Conceptualising knowledge of mathematical concepts or procedures for diagnostic and supporting measures at university entry level

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At the 10th CERME in Dublin, Pinkernell, Düsi and Vogel (2017) introduced a model of proficiency in elementary algebra that serves as a frame of reference for analysing and constructing material for diagnostic and supporting measures for students at the entry level of university. One basis of this model was a systematic analysis of the mathematical concepts and procedures of elementary algebra with regard to five aspects of understanding mathematics. This paper introduces this framework in detail, thus contributing to a genuine mathematics educational conceptualisation of content oriented knowledge for STEM subjects at university entry.

Keywords: STEM Education, Secondary Education, Assessment, Knowledge, Understanding.

Conceptualisations of knowledge at university level

The recent years have shown a growing interest of mathematics educators for the transition from secondary to tertiary phase, which is mainly being characterised as a gap between mathematical levels and institutional cultures. In her analysis Gueudet (2008) describes differences with regard to modes of conceptualising mathematical objects, also different levels of rigour in communication or reasoning, and institutional differences, e.g. concerning the didactics of teaching and learning mathematics. Thomas, de Freitas Druck, Huillet, Ju, Nardi, Rasmussen and Xie (2015) come to similar findings when they analyse the transition from four different theoretical perspectives. One aspect that adds to these cognitive, didactical and institutional differences, however, is the content orientation that appears to dominate discussions about the demands and deficits of students in STEM subjects at university entry. This becomes manifest, for example, in catalogues of minimum mathematical requirements (e.g. cosh, 2014) or in a large-scale delphi study with mathematics teachers in the tertiary sector (Neumann, Pigge, & Heinze, 2017). Yet when it comes to devising diagnostic and supporting measures, what students need to know can only be made clear when there is a notion of what knowing means.

Among the frameworks for conceptualising knowledge that are familiar at university level is the so-called Bloom taxonomy, which in a revised version by Anderson and Krathwohl (2001) differs between various categories of knowledge about subject matter content as well as various forms of cognitive activities on that subject matter content. According to Krathwohl (2002) the reasons for revision were to update to new psychological models of knowledge as well as to meet with terminology use among teachers or educators. This resulted, e. g., in formulating four categories of knowledge, which are factual, procedural, conceptual and metacognitive knowledge. This framework had been designed for trans-disciplinary purposes (Krathwohl, 2002), which is seen as an advantage since it seems suitable for interdisciplinary use (Maier, Kleinknecht, Metz, & Bohl, 2010). While the procedural-conceptual dichotomy is not only part of many models of general knowledge but of mathematical knowledge as well (e.g. Hiebert & LeFevre, 1986; de Jong & Ferguson-Hessler, 1996; Rittle-Johnson & Alibali, 1999), it appears that for analysing mathematical content knowledge, the four categories do not quite fit to how the knowledge of mathematical concepts or procedures is seen from within mathematics education. Firstly, Star and Stylianides (2013) have shown that while psychologists see procedural and conceptual knowledge as merely two different types of knowledge, in mathematics education these two are regarded as of different quality. For many school teachers, conceptual knowledge is preferable over procedural knowledge which is seen as simply recalling facts or applying algorithms "without significant thought." (Star & Stylianides, 2013, p. 178) Secondly, this again seems to be in contrast to how many university teachers view procedural knowledge. They see procedural proficiency as a necessary basis for following or performing symbolic mathematical reasoning. In fact, since the process of learning abstract mathematical concepts can be described as a progression from procedures to concepts (Tall, 1991; Sfard, 1994), the procedural-conceptual dichotomy does not quite grasp the nature of knowledge in mathematics where concepts and processes are seen as part of the very same knowledge entities (see also Kieran, 2013). In fact, Star and Stylianides (2013) suggest "abandon[ing] the conceptual/procedural framework entirely and select new words or phrases to describe knowledge outcomes of interest." (p. 179)

This paper proposes a genuine mathematics educational approach to conceptualising mathematical knowledge at the transition from the secondary to the tertiary level. While procedural and conceptual knowledge is still implicitly present, it addresses specific forms of accessing or understanding a mathematical object, thus acknowledging the content orientation and other characteristics of mathematical knowledge as it is seen at university level.

Content orientation from perspective of mathematics education

The following ideas are rooted in the German Stoffdidaktik (subject-matter didactics) tradition. It shares the conviction that student oriented approaches to abstract mathematical objects are possible without compromising on the mathematical validity. At the centre of didactical efforts of Stoffdidaktik is a thorough analysis of the mathematical concept. In the past this took on the form of mathematical rigour that seemed unsuitable for the use in learning situations (Hußmann, Rezat & Sträßer, 2016). While subject of an ongoing dispute, the ideas of Stoffdidaktik are still present in German mathematics education. Prediger and Hußmann (2016) for example plead for a combination of a thorough content analysis and empirical evaluation, for which they describe four phases: 1. a formal analysis of mathematical conceptualisations of the teaching object, 2. a semantic analysis of meaningful interpretations of the teaching object, 3. a structural orchestration of the findings in the form of a teaching unit, and 4. constant empirical evaluation and subsequent modifications of steps 2 and 3 to adapt to students' needs. Especially the first two phases indicate that formal conceptualisations and meaningful interpretations of mathematical objects can be part of one didactical framework, which seems suitable for purposes at the transition from school to university where formal and educational perspectives meet.

The WiGORA frame of reference

Before going into details, this section starts with an outline of necessary a priori settings that reflect the area where the framework is being used, which is the transition from secondary to tertiary maths: First, it is a concise and summative view on what facets of knowledge of a given mathematical object a student needs to have at his or her disposal once it has been taught, not a formative view on how knowledge should be developed during school-time. Further, the facets of knowledge are considered normative, that is they are meant to cover mathematically sound ways of accessing a mathematical object which include, e.g. explanatory models or visualizations that are structurally equivalent to the object. Moreover, considering the formal level at which mathematics is being taught at university (Gueudet, 2008; Thomas et al., 2015), a correct use of terminology and definition based access to mathematical objects will be addressed explicitly. Simultaneously, since formal and abstract nature of mathematical objects requires a flexible use of representations (Duval, 1999), this aspect will be addressed explicitly, too. And last, this framework is for conceptualising an "intelligent content knowledge base" (Klieme et al., 2007) for developing higher level competencies, it is not a framework for higher level competencies itself.

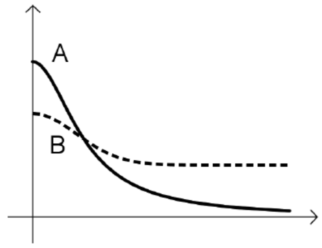
The acronym WiGORA derives from the German labels for the five facets of knowledge that make up the frame of reference. In the following each facet will be introduced and illustrated by tasks that address the concept of integral.

- Declarative knowledge ("Wissen") refers to the ability to recall or identify correct definitions, rules or characteristic properties of a mathematical concept or procedure as well as the necessary terminology associated with it. Declarative knowledge basically is knowledge about facts and information (Anderson, 1976). It seems rather less present in conceptualisations of mathematical knowledge as compared to procedural or conceptual knowledge. While in its strictest sense it does not allow for weighting knowledge regarding significance, declarative knowledge here also comprises prototypical knowledge that characterises, but not necessarily defines, the object (Rosch, 1983; Tall & Bakar, 1992; Weigand, 2004). The following task asks for prototypical knowledge that differs between definite and indefinite integrals.

Which of the four statements are correct?
$\int_a^b f(x)dx$ is a number. $\int_a^b f(x)dx$ is a function.
$\int f(x)dx$ is a number. $\int f(x)dx$ is a function.

- Explanatory models ("Grundvorstellungen" or GV for short) refers to the ability to recall or identify conceptualisations of a mathematical object that "make sense" (vom Hofe & Blum, 2016; Greefrath et al., 2016; Weber, 2017). The concept of GV is one of the key concepts of German Stoffdidaktik, which "should be able to, on the one hand, accurately fit to the cognitive qualifications of students and, on the other hand, also capture the substance of the mathematical content at hand" (vom Hofe & Blum, 2016, p. 227). In international context, GV are also being referred to as "basic ideas", "basic notions" or "conceptual metaphors" (Soto-Andrade & Reyes-Santander, 2011). In its broadest sense, GV comprise normative, descriptive and constructive aspects. Here, at the transition from school to university, the notion of GV is restricted to its normative aspect. As such, a GV could result from a semantic analysis of a mathematical object (Hußmann & Prediger, 2016). For example, "reconstruction" of rates or speed is one of the GV for the concept of the definite integral

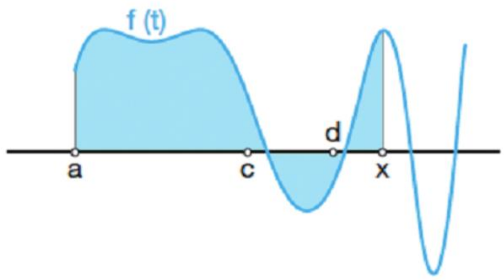
(Greefrath et al., 2016) and is subject of the following task.

The graph shows for any point of time the current fuel consumption for each of two vehicles A and B. Which of the vehicles has a higher consumption overall over the period shown?	

- Operational flexibility ("Operationale Flexibilität") refers to the ability to apply, adapt and modify mathematical procedures for situational needs. Going beyond simply reproducing step-by-step instructions this facet refers to the cognitive construct of operations in the sense of Piaget and Aebli. Characterised for example by reversibility or transitivity of the mental operations involved (Fricke, 1970), corresponding tasks would require reversing procedures or selecting efficient over routine procedures ("strategic flexibility": Rittle-Johnson & Star, 2007). Here, procedures are not restricted to algorithms for calculating numbers or transforming algebraic expressions. A procedure can be any method for solving a mathematical task, which e.g. could also involve switching representation forms. The following example shows a reverse task which can be answered by mentally visualising a graph and/or recalling prototypical information about the periodicity of trigonometric functions.

Specify as many a ≠ b as you can find such that $\int_a^b sin\,(x)dx = 0$.

- Representational flexibility ("Repräsentationale Flexibilität") refers to the ability to switch within and between representational forms or registers of a mathematical object. Following Duval (1999), this ability is specific to understanding higher level mathematics since a mathematical concept, being essentially abstract, can not be addressed otherwise: "From a didactical point of view, only students who can perform register change do not confuse a mathematical object with its representation." (Duval, 1999, p. 318). As to the many possible forms in which a mathematical object can be represented, this framework is restricted to those that are conventionally used in mathematics such as numerical, algebraical, geometrical representations, or verbal paraphrasing.

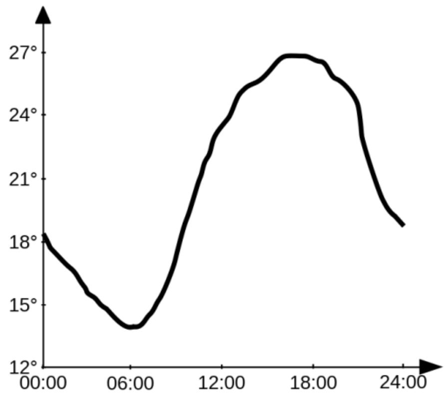
Order by value: $\int_a^a f\,(t)dt,\ \int_a^c f\,(t)dt,\ \int_a^d f\,(t)dt,\ 0,$ $\int_c^d f\,(t)dt$	

by H. Körner (in Pinkernell et al. 2015)

- Knowledge application ("Anwendung") refers to the ability to identify a mathematical concept or procedure as suitable for solving a problem. Here, the given concept or procedure is considered as a potential model for mathematising situations within or outside mathematics ("Mathematisierungsmuster": Tietze, Förster, Klika & Wolpers, 2000). This facet, as all five facets do, focusses on meaning and use of a given mathematical object. It does not refer to the modelling process or parts of it, but it addresses the content knowledge base of modelling. The following example asks for the average of values of a continuous function over an interval $\frac{1}{b-a}\int_a^b f\,(x)dx$ which here is determined by graphical estimation.

The graph shows the temperatures during a day in July.

What is the average temperature of that day? Give a best possible estimate.



Discussion

With WiGORA, this paper proposes a frameworkonfsk for conceptualising mathematical knowledge at the university entry level. With focussing on single concepts or procedures WiGORA follows a similar approach as the familiar taxonomies of Bloom or Anderson and Krautwohl, yet with a genuine theoretical base from mathematics education. When compared with Anderson and Krathwohl (2001), the most significant difference is that the well-known dichotomy of procedural and conceptual knowledge has been abandoned. It has been replaced by facets of knowledge that take specific aspects of objects from formal mathematics into account, which roughly can be characterised as being abstract "by definition". The facet "Grundvorstellungen" (GV) asks for the activation of explanatory models for the abstract mathematical object at hand. Such models can be hands-on activities on real or virtual material or situational interpretations within or outside mathematics. Equally, the facet "Repräsentationale Flexibilität" reflects the abstract nature of mathematical objects by asking for a flexible use of representations of this object. Also, the facet "Operationale Flexibilität" derives from the cognitive nature of mental operations as being abstractions from step-by-step procedures when it asks for heuristic flexibility in adapting mathematical procedures to situational needs. Among these facets, procedural and conceptual aspects of knowledge are still present though not explicitely. From the two other facets, "Wissen" corresponds to the category "factual knowledge" from Anderson and Krauthwohl (2001).

The five facets of the WiGORA framework all stand for viewing the same mathematical concept or procedure from different perspectives. Although the perspectives are different, the tasks that result from operationalisations following the framework are not necessarily disjunct. In fact, the very same task could address several facets. E.g., among the five examples above, three require representation change. And since GV are often associated with actions and concepts from everyday life, they identify obvious applications for the mathematical objects. For example, the GV “average value” (Greefrath et al., 2016) points to applying the integral for measuring the average of a continuous function, as shown in the fifth task. Hence, WiGORA has its main use for analysing or devising test material that is based on a list of given concepts or procedures. It is meant to serve as a model of reference for checking whether the five facets are being covered in a diagnostic tool.

With focussing on single objects, WiGORA does not allow for analysing a whole network of knowledge of a mathematical field, which from an educational perspective is characterised by a meaningful use of the specific language or the mastery of key concepts from that field. In the field of elementary algebra, these would be various aspects of knowledge specific to the structuring, transforming and interpreting of algebraic expressions (Pinkernell et al., 2017), as e.g. aspects of giving meaning to expressions (structure sense: Hoch & Dreyfus, 2006; systemic vs. surface structure: Kieran, 1989), or aspects of interpreting the equation sign (operational vs. relational meaning: Baroody & Ginsburg, 1983). Hence, the object related facets of knowledge from WiGORA would need to be integrated into a larger educational conceptualisation of the knowledge of that area. Presently, as part of the German optes+ project (Mechelke-Schwede, Wörler, Hübl, Küstermann, & Weigand, 2018), this is being done for the areas secondary arithmetic, functions and geometry.

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