Numerical Scheme Impacts on Time Domain Full Waveform Inversion

Pierre Jacquet, Andreas Atle, Hélène Barucq, Henri Calandra, Julien Diaz

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Mathias 2019

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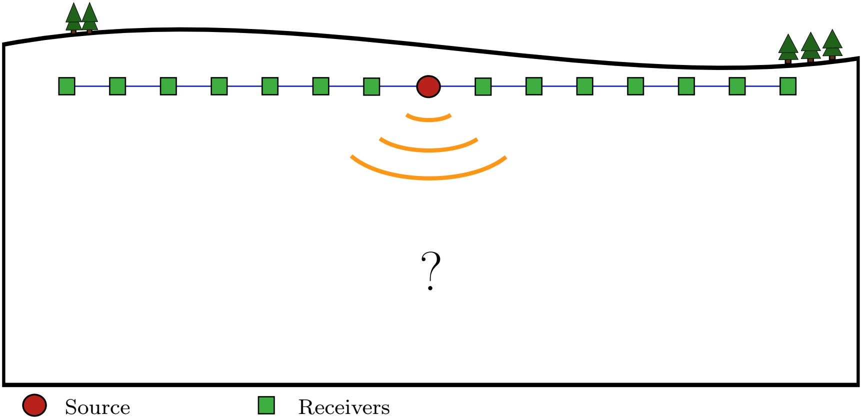
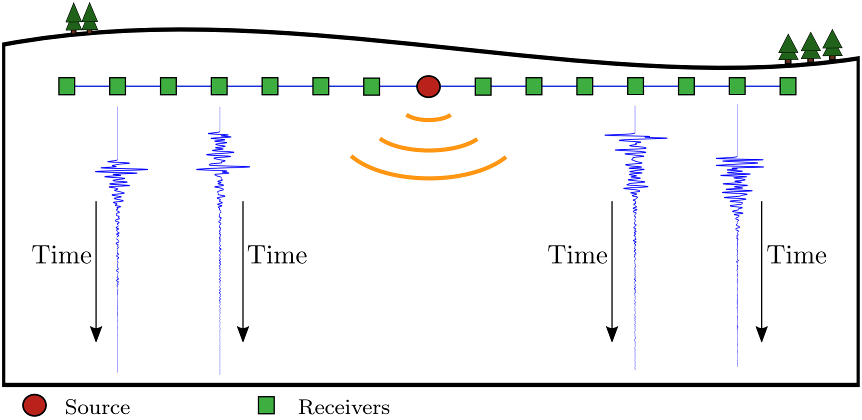
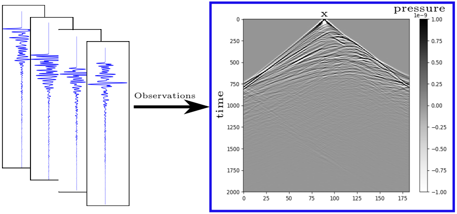
Seismic Acquisition



Seismic Acquisition



FWI Workflow



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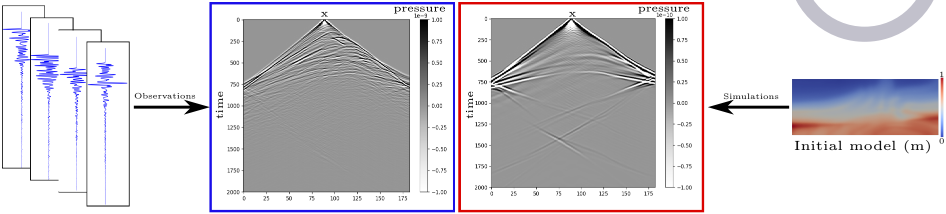

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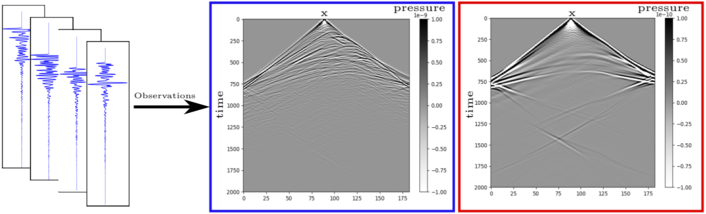


Cost function to minimize :

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}||d_{obs} - \mathcal{F}(\boldsymbol{m})||^2$$

- $\mathcal{F}(m)$ is the restriction on the receivers of the simulated waves in the medium $\boldsymbol{m}$. (With $\boldsymbol{m} = \boldsymbol{c}, \rho, \kappa ...$)
- FWI iterates until $\mathcal{J}(\boldsymbol{m}) \longrightarrow 0$

[1] Patrick Lailly
The seismic inverse problem as a sequence of before stack migrations
Conference on Inverse Scattering

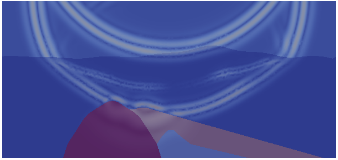
[2] Albert Tarantola
Inversion of seismic reflection data in the acoustic approximation
Geophysics, Vol. 49, 1984

FWI Workflow

Observations (Data)

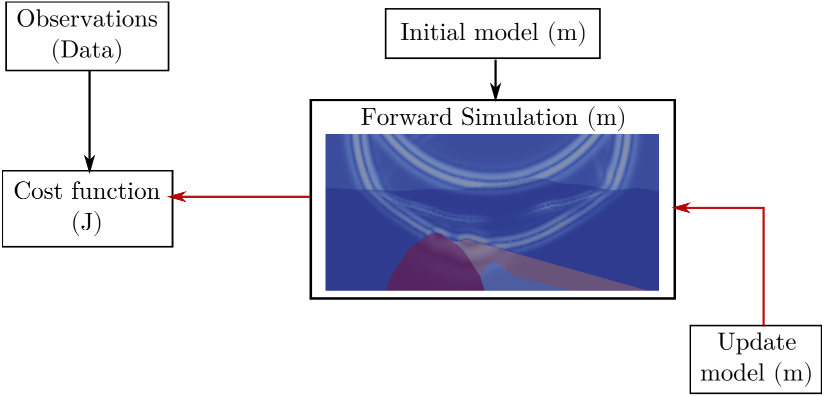
Initial model (m)

Forward Simulation (m)



Cost function (J)

FWI Workflow



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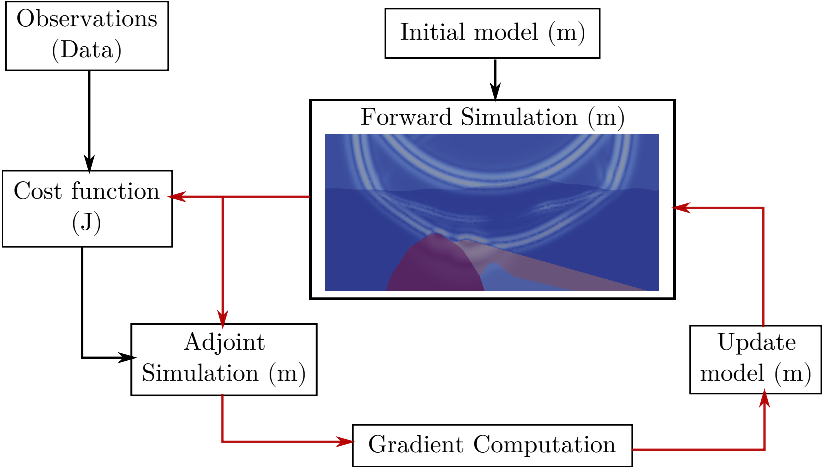

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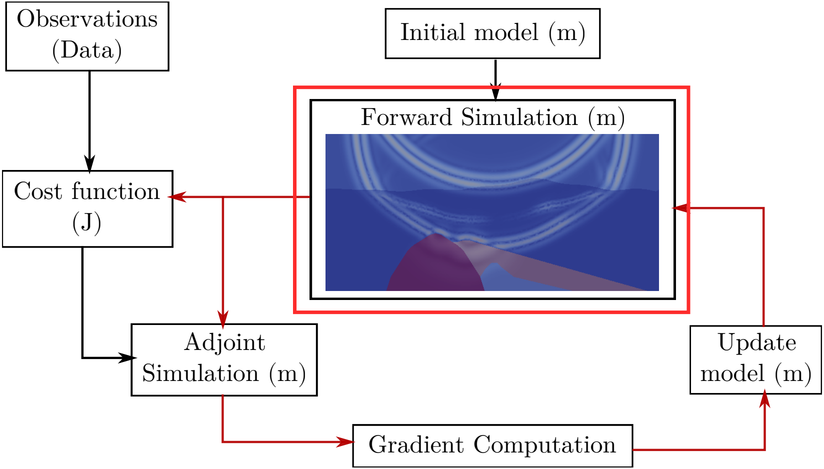
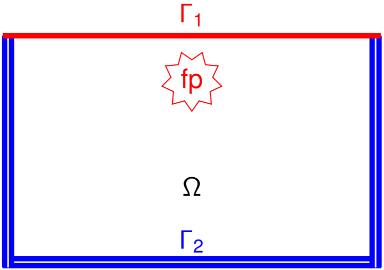


Continuous Forward Model

First order acoustic wave equation

$$
\begin{cases}
\dfrac{1}{\rho \boldsymbol{c}^2}\dfrac{\partial \boldsymbol{p}}{\partial t} + \nabla \cdot \boldsymbol{v} = f_p \quad \text{on } \boldsymbol{\Omega} \\
\rho \dfrac{\partial \boldsymbol{v}}{\partial t} + \nabla \boldsymbol{p} = 0 \quad \text{on } \boldsymbol{\Omega} \\
\boldsymbol{p} = 0 \quad \text{on } \boldsymbol{\Gamma_1} \\
\dfrac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{c} \nabla \boldsymbol{p} \cdot \boldsymbol{n} = 0 \quad \text{on } \boldsymbol{\Gamma_2} \\
\boldsymbol{p}(0) = 0, \quad \boldsymbol{v}(0) = 0
\end{cases}
$$



Domain with Absorbing Boundary Conditions

Discrete Forward Model

Space Discretization :
Discontinuous Galerkin Elements

- Nodal (Lagrangian / Jacobian)
- Modal (Bernstein-Bézier)

Discrete Forward Model

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Semi-discretized model :

$$\frac{\partial}{\partial t}\bar{\boldsymbol{U}}(t) = A\bar{\boldsymbol{U}}(t) + \bar{\boldsymbol{F}}(t)$$

with :

$$\bar{\boldsymbol{U}}(t) = \begin{pmatrix} \bar{\boldsymbol{P}}(t) \\ \bar{\boldsymbol{V}}(t) \end{pmatrix}$$

Discrete Forward Model

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Discontinuous Galerkin Elements

- Nodal (Lagrangian / Jacobian)
- Modal (Bernstein-Bézier)

Time schemes :

- Runge Kutta 2/4
- Adams Bashforth 3

Semi-discretized model :

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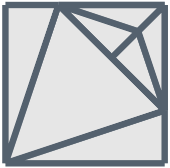
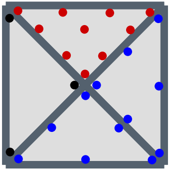
Forward time steps

Assets of Discontinuous Galerkin Methods :

- Unstructured grid (enable to match the topography and medium irregularities)
- Robust to physical discontinuities
- hp-adaptivity
- Massively parallel performance properties



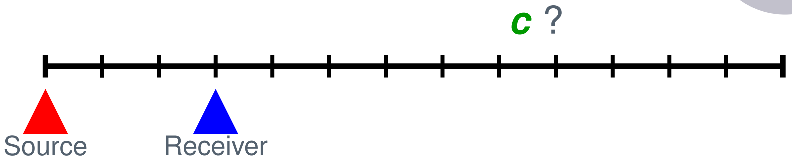
h-adaptivity



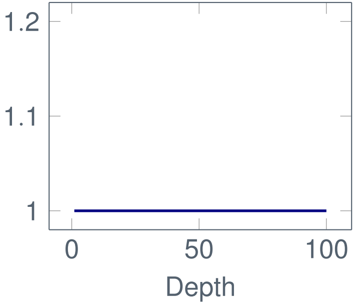
p-adaptivity with P1, P2, P3 elements

Outline

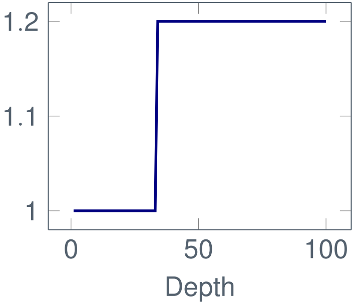
One year ago...



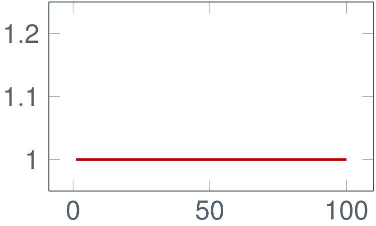
Initial *c* Model



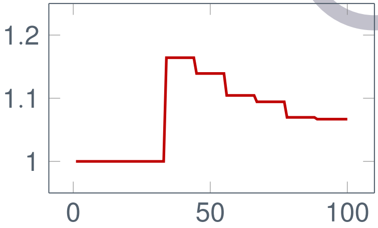
Target *c* Model



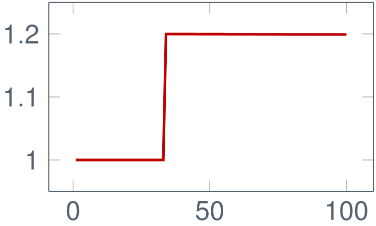
FWI 1D Results



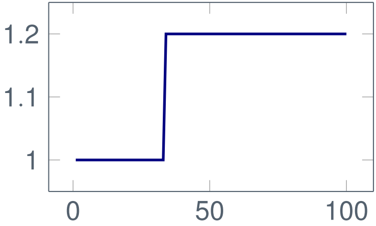
Initial ***c*** model



Itermediate ***c*** model (iter=20)



c model Reconstructed (iter=50)



Target ***c*** model

2D Time Domain Reconstruction

2D FWI :

- Developped in Total environnement (DIP[1])
- Nodal Space Operators (Lagrangian/Jacobian)
- Modal Space Operators (Bernstein-Bézier)
- Runge Kutta 2/4 and Adams Bashforth 3 time-schemes

Gradient expression :

$$\nabla_{\frac{1}{\kappa}} \mathcal{J} = \int_0^T \int_\Omega \frac{\partial \boldsymbol{p}}{\partial t} \boldsymbol{\lambda_1} \, d\Omega dt \qquad \text{with : } \boldsymbol{\kappa} = \boldsymbol{\rho c}^2$$

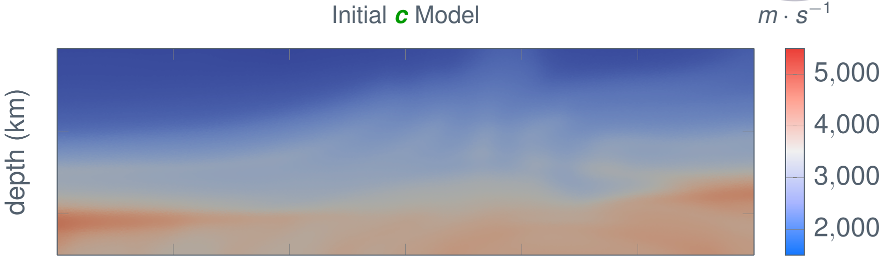
$\boldsymbol{c}$, $\boldsymbol{\rho}$ and $\boldsymbol{\kappa}$ Constant per elements

[1] `http://dip.inria.fr/`

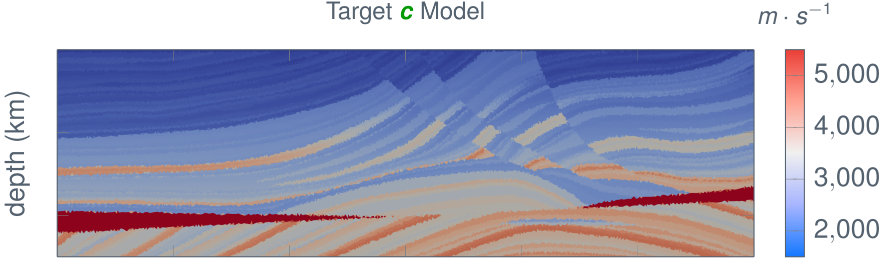
2D Time Domain FWI Reconstructions

Time-schemes comparison

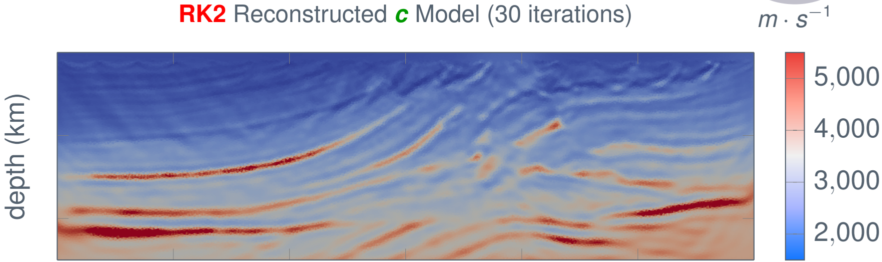
Initial *c* Model



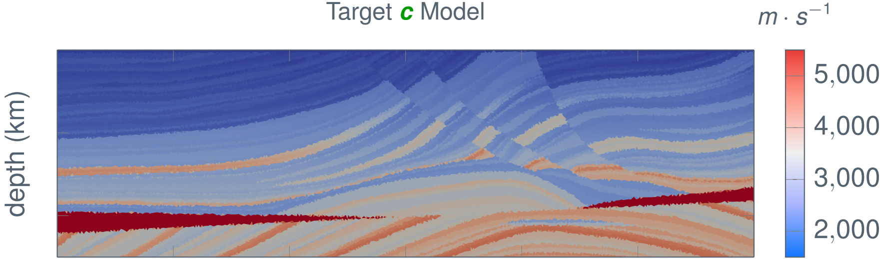
Target *c* Model



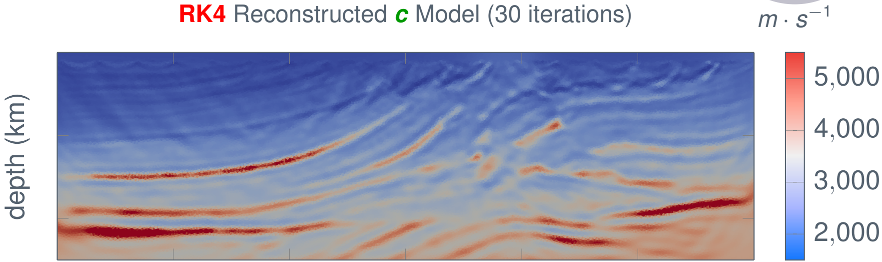
RK2 Reconstructed ***c*** Model (30 iterations)



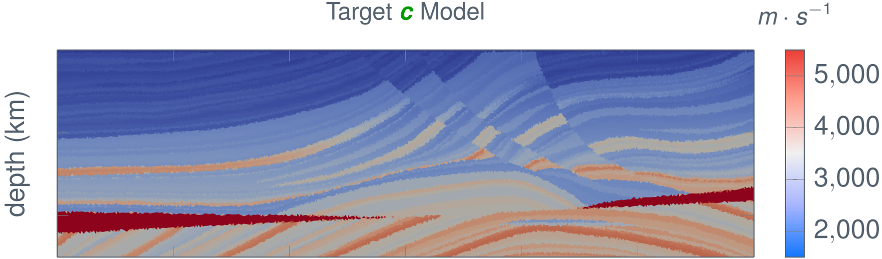
Target ***c*** Model



RK4 Reconstructed ***c*** Model (30 iterations)



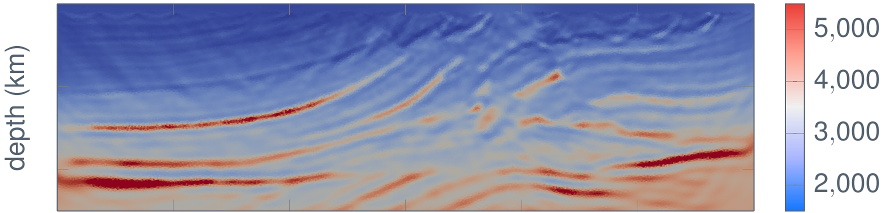
Target ***c*** Model



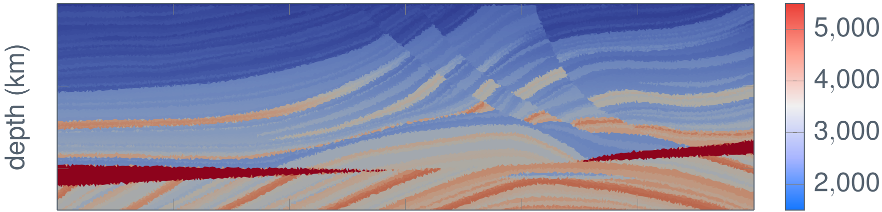
2D Time Domain FWI Reconstructions

Time-schemes comparison

AB3 Reconstructed ***c*** Model (30 iterations)



Target ***c*** Model

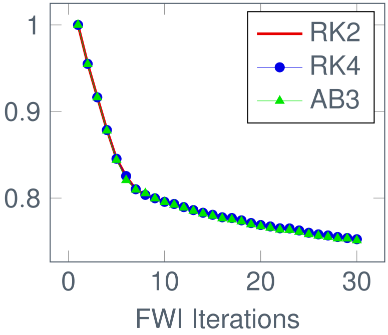


- 47k P1 elements
- Constant ρ model ($\rho = 1$)
- 19 sources / 181 Receivers
- Noise : SNR=10
- 30 iterations
- 120 cores
- Polynomial basis : Nodal

Computational time :

- RK2 : **3h15**
- RK4 : **4h30**
- AB3 : **5h10**

Cost function evolution :



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- 19 sources / 181 Receivers
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Computational time :

- Nodal : **3h15**
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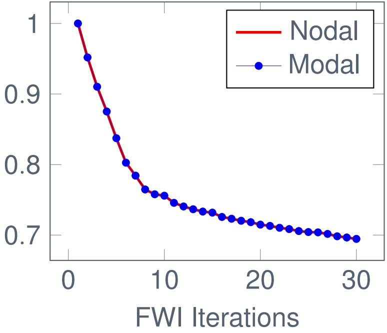
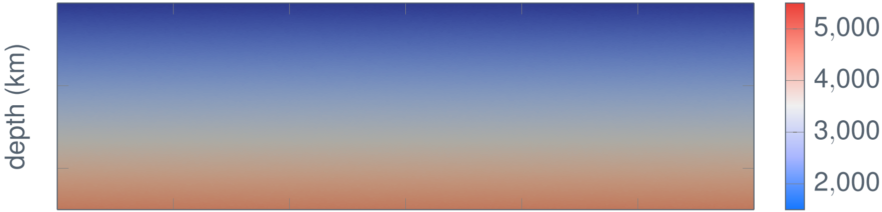
Cost function evolution :



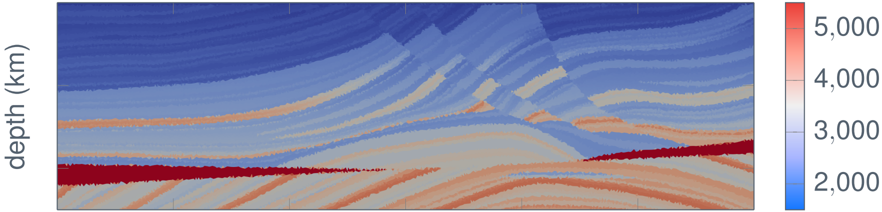
[1] Chan J. and Warburton T.
GPU-Accelerated Bernstein Bézier Discontinuous Galerkin Methods for Wave Problems
SIAM Journal on Scientific Computing 2017

2D Multiscale Reconstructions
Reconstruction with an initial smooth model

Initial $\boldsymbol{c}$ Model $m \cdot s^{-1}$



Target $\boldsymbol{c}$ Model $m \cdot s^{-1}$



2D Multiscale Reconstructions

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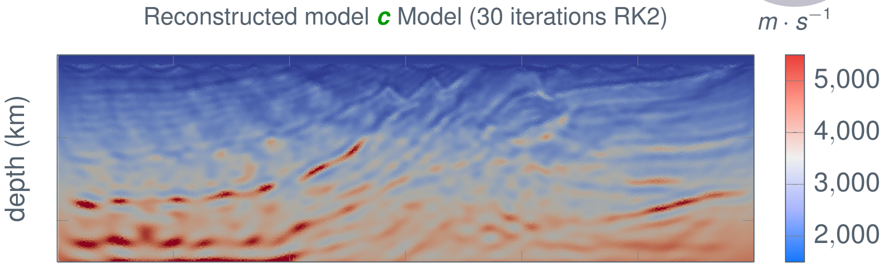
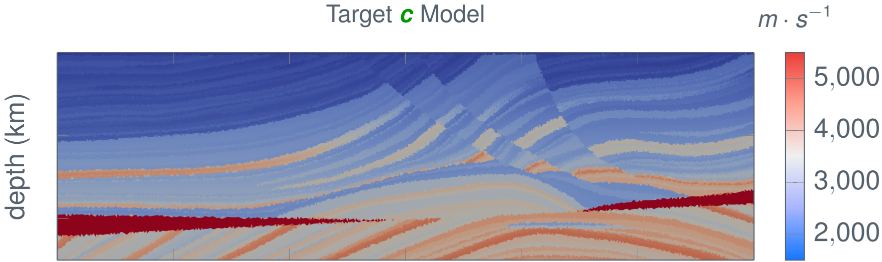




2D Multiscale Reconstructions

Multiscale Principle [1]

Filtered Traces :

Low Frequencies

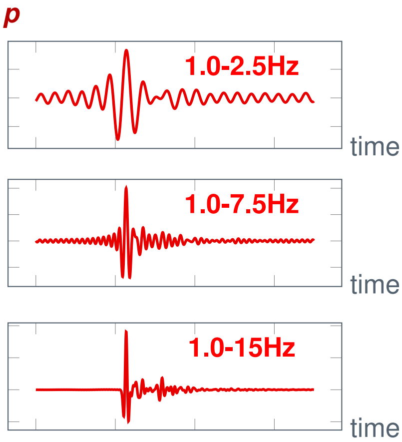
$\Longleftrightarrow$

Reconstruct **coarse** structures

High Frequencies

$\Longleftrightarrow$

Reconstruct **small** structures



[1] C. Bunks, F. M. Saleck, S. Zaleski, and G. Chavent
Multiscale seismic waveform inversion
GEOPHYSICS, Vol. 60, No. 5, 1995

2D Multiscale Reconstructions

Multiscale Principle [1]

Heuristic Illustration :

Low Frequencies

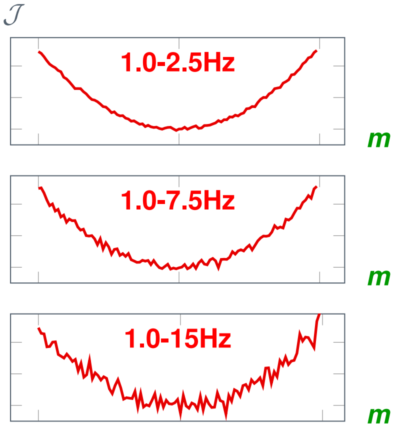
$\Longleftrightarrow$

Reconstruct **coarse** structures

High Frequencies

$\Longleftrightarrow$

Reconstruct **small** structures



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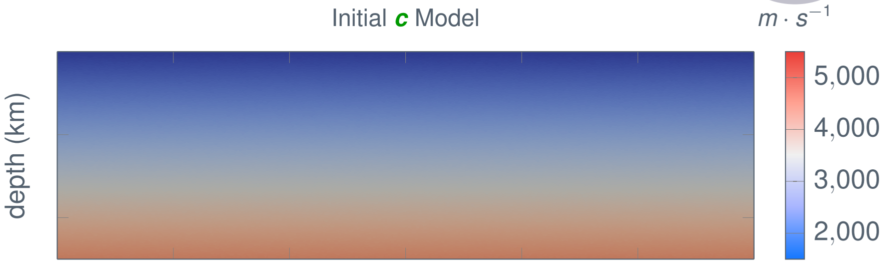
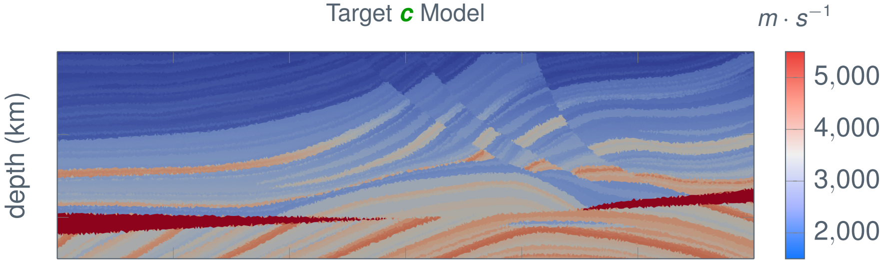
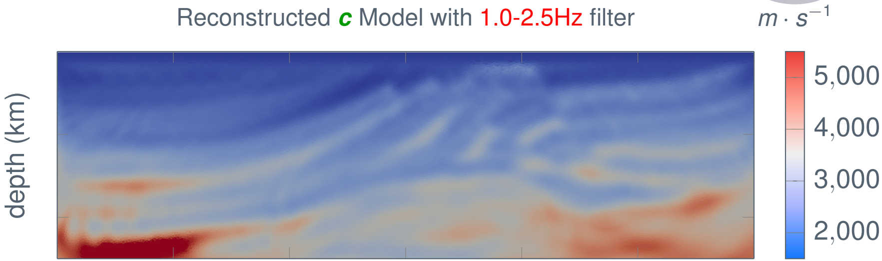




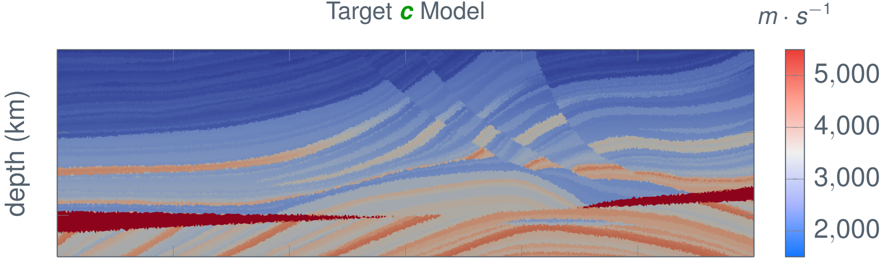
2D Multiscale Reconstructions

Reconstruction with an initial smooth model

Reconstructed ***c*** Model with 1.0-2.5Hz filter



Target ***c*** Model



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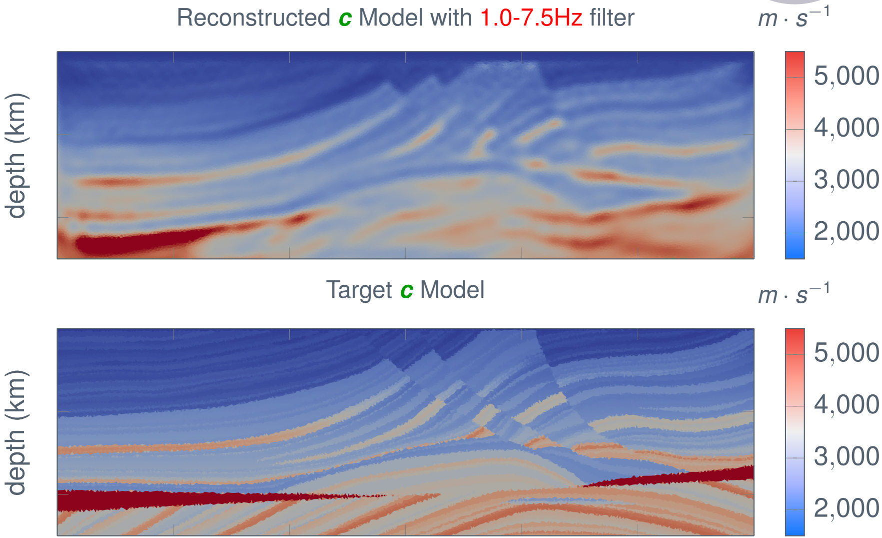
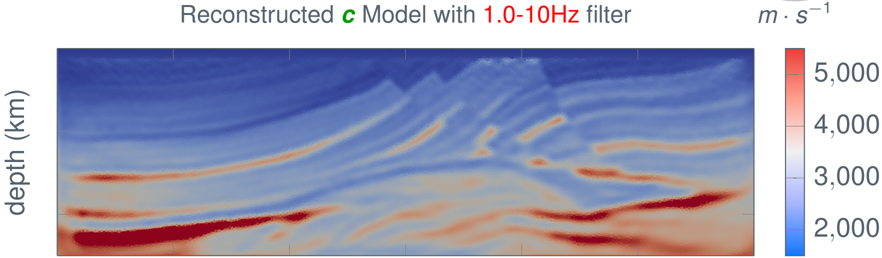


2D Multiscale Reconstructions

Reconstruction with an initial smooth model

Reconstructed ***c*** Model with 1.0-10Hz filter



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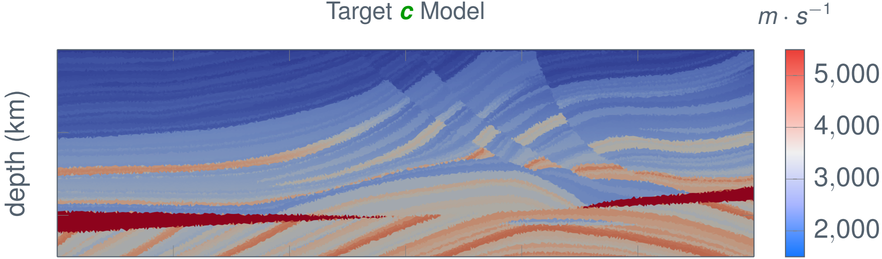
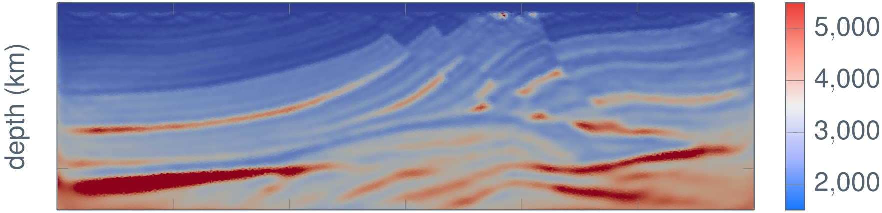


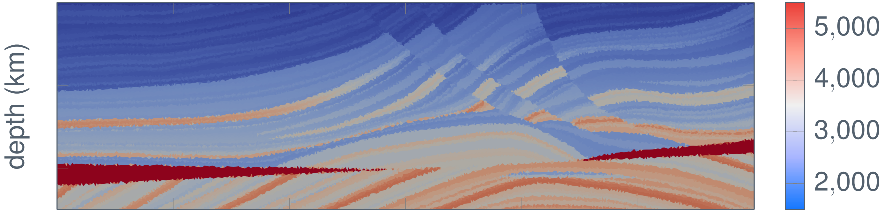
2D Multiscale Reconstructions

Reconstruction with an initial smooth model

Reconstructed ***c*** Model with 1.0-15Hz filter $m \cdot s^{-1}$



Target ***c*** Model $m \cdot s^{-1}$



2D Multiscale Reconstructions

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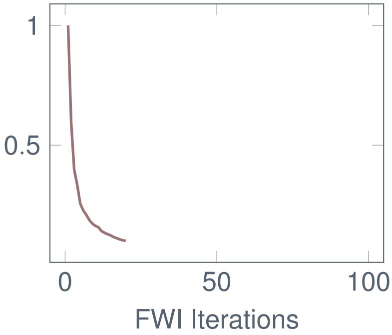
- Computation time : 10h
- Frequencies : 1-2.5Hz

Cost function evolution :



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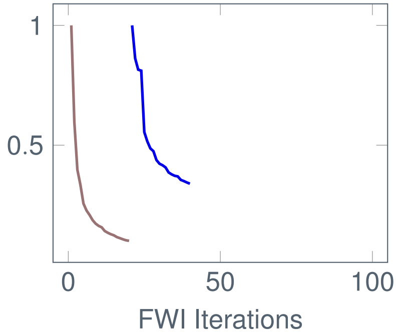

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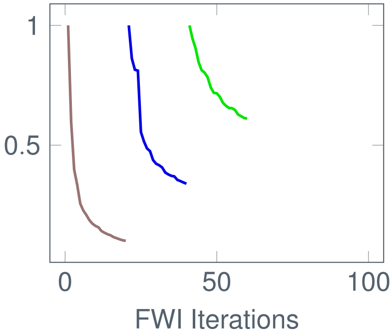

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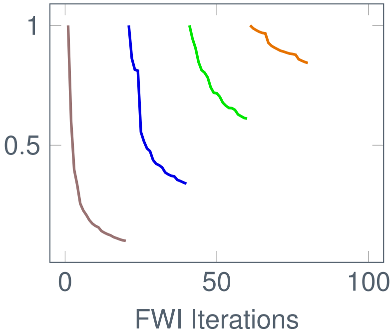

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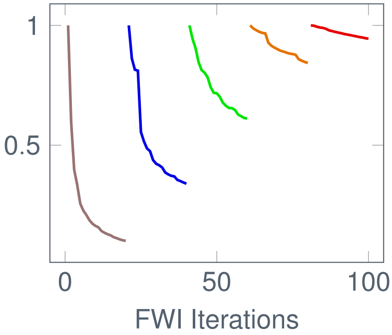
Cost function evolution :



Conclusion

Main Results :

- 2D Acoustic Reconstruction performed with different discretization
- Multiscale FWI implemented and working on Marmousi

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Perspectives :

- Perform reconstruction on other test cases (2D/3D)
- Develop enhanced optimizers (NLCG, Limited BFGS)
- Adapt the code to use High order Model
- Extend the code to elastic and elasto-acoustic propagator
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Thank you.

Annexe

Adjoint State Method

Lagrangian fonctional [1] :

$$\mathcal{L}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\lambda}}, \boldsymbol{m}) = \frac{1}{2}||d_{obs} - \mathcal{R}(\widehat{\boldsymbol{u}})||^2 + < Forward_{\boldsymbol{m}}(\widehat{\boldsymbol{u}}) - f_p, \widehat{\boldsymbol{\lambda}} >$$

If $\widehat{\boldsymbol{u}} = \boldsymbol{u}$ Solution of the Direct Problem $\Longleftrightarrow$ $(Forward_{\boldsymbol{m}}(\boldsymbol{u}) - f_p = 0)$:

$$\mathcal{J}(\boldsymbol{m}) = \mathcal{L}(\boldsymbol{u}, \widehat{\boldsymbol{\lambda}}, \boldsymbol{m})$$

[1] Plessix R-E
A review of the adjoint-state method for computing the gradient of a functional with geophysical applications
Geophysical Journal International, Volume 167, Issue 2, 2006

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$$(\mathcal{R}^* d_{obs} - \boldsymbol{u}) + Forward^*_{\boldsymbol{m}}(\boldsymbol{\lambda}) = 0$$

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For $Forward_{\boldsymbol{m}}(\boldsymbol{u}) - f_p = 0$:

$$\partial_{\boldsymbol{m}_i} \mathcal{J}(\boldsymbol{m}) = \partial_{\boldsymbol{m}_i} \mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{m}) = \partial_{\boldsymbol{m}_i} < Forward_{\boldsymbol{m}}(\boldsymbol{u}), \boldsymbol{\lambda} >$$

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Adjoint Formulation

Continuous
Direct Problem

Adjoint Formulation

Continuous Direct Problem $\xrightarrow{\textbf{Adjoint}}$ *Continuous Adjoint Problem*

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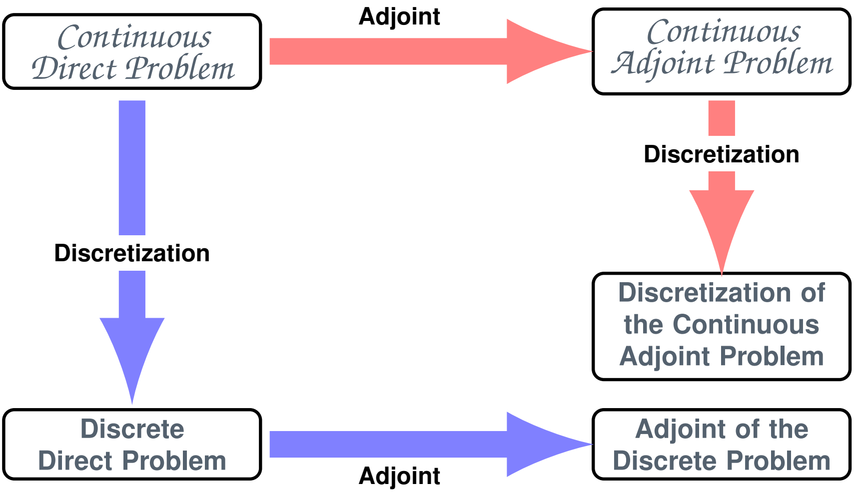
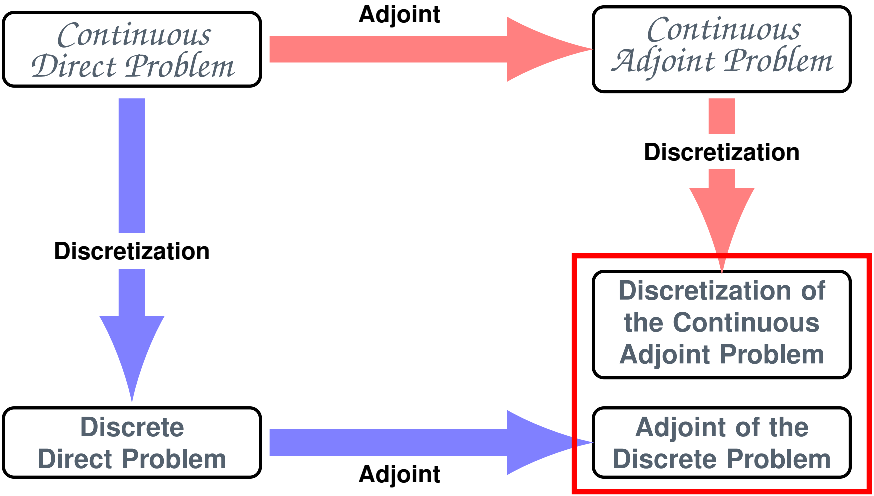
Discretization

Discretization of the Continuous Adjoint Problem

Adjoint Formulation



Adjoint Formulation



AtD : Adjoint then Discretized Strategy

$$\mathcal{J}(\boldsymbol{p}) = \frac{1}{2}||d_{obs} - R\boldsymbol{p}||^2$$

$$\begin{cases} \dfrac{1}{\rho \boldsymbol{c}^2}\dfrac{\partial \boldsymbol{p}}{\partial t} + \nabla \cdot \boldsymbol{v} = f_p \quad \text{on } \boldsymbol{\Omega} \\ \rho \dfrac{\partial \boldsymbol{v}}{\partial t} + \nabla \boldsymbol{p} = 0 \quad \text{on } \boldsymbol{\Omega} \\ \boldsymbol{p} = 0 \quad \text{on } \boldsymbol{\Gamma_1} \\ \dfrac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{c}\nabla \boldsymbol{p} \cdot \boldsymbol{n} = 0 \quad \text{on } \boldsymbol{\Gamma_2} \\ \boldsymbol{p}(0) = 0, \quad \boldsymbol{v}(0) = 0 \end{cases} \qquad \begin{cases} \dfrac{1}{\rho \boldsymbol{c}^2}\dfrac{\partial \boldsymbol{\lambda_1}}{\partial t} + \nabla \cdot \boldsymbol{\lambda_2} = \dfrac{\partial \mathcal{J}}{\partial \boldsymbol{p}} \quad \text{on } \boldsymbol{\Omega} \\ \rho \dfrac{\partial \boldsymbol{\lambda_2}}{\partial t} + \nabla \boldsymbol{\lambda_1} = 0 \quad \text{on } \boldsymbol{\Omega} \\ \boldsymbol{\lambda_1} = 0 \quad \text{on } \boldsymbol{\Gamma_1} \\ \dfrac{\partial \boldsymbol{\lambda_1}}{\partial t} - \boldsymbol{c}\nabla \boldsymbol{\lambda_1} \cdot \boldsymbol{n} = 0 \quad \text{on } \boldsymbol{\Gamma_2} \\ \boldsymbol{\lambda_1}(T) = 0, \quad \boldsymbol{\lambda_2}(T) = 0 \end{cases}$$

$$t \in [0, T] \qquad\qquad t \in [T, 0]$$

AtD : Adjoint then Discretized Strategy

$$\mathcal{J}(\boldsymbol{p}) = \frac{1}{2}||d_{obs} - R\boldsymbol{p}||^2$$

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{U}}^n}{\partial t} = A\bar{\boldsymbol{U}}^n + \bar{\boldsymbol{F}}^n \\ \text{With :} \quad \bar{\boldsymbol{U}}^n = \begin{pmatrix} \bar{\boldsymbol{P}}^n \\ \bar{\boldsymbol{V}}^n \end{pmatrix} \end{cases}$$

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{\Lambda}}^n}{\partial t} = A\bar{\boldsymbol{\Lambda}}^n + R^*(R\bar{\boldsymbol{U}}^n - d_{obs}) \\ \text{With :} \quad \bar{\boldsymbol{\Lambda}}^n = \begin{pmatrix} \bar{\boldsymbol{\Lambda}}_1{}^n \\ \bar{\boldsymbol{\Lambda}}_2{}^n \end{pmatrix} \end{cases}$$

Time-steps going Forward

Time-steps going Backward

All time scheme can be summed-up such as :

$$\boldsymbol{L}\bar{\boldsymbol{U}} = \boldsymbol{E}\bar{\boldsymbol{F}}$$

RK4 time-scheme leads to :

$$\bar{\boldsymbol{U}}^{n+1} = B\bar{\boldsymbol{U}}^{n} + \boldsymbol{C_0}\bar{\boldsymbol{F}}^{n} + \boldsymbol{C_{\frac{1}{2}}}\bar{\boldsymbol{F}}^{n+\frac{1}{2}} + \boldsymbol{C_1}\bar{\boldsymbol{F}}^{n+1}$$

$$\boldsymbol{L}\bar{\boldsymbol{U}} = \boldsymbol{E}\bar{\boldsymbol{F}} = \bar{\boldsymbol{G}}$$

$$\begin{pmatrix} I & & & & \\ -B & I & & & \\ & -B & I & & \\ & & \ddots & \ddots & \\ & & & -B & I \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{U}}^{0} \\ \bar{\boldsymbol{U}}^{1} \\ \bar{\boldsymbol{U}}^{2} \\ \vdots \\ \bar{\boldsymbol{U}}^{n} \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{G}}^{0} \\ \bar{\boldsymbol{G}}^{1} \\ \bar{\boldsymbol{G}}^{2} \\ \vdots \\ \bar{\boldsymbol{G}}^{n} \end{pmatrix}$$

DtA : Discretize then Adjoint Strategy

All time scheme can be summed-up such as :

$$\boldsymbol{L}\bar{\boldsymbol{U}} = \boldsymbol{E}\bar{\boldsymbol{F}}$$

We are looking for a Discrete Adjoint state satisfying :

$$\boldsymbol{L}^{*}\bar{\boldsymbol{\Lambda}} = -R^{*}(d_{obs} - R\bar{\boldsymbol{U}})$$

With the adjoint operator $\boldsymbol{L}^{*}$ satisfying :

$$< \boldsymbol{L}\bar{\boldsymbol{U}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, \boldsymbol{L}^{*}\bar{\boldsymbol{\Lambda}} >$$

DtA : Discretize then Adjoint Strategy

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We are looking for a Discrete Adjoint state satisfying :

$$\boldsymbol{L}^*\bar{\boldsymbol{\Lambda}} = -R^*(d_{obs} - R\bar{\boldsymbol{U}}) = \bar{\boldsymbol{D}}$$

With the adjoint operator $\boldsymbol{L}^*$ satisfying :

$$< \boldsymbol{L}\bar{\boldsymbol{U}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, \boldsymbol{L}^*\bar{\boldsymbol{\Lambda}} >$$

$$< \bar{\boldsymbol{G}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, \bar{\boldsymbol{D}} > \quad \text{(Adjoint Test)}$$

Adjoint test succeeds $\Longleftrightarrow$ operator $\boldsymbol{L}^*$ well established

DtA : Discretize then Adjoint Strategy

Example with RK4

RK4 time-scheme leads to :

$$\bar{\boldsymbol{U}}^{n+1} = B\bar{\boldsymbol{U}}^{n} + \boldsymbol{C_0}\bar{\boldsymbol{F}}^{n} + \boldsymbol{C_{\frac{1}{2}}}\bar{\boldsymbol{F}}^{n+\frac{1}{2}} + \boldsymbol{C_1}\bar{\boldsymbol{F}}^{n+1}$$

$$\boldsymbol{L}\bar{\boldsymbol{U}} = \boldsymbol{E}\bar{\boldsymbol{F}} = \bar{\boldsymbol{G}}$$

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So :

$$\boldsymbol{L}^{*} = \begin{pmatrix} I & -B^{*} & & & \\ & I & -B^{*} & & \\ & & \ddots & \ddots & \\ & & & I & -B^{*} \\ & & & & I \end{pmatrix}$$

Adjoint Strategies Comparison

Adjoint Then Discretize

- \+ Physical approach
- \+ Same discrete operators for Forward and Backward
- \- - Approximate gradient [1]
- Consistent with the discretization

Discretize then Adjoint

- \+ Numerical approach
- \+ Has an Adjoint Test
- \- Tremendous work to develop the adjoint operators
- Non-consistency of the adjoint state [2]

[1] Sirkes, Ziv and Tziperman, Eli
Finite Difference of Adjoint or Adjoint of Finite Difference ?
1997

[2] Sei Alain and Symes William
A Note on Consistency and Adjointness for Numerical Schemes
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Discretize then Adjoint

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- \+ / - Has an Adjoint Test **(in theory)**
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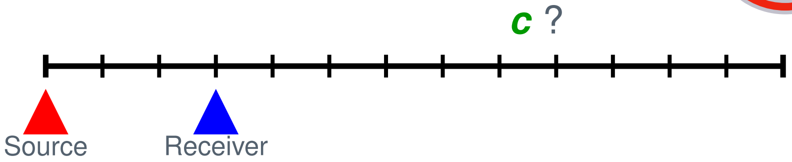
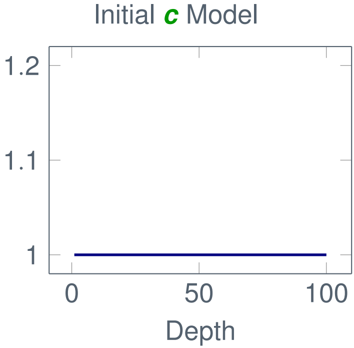
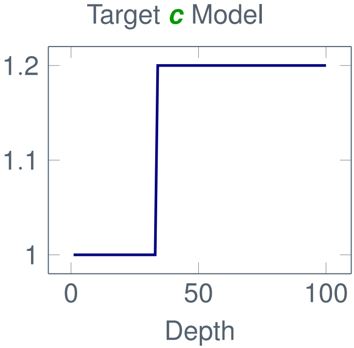
1D Preliminary tests







1D Preliminary tests :

1D FWI :

- Lagrange / B-Bézier Operators
- RK4 / AB3 time-schemes

Gradient expression :

$$\nabla_{\boldsymbol{c}} \mathcal{J} = -\int_0^T \int_\Omega \frac{2}{\rho \boldsymbol{c}^3} \frac{\partial \boldsymbol{p}}{\partial t} \boldsymbol{\lambda_1} \, d\Omega dt$$

1D Preliminary tests :

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Adjoint test passed with :

- With a canonical space inner-product ($< u, v >_X = \sum_i u_i v_i$)
- With a M-space inner product ($< u, v >_X^M = < Mu, v >_X$)

Gradient expression :

$$\nabla_{\mathbf{c}} \mathcal{J} = -\int_0^T \int_\Omega \frac{2}{\rho \mathbf{c}^3} \frac{\partial \mathbf{p}}{\partial t} \boldsymbol{\lambda}_1 d\Omega dt$$

```
./run
--- Adjoint test ----
 inner product U/D 553123.57586755091
 inner product G/Q 553123.57586756046
```

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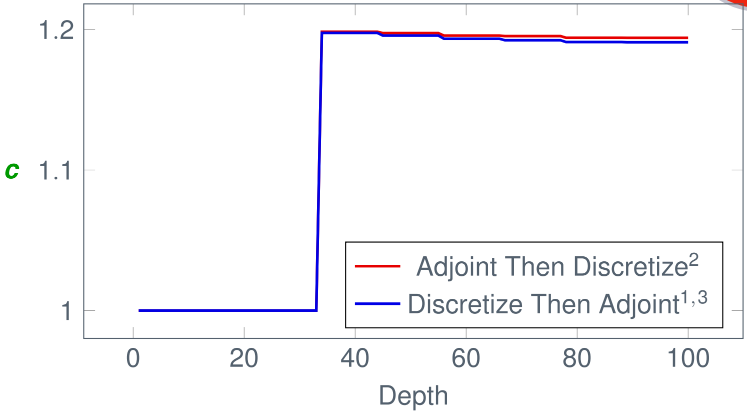
Gradient expression :

$$\nabla_{c}\mathcal{J} = -\int_0^T \int_\Omega \frac{2}{\rho c^3} \frac{\partial p}{\partial t} \lambda_1 \, d\Omega dt$$

```
./run
--- Adjoint test ----
 inner product U/D 553123.57586755091
 inner product G/Q 553123.57586756046
./run
--- Adjoint test ----
 inner product U/D -75077.332007383695
 inner product G/Q -75077.332007386358
./run
--- Adjoint test ----
 inner product U/D 125669.89223600870
 inner product G/Q 125669.89223600952
```



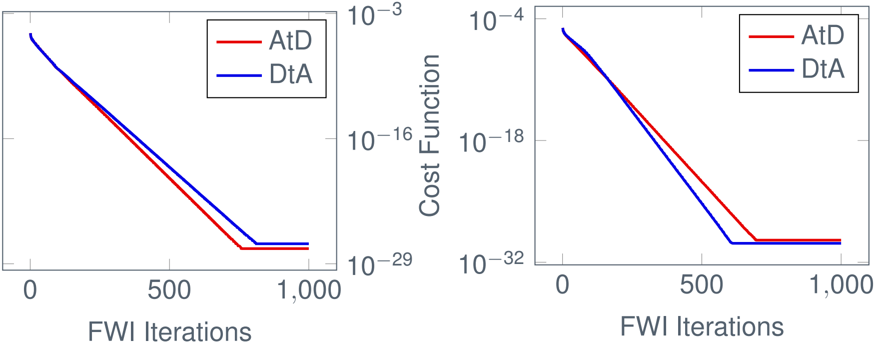
c Model at the 100th FWI iteration

[2] With Bernstein-Bézier elements and AB3 time scheme
[3] With canonical scalar product

1D Velocity Model Reconstructions

With RK4 :

With AB3 :



1D Velocity Model Reconstructions

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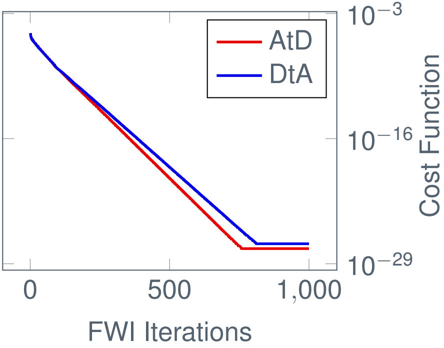
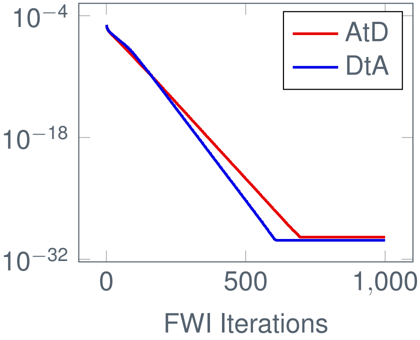


With AB3 :



- For RK4 scheme : AtD is slighly better than DtA
- For AB3 scheme : DtA is slighly better than AtD
- No predominant behaviour