) = 0 \end{array} \right.$$

$$t \in [0, T]$$

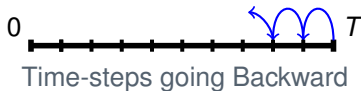
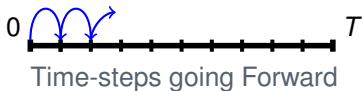
$$\left\{ \begin{array}{l} \frac{1}{\rho c^2} \frac{\partial \lambda_1}{\partial t} + \nabla \cdot \lambda_2 = \frac{\partial \mathcal{J}}{\partial \mathbf{p}} \quad \text{on } \Omega \\ \rho \frac{\partial \lambda_2}{\partial t} + \nabla \lambda_1 = 0 \quad \text{on } \Omega \\ \lambda_1 = 0 \quad \text{on } \Gamma_1 \\ \frac{\partial \lambda_1}{\partial t} - \mathbf{c} \nabla \lambda_1 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_2 \\ \lambda_1(T) = 0, \quad \lambda_2(T) = 0 \end{array} \right.$$

$$t \in [T, 0]$$

$$\mathcal{J}(\mathbf{p}) = \frac{1}{2} \|d_{obs} - R\mathbf{p}\|^2$$

$$\left\{ \begin{array}{l} \frac{\partial \bar{\mathbf{U}}^n}{\partial t} = A\bar{\mathbf{U}}^n + \bar{\mathbf{F}}^n \\ \text{With : } \bar{\mathbf{U}}^n = \begin{pmatrix} \bar{\mathbf{P}}^n \\ \bar{\mathbf{V}}^n \end{pmatrix} \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \bar{\boldsymbol{\Lambda}}^n}{\partial t} = A\bar{\boldsymbol{\Lambda}}^n + R^*(R\bar{\mathbf{U}}^n - d_{obs}) \\ \text{With : } \bar{\boldsymbol{\Lambda}}^n = \begin{pmatrix} \bar{\boldsymbol{\Lambda}}_1^n \\ \bar{\boldsymbol{\Lambda}}_2^n \end{pmatrix} \end{array} \right.$$



# DtA : Discretize then Adjoint Strategy

Example With RK4



All time scheme can be summed-up such as :

$$L\bar{U} = E\bar{F}$$

RK4 time-scheme leads to :

$$\bar{U}^{n+1} = B\bar{U}^n + C_0\bar{F}^n + C_{\frac{1}{2}}\bar{F}^{n+\frac{1}{2}} + C_1\bar{F}^{n+1}$$

$$L\bar{U} = E\bar{F} = \bar{G}$$

$$\begin{pmatrix} I & & & & & \\ -B & I & & & & \\ & -B & I & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & -B & I \end{pmatrix} \begin{pmatrix} \bar{U}^0 \\ \bar{U}^1 \\ \bar{U}^2 \\ \vdots \\ \bar{U}^n \end{pmatrix} = \begin{pmatrix} \bar{G}^0 \\ \bar{G}^1 \\ \bar{G}^2 \\ \vdots \\ \bar{G}^n \end{pmatrix}$$

All time scheme can be summed-up such as :

$$\mathbf{L}\bar{\mathbf{U}} = \mathbf{E}\bar{\mathbf{F}}$$

We are looking for a Discrete Adjoint state satisfying :

$$\mathbf{L}^*\bar{\boldsymbol{\Lambda}} = -\mathbf{R}^*(d_{obs} - \mathbf{R}\bar{\mathbf{U}})$$

With the adjoint operator  $\mathbf{L}^*$  satisfying :

$$\langle \mathbf{L}\bar{\mathbf{U}}, \bar{\boldsymbol{\Lambda}} \rangle = \langle \bar{\mathbf{U}}, \mathbf{L}^*\bar{\boldsymbol{\Lambda}} \rangle$$



All time scheme can be summed-up such as :

$$\mathbf{L}\bar{\mathbf{U}} = \mathbf{E}\bar{\mathbf{F}} = \bar{\mathbf{G}}$$

We are looking for a Discrete Adjoint state satisfying :

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With the adjoint operator  $\mathbf{L}^*$  satisfying :

$$\langle \mathbf{L}\bar{\mathbf{U}}, \bar{\boldsymbol{\lambda}} \rangle = \langle \bar{\mathbf{U}}, \mathbf{L}^*\bar{\boldsymbol{\lambda}} \rangle$$

$$\langle \bar{\mathbf{G}}, \bar{\boldsymbol{\lambda}} \rangle = \langle \bar{\mathbf{U}}, \bar{\mathbf{D}} \rangle \quad (\text{Adjoint Test})$$

Adjoint test succeeds  $\iff$  operator  $\mathbf{L}^*$  well established

# DtA : Discretize then Adjoint Strategy

Example with RK4



RK4 time-scheme leads to :

$$\bar{U}^{n+1} = B\bar{U}^n + C_0\bar{F}^n + C_{\frac{1}{2}}\bar{F}^{n+\frac{1}{2}} + C_1\bar{F}^{n+1}$$

$$L\bar{U} = E\bar{F} = \bar{G}$$
$$\begin{pmatrix} I & & & & \\ -B & I & & & \\ & -B & I & & \\ & & \ddots & \ddots & \\ & & & -B & I \end{pmatrix} \begin{pmatrix} \bar{U}^0 \\ \bar{U}^1 \\ \bar{U}^2 \\ \vdots \\ \bar{U}^n \end{pmatrix} = \begin{pmatrix} \bar{G}^0 \\ \bar{G}^1 \\ \bar{G}^2 \\ \vdots \\ \bar{G}^n \end{pmatrix}$$

So :

$$L^* = \begin{pmatrix} I & -B^* & & & \\ & I & -B^* & & \\ & & \ddots & \ddots & \\ & & & I & -B^* \\ & & & & I \end{pmatrix}$$

## Adjoint Then Discretize

- + Physical approach
- + Same discrete operators for Forward and Backward
- - Approximate gradient [1]
- Consistent with the discretization

## Discretize then Adjoint

- + Numerical approach
- + Has an Adjoint Test
- Tremendous work to develop the adjoint operators
- Non-consistency of the adjoint state [2]

[1] Sirkes, Ziv and Tziperman, Eli  
Finite Difference of Adjoint or Adjoint of Finite Difference ?  
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[2] Sei Alain and Symes William  
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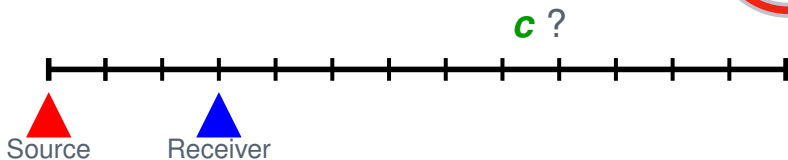
## Discretize then Adjoint

- + Numerical approach
- + / - Has an Adjoint Test (**in theory**)
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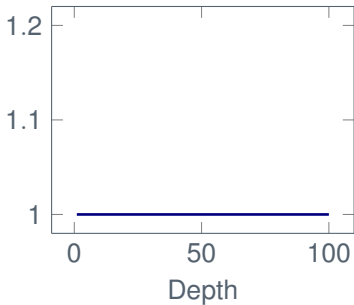
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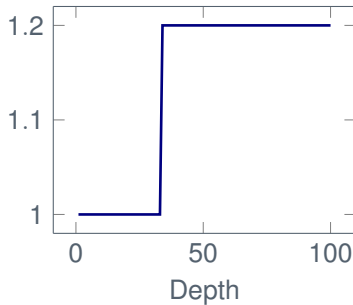
# 1D Preliminary tests



Initial  $c$  Model



Target  $c$  Model



1D FWI :

- ▶ Lagrange / B-Bézier Operators
- ▶ RK4 / AB3 time-schemes

Gradient expression :

$$\nabla_{\mathbf{c}} \mathcal{J} = - \int_0^T \int_{\Omega} \frac{2}{\rho \mathbf{c}^3} \frac{\partial \mathbf{p}}{\partial t} \lambda_1 d\Omega dt$$

## 1D FWI :

- ▶ Lagrange / B-Bézier Operators
- ▶ RK4 / AB3 time-schemes

## Adjoint test passed with :

- ▶ With a canonical space inner-product  
 $\langle u, v \rangle_X = \sum_i u_i v_i$
- ▶ With a M-space inner product  
 $\langle u, v \rangle_X^M = \langle Mu, v \rangle_X$

## Gradient expression :

$$\nabla_{\mathbf{c}} \mathcal{J} = - \int_0^T \int_{\Omega} \frac{2}{\rho \mathbf{c}^3} \frac{\partial \mathbf{p}}{\partial t} \lambda_1 d\Omega dt$$

```
./run
```

```
--- Adjoint test ----
```

```
inner product U/D 553123.57586755091
```

```
inner product G/Q 553123.57586756046
```

## 1D FWI :

- ▶ Lagrange / B-Bézier Operators
- ▶ RK4 / AB3 time-schemes

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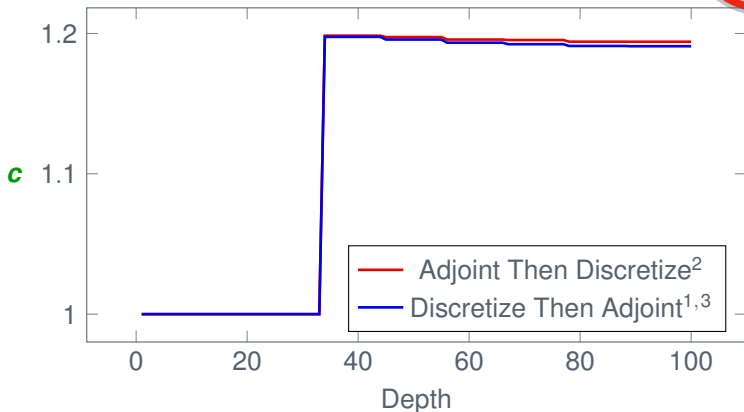
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```
./run
--- Adjoint test ----
inner product U/D 553123.57586755091
inner product G/Q 553123.57586756046
./run
--- Adjoint test ----
inner product U/D -75077.332007383695
inner product G/Q -75077.332007386358
./run
--- Adjoint test ----
inner product U/D 125669.89223600870
inner product G/Q 125669.89223600952
```



# 1D Velocity Model Reconstructions



**c** Model at the 100th FWI iteration

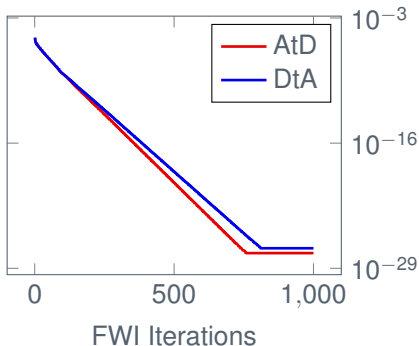
<sup>2</sup>With Bernstein-Bézier elements and AB3 time scheme

<sup>3</sup>With canonical scalar product

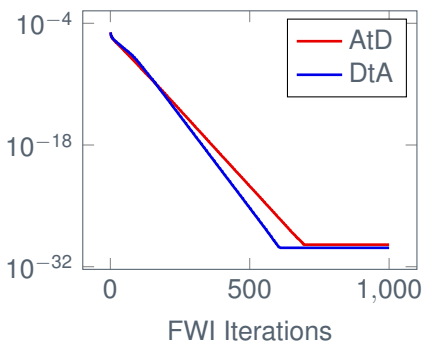
# 1D Velocity Model Reconstructions



With RK4 :



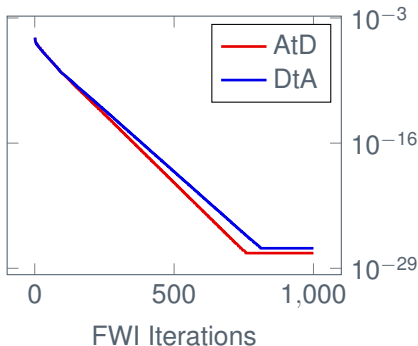
With AB3 :



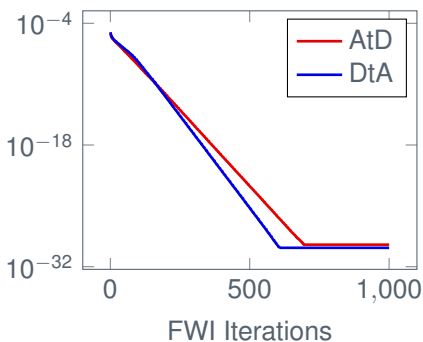
# 1D Velocity Model Reconstructions



With RK4 :



With AB3 :



- ▶ For RK4 scheme : **AtD** is slightly better than **DtA**
- ▶ For AB3 scheme : **DtA** is slightly better than **AtD**
- ▶ No predominant behaviour