Spectral dominance of complex roots for single-delay linear equations ⋆

Guilherme Mazanti [*,**] **Islam Boussaada** [*,**]
Silviu-Iulian Niculescu [*] **Tomáš Vyhlídal** [***]

[*] *University Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes (L2S), Inria Saclay, DISCO Team, 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France.*
{first name.last name}@l2s.centralesupelec.fr
[**] *Institut Polytechnique des Sciences Avancées (IPSA) 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France.*
[***] *Dept. of Instrumentation and Control Eng., and Center of Advanced Aerospace Technology, Faculty of Mechanical Engineering, Czech Technical University in Prague.*
Technická 4, 166 07, Praha 6, Czechia.
tomas.vyhlidal@fs.cvut.cz

Abstract: This paper provides necessary and sufficient conditions for the existence of a pair of complex conjugate roots, each of multiplicity two, in the spectrum of a linear time-invariant single-delay equation of retarded type. This pair of roots is also shown to be always strictly dominant, determining thus the asymptotic behavior of the system. The proof of this result is based on the corresponding result for real roots of multiplicity four, continuous dependence of roots with respect to parameters, and the study of crossing imaginary roots. We also present how this design can be applied to vibration suppression and flexible mode compensation.

Keywords: Time-delay equations, stability analysis, spectral methods, root assignment, crossing imaginary roots.

1. INTRODUCTION

In this paper, we consider a linear time-invariant equation with a single delay of the form

$$y''(t)+a_1y'(t)+a_0y(t)+\alpha_1y'(t-\tau)+\alpha_0y(t-\tau)=0, \quad (1)$$

where the coefficients $a_1, a_0, \alpha_1, \alpha_0$ are real numbers and the delay τ is a positive real number. Equations of the form (1) are said to be delayed equations of *retarded* type since the derivative of highest order only appears in the non-delayed term $y''(t)$.

Time delays are useful for modeling propagation phenomena, such as of material, energy, or information, with a finite propagation speed, this propagation taking place typically between parts of a complex system. For this reason, equations and systems with time delays have been widely used in several scientific and technological domains in which modeling such propagation phenomena is important, such as in biology, chemistry, economics, physics, or engineering. Due to these applications and the challenging mathematical problems arising in their analysis, time-delay systems have been the subject of much attention by researchers in several fields, in particular since the 1950s and 1960s, such as, for instance, in Bellman and Cooke (1963); Halanay (1966); Pinney (1958). We refer to Diekmann et al. (1995); Gopalsamy (1992); Gu et al. (2003); Hale and Verduyn Lunel (1993); Insperger and Stépán (2011); Li et al. (2017); Michiels and Niculescu (2014); Stépán (1989) for details on time-delay systems and their applications.

The stability analysis of time-delay systems has attracted much research effort and is an active field (see, e.g., Abdallah et al. (1993); Chen et al. (1995); Cooke and van den Driessche (1986); Gu et al. (2003); Michiels and Niculescu (2014); Olgac and Sipahi (2002); Sipahi et al. (2011)). A usual technique for addressing stability of linear time-invariant systems in the delay-free situation is based on spectral methods and consists in considering the corresponding characteristic polynomial, whose complex roots determine the asymptotic behavior of solutions of the system. This technique also carries over for linear time-invariant systems with delays, whose asymptotic behavior can also be characterized in terms of complex roots of a certain characteristic function (see, e.g., (Michiels and Niculescu, 2014, Proposition 1.13)). For (1), this characteristic function is

$$\Delta(s)=s^2+a_1s+a_0+e^{-s\tau}(\alpha_1s+\alpha_0). \quad (2)$$

Similarly to the delay-free case, all solutions of (1) converge exponentially fast to 0 if and only if $\operatorname{Re} s<0$ for every $s\in\mathbb{C}$ such that $\Delta(s)=0$, and the asymptotic behavior of solutions of (1) is determined by the real number $\gamma_0=\sup\{\operatorname{Re} s \mid s\in\mathbb{C},\ \Delta(s)=0\}$, called the *spectral abscissa* of Δ.

Entire functions such as Δ that can be written under the form $Q(s)=\sum_{k=1}^{\ell}p_k(s)e^{\lambda_k s}$ for some polynomials with

⋆ Corresponding author: Guilherme Mazanti (guilherme.mazanti@l2s.centralesupelec.fr)

real coefficients $p_1, \dots, p_\ell$ and pairwise distinct real numbers $\lambda_1, \dots, \lambda_\ell$ are called *quasipolynomials*. The interest in studying quasipolynomials come from the fact that, when $\lambda_k \leq 0$ for every k, they are characteristic equations of linear time-invariant delayed equations.

One usually defines the *degree* of a quasipolynomial Q as above to be $D = \ell + \delta - 1$, where δ is the sum of the degrees of $p_1, \dots, p_\ell$ (see, e.g., Wielonsky (2001); Berenstein and Gay (1995)). In particular, the degree of Δ in (2) is $D = 2 + 3 - 1 = 4$. Contrarily to the case of polynomials, the degree of a quasipolynomial does not determine the number of roots of the quasipolynomial, which is infinite except in trivial cases. However, similarly to polynomials, the degree does have a link with multiplicities of roots. More precisely, a classical result on quasipolynomials provided in (Pólya and Szegő, 1998, Problem 206.2), known as the *Pólya–Szegő bound*, implies that, given a quasipolynomial Q of degree $D \geq 0$, the multiplicity of any root of Q does not exceed D. For the quasipolynomial Δ from (2), this means that any of its roots has multiplicity at most 4. Recent works such as Boussaada and Niculescu (2016a,b) have provided characterizations of multiple roots of quasipolynomials using approaches based on Birkhoff and Vandermonde matrices.

When studying the roots of a quasipolynomial in order to analyze the stability of a time-delay system, only the rightmost roots on the complex plane are important for determining the system's asymptotic behavior. These roots are usually called *dominant roots* and can be defined as follows.

Definition 1. Let $Q : \mathbb{C} \to \mathbb{C}$ and $s_0 \in \mathbb{C}$.

(a) We say that s_0 is a *dominant* (respectively, *strictly dominant*) root of Q if $Q(s_0) = 0$ and, for every $s \in \mathbb{C} \setminus \{s_0\}$ such that $Q(s) = 0$, one has $\operatorname{Re} s \leq \operatorname{Re} s_0$ (respectively, $\operatorname{Re} s < \operatorname{Re} s_0$).

(b) We say that s_0 and its complex conjugate $\overline{s}_0$ are a pair of *dominant* (respectively, *strictly dominant*) roots of Q if $Q(s_0) = Q(\overline{s}_0) = 0$ and, for every $s \in \mathbb{C} \setminus \{s_0, \overline{s}_0\}$ such that $Q(s) = 0$, one has $\operatorname{Re} s \leq \operatorname{Re} s_0$ (respectively, $\operatorname{Re} s < \operatorname{Re} s_0$).

Dominant roots may not exist in general, but they always exist for functions of the form (2) (see, e.g., (Hale and Verduyn Lunel, 1993, Chapter 1, Lemma 4.1)). Exponential stability of (1) is equivalent to the dominant roots of Δ having negative real part.

It has been observed in several works that real roots of high multiplicity tend to be dominant, a property known as *multiplicity-induced dominance* (MID for short). We refer the reader, for instance, to Boussaada et al. (2018), in which MID was proved for (2) in the case $\alpha_1 = 0$ for a real root of multiplicity 3 thanks to a suitable factorization of Δ, and to Boussaada et al. (2020), which considers the case $\alpha_1 \neq 0$ and proves dominance of a real root of multiplicity 4 using Cauchy's argument principle. MID is also reminiscent of the fact that, for delay-free systems with an affine constraint on their coefficients, the spectral abscissa is minimized on a polynomial with a single root of maximal multiplicity (see Blondel et al. (2012); Chen (1979)), with similar properties for some time-delay systems obtained in Michiels et al. (2002); Ramírez et al. (2016); Vanbiervliet et al. (2008). The interest in considering multiple roots does not rely on the multiplicity itself, but rather on its connection with dominance and the corresponding implications for stability analysis and control design.

The main goal of this paper is to investigate whether MID holds for Δ when assigning a pair of complex conjugate roots instead of a real root. Designing a system to have a pair of dominant complex conjugate roots may have several practical interests, as highlighted in Kuře et al. (2018), in which a robust delayed resonator is designed by assigning double imaginary roots, and as we also illustrate in Section 3.2. The questions we address in this paper are the following.

(Q1) Is it possible to choose $a_1, a_0, \alpha_1, \alpha_0 \in \mathbb{R}$ in such a way that a given complex number s_0 and its complex conjugate $\overline{s}_0$ are roots of multiplicity 2 of Δ?

(Q2) Under the above choice, do s_0 and $\overline{s}_0$ form a pair of (strictly) dominant roots?

Our main result, Theorem 2, in addition to recalling the situation for real root assignment, also provides affirmative answers to both questions. Question (Q1) can be addressed in a straightforward manner, whereas the answer to (Q2) relies on the continuity of the other roots of Δ with respect to the assigned root and a study of crossing imaginary roots, using techniques similar in spirit to those of Boussaada and Niculescu (2016b).

The paper is organized as follows: Notations used in the paper are standard. Section 2 provides the statement of our main result, Theorem 2, as well as a sketch of its proof, while Section 3 contains illustrative examples.

2. MAIN RESULT

The main result of this paper is the following.

Theorem 2. Consider the quasipolynomial Δ given by (2) and let $s_0 \in \mathbb{C}$, $\sigma_0 = \operatorname{Re} s_0$, and $\theta_0 = \operatorname{Im} s_0$.

(a) Assume that $\theta_0 = 0$. Then s_0 is a root of multiplicity 4 of Δ if and only if the coefficients $a_0, a_1, \alpha_0, \alpha_1$, the value σ_0, and the delay τ satisfy the relations

$$a_1 = -\frac{4}{\tau} - 2\sigma_0, \qquad a_0 = \frac{6}{\tau^2} + \frac{4}{\tau}\sigma_0 + \sigma_0^2, \tag{3a}$$

$$\alpha_1 = -\frac{2}{\tau} e^{\sigma_0 \tau}, \qquad \alpha_0 = \frac{2}{\tau} e^{\sigma_0 \tau} \left(\sigma_0 - \frac{3}{\tau}\right). \tag{3b}$$

(b) Assume that $\theta_0 \neq 0$. Then s_0 and $\overline{s}_0$ are roots of multiplicity 2 of Δ if and only if the coefficients $a_0, a_1, \alpha_0, \alpha_1$, the values σ_0 and θ_0, and the delay τ satisfy the relations

$$a_1 = -2\sigma_0 - 2\theta_0 \frac{\tau\theta_0 - \sin(\tau\theta_0)\cos(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}, \tag{4a}$$

$$a_0 = \sigma_0^2 + 2\sigma_0\theta_0 \frac{\tau\theta_0 - \sin(\tau\theta_0)\cos(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)} + \theta_0^2 \frac{\tau^2\theta_0^2 + \sin^2(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}, \tag{4b}$$

$$\alpha_1 = 2\theta_0 e^{\sigma_0\tau} \frac{\tau\theta_0 \cos(\tau\theta_0) - \sin(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}, \tag{4c}$$

$$\alpha_0 = 2\theta_0 e^{\sigma_0 \tau}\left(\sigma_0 \frac{\sin(\tau\theta_0) - \tau\theta_0\cos(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)} - \frac{\tau\theta_0^2 \sin(\tau\theta_0)}{\tau^2\theta_0^2 - \sin^2(\tau\theta_0)}\right). \tag{4d}$$

(c) If (3) is satisfied, then s_0 is a strictly dominant root of Δ.

(d) If (4) is satisfied, then s_0 and $\overline{s}_0$ are a pair of strictly dominant roots of Δ.

Remark 3. The expressions of $a_1, a_0, \alpha_1, \alpha_0$ in (3) and (4) are singular with respect to τ as $\tau \to 0$. If one is interested in studying the behavior of the roots of Δ as $\tau \to 0$ when (3) or (4) is satisfied, one may consider instead the quasipolynomial $s \mapsto \tau^2\Delta(s)$, which has the same roots as Δ but whose coefficients are regular with respect to τ.

Remark 4. The expressions of $a_1, a_0, \alpha_1, \alpha_0$ in (4) are well-defined for every $\theta_0 \in \mathbb{R} \setminus \{0\}$ and $\tau > 0$, since $\sin^2(\tau\theta_0) = \tau^2\theta_0^2$ if and only if $\tau\theta_0 = 0$. Moreover, these expressions are even functions of θ_0 — as one might expect by symmetry since one is placing both roots s_0 and $\overline{s}_0$ — and they converge to the corresponding expressions in (3) as $\theta_0 \to 0$.

Up to a translation and a scaling of the spectrum represented by the change of variables $z = \tau(s - \sigma_0)$, one may reduce to the case $\sigma_0 = 0$ and $\tau = 1$, in which (3) reduces to $a_1 = -4$, $a_0 = 6$, $\alpha_1 = -2$, $\alpha_0 = -6$, yielding the quasipolynomial

$$\widehat{\Delta}_{\mathrm{R}}(z) = z^2 - 4z + 6 - e^{-z}(2z + 6), \tag{5}$$

and (4) reduces to

$$a_1 = -2\theta_0 \frac{\theta_0 - \sin\theta_0\cos\theta_0}{\theta_0^2 - \sin^2\theta_0}, \quad a_0 = \theta_0^2 \frac{\theta_0^2 + \sin^2\theta_0}{\theta_0^2 - \sin^2\theta_0}, \tag{6a}$$

$$\alpha_1 = 2\theta_0 \frac{\theta_0\cos\theta_0 - \sin\theta_0}{\theta_0^2 - \sin^2\theta_0}, \quad \alpha_0 = -\frac{2\theta_0^3\sin\theta_0}{\theta_0^2 - \sin^2\theta_0}, \tag{6b}$$

yielding the quasipolynomial

$$\widehat{\Delta}_{\mathrm{C}}(z;\theta_0) = z^2 - 2\theta_0 \frac{\theta_0 - \sin\theta_0\cos\theta_0}{\theta_0^2 - \sin^2\theta_0} z + \theta_0^2 \frac{\theta_0^2 + \sin^2\theta_0}{\theta_0^2 - \sin^2\theta_0} + e^{-z}\left(2\theta_0 \frac{\theta_0\cos\theta_0 - \sin\theta_0}{\theta_0^2 - \sin^2\theta_0} z - \frac{2\theta_0^3\sin\theta_0}{\theta_0^2 - \sin^2\theta_0}\right). \tag{7}$$

In the sequel of the paper, we use the convention, in accordance with Remark 4, that $\widehat{\Delta}_{\mathrm{C}}(\cdot;0) = \widehat{\Delta}_{\mathrm{R}}(\cdot)$.

We now provide the main ideas for the proof of Theorem 2. The complete proof can be found in an upcoming extended version of this paper.

Sketch of the proof. We consider only the case $s_0 = i\theta_0$ for $\theta_0 \geq 0$, since the general case can be reduced to it by the above change of variables. Assertions (a) and (c) have already been proved in Boussaada et al. (2020); Mazanti et al. (2020b). To prove assertion (b), we notice that, for real coefficients $a_1, a_0, \alpha_1, \alpha_0$, $s_0 = i\theta_0$ with $\theta_0 > 0$ satisfies $\Delta(s_0) = \Delta'(s_0) = 0$ if and only if (6) holds. In this case, one verifies that $\Delta''(s_0) \neq 0$, showing that the multiplicity of s_0 is indeed 2.

To prove (d), note that, by (c), 0 is a strictly dominant root of $\widehat{\Delta}_{\mathrm{C}}(\cdot;0)$ of multiplicity 4 and, as θ_0 increases, this root splits into two roots $\pm i\theta_0$ of $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ of multiplicity 2 each. By continuity of the roots of $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ with respect to θ_0, the roots $\pm i\theta_0$ will cease to be strictly dominant as θ_0 increases if and only if one root coming from ∞ appears at the right half-plane or one root with negative real part crosses the imaginary axis. The proof is completed by arguing by contradiction to show that none of these two cases may occur. □

3. ILLUSTRATIVE EXAMPLES

3.1 Roots of $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ as a function of θ_0

The quasipolynomial $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ from (7) is obtained by applying Theorem 2 to $s_0 = i\theta_0$ for some $\theta_0 \in \mathbb{R}$. Theorem 2 guarantees that the multiple roots $\pm i\theta_0$ are strictly dominant, but says nothing about how the roots on the open left half-plane behave. In order to get a grasp on their behavior, we have performed numerical computations of all roots of $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ on the region $\{s \in \mathbb{C} \mid -4.75 \leq \operatorname{Re} s \leq 0.25 \text{ and } -25 \leq \operatorname{Im} s \leq 25\}$ for several values of $\theta_0 \in [0, 8]$. The results are provided in Fig. 1, with different values of θ_0 being represented with different colors. All numerical computations have been performed using Python `cxroots` package, which implements numerical methods described in Kravanja and Van Barel (2000).

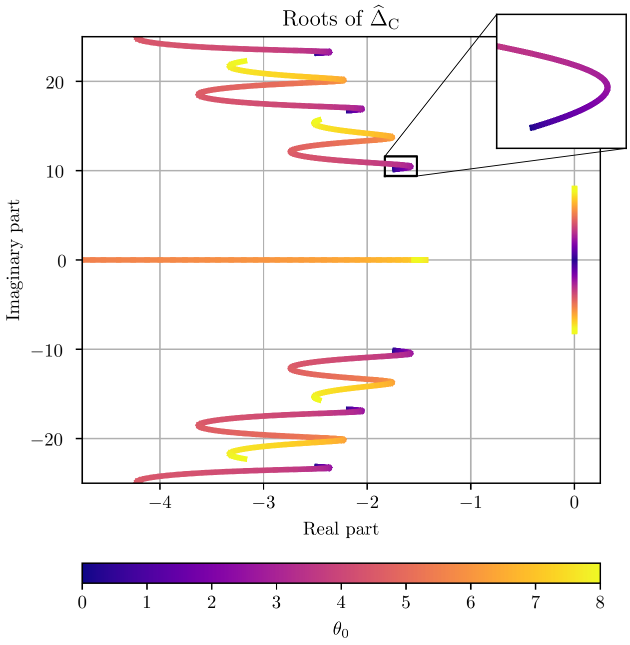

Fig. 1. Roots of $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ for $\theta_0 \in [0, 8]$, with a detailed view of the region $\{s \in \mathbb{C} \mid -1.80 \leq \operatorname{Re} s \leq -1.55 \text{ and } 10 \leq \operatorname{Im} s \leq 11\}$.

One observes in Fig. 1 the movement of the dominant roots $\pm i\theta_0$ along the imaginary axis. Concerning the other roots, as θ_0 increases, the imaginary parts of the non-real roots represented in the figure increase in absolute value, while the real parts oscillate. Table 1 synthesizes this oscillatory behavior for the first pair of non-dominant complex conjugate roots in Fig. 1 (in order of increasing absolute value of imaginary part) by presenting the values of θ_0 and the corresponding roots at local extrema of their real part, as well as for the initial and final values $\theta_0 = 0$ and $\theta_0 = 8$ used in the numerical computations.

Table 1. First pair of non-dominant complex conjugate roots in Fig. 1 at their initial and final positions and on local extrema of their real parts.

	θ_0	Roots
Initial	0	$-1.731 \pm 10.16i$
First local maximum	2.51	$-1.586 \pm 10.46i$
First local minimum	4.59	$-2.735 \pm 12.14i$
Second local maximum	6.19	$-1.764 \pm 13.74i$
Second local minimum	7.83	$-2.508 \pm 15.32i$
Final	8	$-2.466 \pm 15.66i$

We also notice in Fig. 1 the presence of a real-valued root. Its detailed behavior obtained from numerical computations for $\theta_0 \in [5.49, 10.00]$ is provided in Fig. 2, which is split in three different ranges for θ_0 corresponding to different observed behaviors of the root. This root first appears in the domain under consideration for $\theta_0 \approx 5.49$ and moves to the right, reaching a local maximum at $\theta_0 \approx 7.54$, at which point its value is approximately -1.437. It then starts moving to the left for $\theta_0 \in [7.54, 8.85]$. At $\theta_0 \approx 8.85$, a second real root appears in the domain under consideration, coming from $-\infty$ and moving to the right, and both roots meet, giving rise, when $\theta_0 \approx 8.88$, to a real root of multiplicity 2 whose value is approximately -3.927. For $\theta_0 \in [8.88, 10]$, these roots become a pair of complex conjugate roots which start moving to the right. As θ_0 increases beyond 10 (not represented in Fig. 2), one observes that this pair of roots oscillates like the other pairs of complex conjugate roots from Fig. 1.

3.2 Applications: vibration suppression and flexible mode compensation

We provide two engineering applications, with a common requirement for having a double root on the imaginary axis. The first application is *active vibration suppression* (AVS) and the second application is *flexible mode compensation* (FMC). The common feature of these two methods is that the purely imaginary roots $\pm i\omega$ of (2) are turned to imaginary zeros of the overall system. In AVS, ω is the frequency of an excitation force, while, for FMC, ω is the natural frequency of the flexible mode to be compensated. In both cases, the overall system magnitude at frequency ω is zero. The multiplicity two of the zero then increases the robustness in the vibration suppression or mode compensation. Before explaining these two applications in more detail, let us propose delay values τ. From the practical point of view, an intuitive choice for the delay is given by

$$\tau_k = \frac{k\pi}{\omega}, \qquad k = 1, 2, 3, \dots, \tag{8}$$

for which (4) gives $a_1 = -\frac{2\omega}{k\pi}$, $a_0 = \omega^2$, $\alpha_1 = (-1)^k \frac{2\omega}{k\pi}$, and $\alpha_0 = 0$. Thus, the characteristic function (2) turns to

$$\Delta_\omega(s) = s^2 + \frac{2}{\tau_k} \left((-1)^k e^{-s\tau_k} - 1 \right) s + \omega^2. \tag{9}$$

In the AVS application, we adapt the *delayed resonator scheme* proposed by Olgac and Holm-Hansen (1994) with a single root at $\pm i\omega$. Recently the concept has been adjusted by Kuře et al. (2018) with double roots at $\pm i\omega$ in order to enhance the robustness. Let us note that the solution in Kuře et al. (2018) required two time delays. Here, we provide a solution with a single delay. The scheme of the set-up is shown in Fig. 3. The system main body is a vibrating platform P excited by a periodic external force $f(t) = F\cos(\omega t)$, F denoting the force amplitude. In order to compensate fully the vibrations, the absorber A is actuated with the active feedback $u(t)$. The absorber dynamics is then given by

$$x_a''(t) + 2\zeta\Omega x_a'(t) + \Omega^2 x_a(t) = \frac{1}{m_a} u(t). \tag{10}$$

where ζ, Ω, m_a are the damping, natural frequency and mass of the physical absorber. Introducing the active feedback in the form

$$u(t) = m_a(\Omega^2 - \omega^2)x_a(t) + 2m_a \left(\zeta\Omega + \frac{1}{\tau_k} \right) x_a'(t) - 2m_a \frac{(-1)^k}{\tau_k} x_a'(t - \tau_k), \tag{11}$$

the characteristic function of the active absorber (10)–(11) is given by (9) with a double root at $\pm i\omega$. As demonstrated e.g. in Kuře et al. (2018), the transfer function $f \to x_p$ is in the form

$$G_{x_a f}(s) = \frac{\Delta_\omega(s)}{M(s)} \tag{12}$$

where $M(s)$ is a characteristic function of the closed loop system. Therefore, as required, the double roots at $\pm i\omega$ become double zeros of (12). This implies that no vibrations are transferred from f to x_p and the platform is fully silenced.

The scheme of the second application, FMC, is in Fig. 4. The proposed concept adapts an *inverse shaper* application elaborated in Vyhlídal et al. (2016). A typical application of this concept is position-control of a crane trolley (G: main body) with the aim to compensate the oscillatory modes of the suspended payload (F: flexible subsystem), i.e., the payload should not sway once the main body position y reaches the set-point value w. The architecture in Fig. 4 ensures the mode compensation also in the responses to the main-body disturbance d. For the crane application the mode of $F(s)$ to be compensated is assumed $\pm i\omega$, where $\omega = \sqrt{\frac{g}{L}}$, L is the length of the payload and g is gravitational acceleration.

The adaptation of the concept is in substituting the inverse shaper by the transfer function $\frac{1}{\Delta_\omega(s)}$. As can be seen from the transfer functions

$$T_{y_s w} = \frac{C(s)G(s)\Delta_\omega(s)}{\Delta_\omega(s) + C(s)G(s)} F(s), \tag{13}$$

$$T_{y_s d} = \frac{\Delta_\omega(s)}{\Delta_\omega(s) + C(s)G(s)} F(s), \tag{14}$$

with $C(s)$ denoting the feedback controller, the double root at $\pm i\omega$ compensates the oscillatory pole of $F(s)$. Analogously to the previous application, the root multiplicity two enhances the robustness in mode compensation. Let us note that if the mode to be compensated is damped, i.e. given by $-\zeta\omega \pm i\omega\sqrt{1-\zeta^2}$, the parameters of $\Delta_\omega(s)$ can be adapted according to (4).

For both above potential applications, only the concept was outlined with the simplest possible structure of $\Delta(s)$

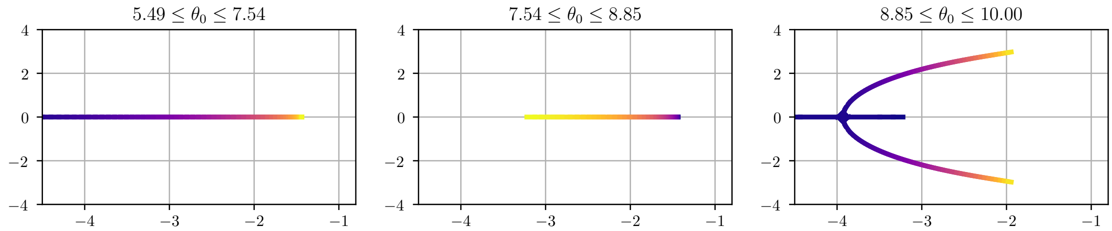

Fig. 2. Detailed view of the roots of $\widehat{\Delta}_{\mathrm{C}}(\cdot;\theta_0)$ around the real axis for different ranges of θ_0. In each figure, darker colors correspond to smaller values of θ_0.

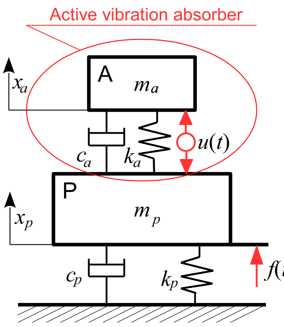

Fig. 3. Primary Structure (P), with an active vibration absorber (A) to suppress displacement x_p induced by harmonic disturbance force $f(t)$

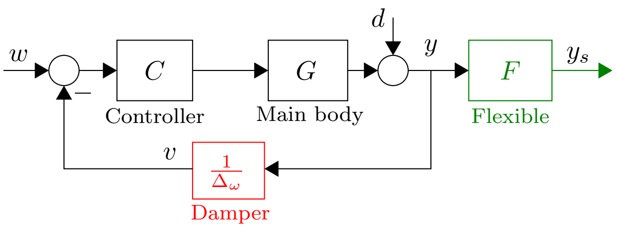

Fig. 4. Feedback interconnection with feedback damper

obtained for a purposefully selected delay value τ. A more detailed analysis is needed, mainly in studying the stability posture/margin of the overall systems with respect to the delay length. Possibly, selection of delay satisfying $0 < \tau < \frac{\pi}{\omega}$ can be beneficial. Then the parameter determining rules (4) are needed in their full complexity.

3.3 Equations of higher order

One may consider, instead of (1), a n-th order equation with derivatives of order up to $n-1$ in the delays, and the corresponding quasipolynomial Δ of degree $2n$ made of a n-th degree polynomial and a polynomial of degree $n-1$ multiplied by $e^{-s\tau}$. The problem of assigning a real root of multiplicity $2n$ and proving its dominance has already been considered in Mazanti et al. (2020a,b). As for the assignment of complex conjugate roots of multiplicity n each and proving their dominance, several arguments used in the present paper still hold with only minor modifications. For instance, the proof of Theorem 2(d) only requires continuity of the coefficients (4) of the quasipolynomial with respect to θ_0 as well as the MID property for the case of a real root of multiplicity $2n$. The main difficulty in generalizing the results of this paper to equations of higher order relies on providing suitable characterizations of the coefficients. Explicit characterizations such as (4) seem intractable in the general case, but one may still rely on implicit characterizations, such as those in (Boussaada and Niculescu, 2016b, Lemma 1).

4. CONCLUDING REMARKS

We have considered in this paper the multiplicity-induced-dominancy property for the linear time-invariant delay differential equation (1) when placing a pair of complex conjugate roots of maximal multiplicity of its characteristic quasipolynomial (2). Our main result, Theorem 2, provides necessary and sufficient conditions for a pair of complex conjugate numbers $\sigma_0 \pm i\theta_0$ being roots of maximal multiplicity of (2) and shows that, under these conditions, these roots are necessarily strictly dominant.

We have also presented, in Section 3.1, how other roots of (2) behave as the chosen roots $\sigma_0 \pm i\theta_0$ move away from the imaginary axis. Section 3.2 has illustrated the utility of our main result by presenting two engineering applications in which the proposed design is useful. Finally, we have discussed in Section 3.3 how the results of this paper can be generalized to higher-order equations.

ACKNOWLEDGEMENTS

This work is supported by a public grant overseen by the French National Research Agency (ANR) as part of the “Investissement d’Avenir” program, through the iCODE project funded by the IDEX Paris-Saclay, ANR-11-IDEX-0003-02. The authors also acknowledge the support of Institut Polytechnique des Sciences Avancées (IPSA). The fourth author acknowledges support from the ESIF, EU Operational Programme Research, Development and Education, and from the Center of Advanced Aerospace Technology (CZ.02.1.01/0.0/0.0/16_019/0000826), Faculty of Mechanical Engineering, Czech Technical University in Prague

REFERENCES

Abdallah, C., Dorato, P., Benites-Read, J., and Byrne, R. (1993). Delayed positive feedback can stabilize oscillatory systems. In *1993 American Control Conference*, 3106–3107. IEEE.

Bellman, R. and Cooke, K.L. (1963). *Differential-difference equations.* Academic Press, New York-London.

Berenstein, C.A. and Gay, R. (1995). *Complex analysis and special topics in harmonic analysis.* Springer-Verlag, New York.

Blondel, V.D., Gürbüzbalaban, M., Megretski, A., and Overton, M.L. (2012). Explicit solutions for root optimization of a polynomial family with one affine constraint. *IEEE Trans. Automat. Control*, 57(12), 3078–3089.

Boussaada, I. and Niculescu, S.I. (2016a). Characterizing the codimension of zero singularities for time-delay systems: a link with Vandermonde and Birkhoff incidence matrices. *Acta Appl. Math.*, 145, 47–88.

Boussaada, I. and Niculescu, S.I. (2016b). Tracking the algebraic multiplicity of crossing imaginary roots for generic quasipolynomials: a Vandermonde-based approach. *IEEE Trans. Automat. Control*, 61(6), 1601–1606.

Boussaada, I., Niculescu, S.I., El Ati, A., Perez-Ramos, R., and Trabelsi, K.L. (2020). Multiplicity-induced-dominancy in parametric second-order delay differential equations: Analysis and application in control design. *ESAIM Control Optim. Calc. Var.* To appear.

Boussaada, I., Tliba, S., Niculescu, S.I., Ünal, H.U., and Vyhlídal, T. (2018). Further remarks on the effect of multiple spectral values on the dynamics of time-delay systems. Application to the control of a mechanical system. *Linear Algebra Appl.*, 542, 589–604.

Chen, J., Gu, G., and Nett, C.N. (1995). A new method for computing delay margins for stability of linear delay systems. *Systems Control Lett.*, 26(2), 107–117.

Chen, R. (1979). *Output feedback stabilization of linear systems.* Ph.D. thesis, University of Florida.

Cooke, K.L. and van den Driessche, P. (1986). On zeroes of some transcendental equations. *Funkcial. Ekvac.*, 29(1), 77–90.

Diekmann, O., van Gils, S.A., Verduyn Lunel, S.M., and Walther, H.O. (1995). *Delay equations: Functional-, complex-, and nonlinear analysis*, volume 110 of *Applied Mathematical Sciences.* Springer-Verlag, New York.

Gopalsamy, K. (1992). *Stability and oscillations in delay differential equations of population dynamics*, volume 74 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht.

Gu, K., Kharitonov, V.L., and Chen, J. (2003). *Stability of time-delay systems.* Control Engineering. Birkhäuser Boston, Inc., Boston, MA.

Halanay, A. (1966). *Differential equations: Stability, oscillations, time lags.* Academic Press, New York-London.

Hale, J.K. and Verduyn Lunel, S.M. (1993). *Introduction to functional differential equations*, volume 99 of *Applied Mathematical Sciences.* Springer-Verlag, New York.

Insperger, T. and Stépán, G. (2011). *Semi-discretization for time-delay systems*, volume 178 of *Applied Mathematical Sciences.* Springer, New York. Stability and engineering applications.

Kravanja, P. and Van Barel, M. (2000). *Computing the zeros of analytic functions*, volume 1727 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin.

Kuře, M., Vyhlídal, T., Michiels, W., and Boussaada, I. (2018). Spectral design of robust delayed resonator by double-root assignment. *IFAC-PapersOnLine*, 51(14), 72–77. 14th IFAC Workshop on Time Delay Systems TDS 2018.

Li, X.G., Niculescu, S.I., Çela, A., Zhang, L., and Li, X. (2017). A frequency-sweeping framework for stability analysis of time-delay systems. *IEEE Trans. Automat. Control*, 62(8), 3701–3716.

Mazanti, G., Boussaada, I., and Niculescu, S.I. (2020a). Multiplicity-induced-dominancy for delay-differential equations of retarded type. Preprint arXiv:2002.06128.

Mazanti, G., Boussaada, I., and Niculescu, S.I. (2020b). On qualitative properties of single-delay linear retarded differential equations: Characteristic roots of maximal multiplicity are necessarily dominant. In *Proceedings of the 21st IFAC World Congress 2020 (to appear).*

Michiels, W., Engelborghs, K., Vansevenant, P., and Roose, D. (2002). Continuous pole placement for delay equations. *Automatica J. IFAC*, 38(5), 747–761.

Michiels, W. and Niculescu, S.I. (2014). *Stability, control, and computation for time-delay systems: An eigenvalue-based approach*, volume 27 of *Advances in Design and Control.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition.

Olgac, N. and Holm-Hansen, B. (1994). A novel active vibration absorption technique: delayed resonator. *Journal of Sound and Vibration*, 176(1), 93–104.

Olgac, N. and Sipahi, R. (2002). An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. *IEEE Trans. Automat. Control*, 47(5), 793–797.

Pinney, E. (1958). *Ordinary difference-differential equations.* University of California Press, Berkeley-Los Angeles.

Pólya, G. and Szegő, G. (1998). *Problems and theorems in analysis. I.* Classics in Mathematics. Springer-Verlag, Berlin. Series, integral calculus, theory of functions, Translated from the German by Dorothee Aeppli, Reprint of the 1978 English translation.

Ramírez, A., Mondié, S., Garrido, R., and Sipahi, R. (2016). Design of proportional-integral-retarded (PIR) controllers for second-order LTI systems. *IEEE Trans. Automat. Control*, 61(6), 1688–1693.

Sipahi, R., Niculescu, S.I., Abdallah, C.T., Michiels, W., and Gu, K. (2011). Stability and stabilization of systems with time delay: limitations and opportunities. *IEEE Control Syst. Mag.*, 31(1), 38–65.

Stépán, G. (1989). *Retarded dynamical systems: stability and characteristic functions*, volume 210 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York.

Vanbiervliet, J., Verheyden, K., Michiels, W., and Vandewalle, S. (2008). A nonsmooth optimisation approach for the stabilisation of time-delay systems. *ESAIM Control Optim. Calc. Var.*, 14(3), 478–493.

Vyhlídal, T., Hromčík, M., Kučera, V., and Anderle, M. (2016). On feedback architectures with zero-vibration signal shapers. *IEEE Trans. Automat. Control*, 61(8), 2049–2064.

Wielonsky, F. (2001). A Rolle's theorem for real exponential polynomials in the complex domain. *J. Math. Pures Appl. (9)*, 80(4), 389–408.