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An exploratory study of calculus students' understanding of multivariable optimization problems

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Contributing to the growing body of research on students' quantitative reasoning at the undergraduate level, this study reports on students' reasoning about a multivariable optimization problem that involves finding the dimensions of the largest volume bag a passenger can carry on an American Airlines flight. Analysis of verbal responses and work written by 11 students when solving the problem revealed that finding the required dimensions was easy for nearly all the students. However, showing that these are the dimensions of the largest volume bag was problematic for a majority of the students. Implications for instruction and directions for future research are discussed.

Keywords: Optimization problems, quantitative reasoning, problem solving, calculus education.

Introduction

Optimization problems form an essential part of the study of differential calculus at the undergraduate level in the United States. Findings of a comprehensive review of literature on students' understanding of various topics in university calculus by Speer and Kung (2016) indicate that research on students' reasoning about optimization problems is lacking. A growing body of research (e.g., Borgen & Manu, 2002; Dominguez, 2010; LaRue & Infante, 2015) has examined students' reasoning about univariate optimization problems (UOPs). We define UOPs as optimization problems where the objective function, that is, the function whose minimum/maximum value(s) is to be found is a real-valued function of a single variable. Only one study (Heid, 1988) has reported on students' understanding of multivariable optimization problems (MOPs), that is, optimization problems where the objective function is a real-valued function of two variables. Heid found that determining minimum/maximum values of objective functions when solving MOPs where algebraic forms of the objective functions were given was problematic for a majority of the 135 students who participated in her study. We remark that the focus of Heid's study was more on the impact of resequencing skills and concepts on students' understanding of fundamental concepts in calculus, and less on students' understanding of MOPs.

While Heid's study has provided beneficial information about how students calculate minimum/maximum values when solving MOPs, there is still much to be explored about how students set up and solve MOPs in situations where an algebraic form of the objective function is not given, which is the motivation for this study. Thus, to build on Heid's study, we intend to explicitly examine students' understanding of the key steps (e.g., setting up the objective function, finding critical points of the objective function, and using the critical points to find the minimum/maximum value(s) of the objective function) that are generally involved in the process of solving MOPs algebraically.

Research on students' understanding of UOPs

While the focus of this study is on students' understanding of MOPs, it is important to discuss the research literature on students' understanding of UOPs for comparison. Specifically, we use the literature base on students' understanding of UOPs to identify trends in students' reasoning about UOPs that may be comparable to the results found in this study. There are three themes that emerge from the research that has looked at students' understanding of UOPs: (1) students' difficulties with setting up the objective function, (2) students' difficulties with determining and interpreting critical numbers and/or extrema, and (3) students' difficulties with justifying/verifying extrema (the minimum/maximum value(s) of the objective function). Following is a brief discussion of each of the aforementioned themes.

In the first theme, several researchers have found that setting up the objective function is problematic for high school and undergraduate students (e.g., LaRue & Infante, 2015; Swanagan, 2012). Swanagan (2012) reported on a high school student who used the equation of the parabola, $y = x^2$, as the objective function in a task about finding a point on the parabola closest to the point (1,0). In the same study, another high school student used the perimeter function as the objective function in a task about minimizing the cost of fencing a rectangular plot of land. LaRue and Infante (2015) found that formulating the objective function when solving contextualized optimization problems is problematic for undergraduate students even if the objective function is simple and the context of the problem is familiar to the students.

In the second theme, evidence from research (e.g., Dominguez, 2010; Mkhathshwa, 2018; Swanagan, 2012) shows that determining and interpreting critical numbers and/or extrema when solving optimization problems that have real-world contexts is problematic for high school and undergraduate students. Two of the five high school students in Swanagan's (2012) study occasionally relied on guesswork to determine extrema in a cost minimization context. Interpreting extrema in a profit maximization context was problematic for a majority of the 94 undergraduate calculus students who participated in Dominguez's (2010) study. In another profit maximization context, Mkhathshwa (2018) reported on three undergraduate business calculus students who conflated extrema (maximum profit) with a critical number (profit maximizing quantity).

In the third theme, research by Borgen and Manu (2002) and Mkhathshwa (2018) shows that justifying/verifying extrema is problematic for undergraduate students. Borgen and Manu (2002) reported on a student who presented a perfect solution to the problem of finding the minimum value of the function $y = 2x^2 - x + 1$. However, when asked for a justification on why the function has a minimum value and not a maximum value, the student incorrectly claimed that since the coefficient of the linear term of the function is negative, the function must have a minimum value. Mkhathshwa (2018) found that 16 of the 24 students who participated in his study had difficulty verifying extrema (maximum profit) while reasoning about a profit maximization task.

Relationship between UOPs and MOPs

Solving MOPs algebraically may be more difficult for students than solving UOPs for several reasons. First, setting up the objective function for a MOP is more difficult than for a UOP. Part of the difficulty stems from the fact that in the case of a MOP the objective function is a real-valued

function of two variables, where as in the case of a UOP the objective function is a real-valued function of a single variable. Second, the algebra involved when solving a MOP may be more challenging than the algebra involved when solving a UOP. In particular, to find a critical point(s) for a MOP, students have to simultaneously solve a system of equations, where as in the case of a UPO, they only have to solve a single equation (the derivative of the objective function) to find a critical number(s). Third, algebraic procedures such as the Second Derivative Test (Stewart, 2016) for verifying extrema when solving a MOP are much more complex than algebraic procedures for verifying extrema in the case of a UOP. Thus, one can expect the aforementioned students' difficulties with UOPs to carry over to MOPs.

Theoretical perspective

This study draws on the theory of quantitative reasoning (Thompson, 1993; Thompson, 1994b; Thompson, 2011). Quantitative reasoning (hereafter, QR) is the act of analyzing a problem situation in terms of the quantities and relationships among the quantities involved in the situation (Thompson, 1993). In this study, QR refers to how students: (1) constructed formulas for quantities, (2) how they evaluated these formulas to determine numeric measures for quantities, (3) how they interpreted quantities, and (4) and how they reasoned about relationships between or among quantities while solving a MOP situated in a volume maximization context. Thompson (2011) identified several tenets that are central to the theory of QR. We describe three of those tenets that are related to the study reported in this paper. These tenets are: a quantity, a quantitative operation, and quantification. A quantity is a measurable attribute of an object. Examples of quantities (i.e., measurable attributes) in this study include the volume, length, width, and height of the carry-on bag (i.e., the object) mentioned in the task that appears in the methods section. Thompson (1993) distinguished between a quantity and a numerical value: A quantity has a unit of measurement but a numerical value does not.

A quantitative operation is the process of creating a new quantity from other quantities (Thompson, 1994b). The task used in this study provided a number of opportunities for students to create new quantities (partial derivatives) through the process of differentiation, that is, to perform quantitative operations. More precisely, one way students could use to solve this task successfully is to find a model (algebraic formula) that relates the quantities in the task. In this task, an appropriate model would be the formula for calculating the volume of the carry-on bag (hereafter, bag), i.e., $V = lwh$ where V is the volume of the bag, l is the length of the bag, w is the width of the bag, and h is the height of the bag. Multiplying the length, width, and height of the bag would result in the creation of a formula for the quantity of volume for the bag. In addition, differentiating this model (preferably, after first writing it as a function of two variables by taking into consideration the constraint on the dimensions of the bag given in the task) partially with respect to two of the dimensions (e.g., width and height) of the bag will result in the creation of new quantities (partial derivatives). Quantification is the process of assigning numerical values to quantities (Thompson, 2011). The task used in this study provided several opportunities for students to engage in the process of quantification. For example, evaluating the formula for the quantity of volume, $V = lwh$, after the dimensions of the length, width, and height that will maximize the volume of the bag have been found would lead to the determination of a numerical value for the quantity V . Our study was

guided by the following research question: How can we interpret the theory of quantitative reasoning in the context of students solving MOPs?

Methods

This qualitative study used task-based interviews (Goldin, 2000) with 11 students. The interviews lasted for about 34 minutes, on average, and contained two tasks. In this paper, we report on how the students reasoned quantitatively about one of the tasks:

American Airlines requires that the total outside dimensions (*length + width + height*) of a carry-on bag not exceed 45 inches. What are the dimensions of the largest volume bag that a passenger can carry on an American Airlines flight?

The students worked through the task while the interviewer asked clarifying questions about their work. After the student concluded their work on the task, the interviewer asked the following questions about the task and the content of their solutions: (a) Have you seen a problem like this before? (b) What did you do to solve the problem? (c) What does your answer tell you? (d) What does each quantity you used throughout your solution mean? (e) What are the units of each quantity that you used in your solution? (f) What was the easiest part when solving this problem? (g) What was the challenging part when solving this problem? Eight students acknowledged having seen or even solved a similar task prior to participating in this study.

Setting, participants, and data collection

The study participants were undergraduate students at a research university in the United States. These students represented most of the high-performing students who were enrolled in two sections of a traditional calculus III course taught in the spring semester of 2018. The students were chosen based on their willingness to participate in the study, and on their ability to explain their thinking when engaged in problem solving. In addition, drawing on my my past experience with inviting students to participate in research studies similar to the one reported in this paper, high-performing students are more likely to accept the invitation than average or low-performing students. The students were familiar with formal techniques (e.g., the Lagrange multiplier method) that can be used to solve the problem posed in the task from course lectures and the course textbook (Stewart, 2016). We state as a remark that the students in this study had limited exposure to optimization problems that are situated in real-world contexts through classroom instruction and/or homework assignments. At the time of the study, four students were mathematics majors, another four students were engineering majors, two students were physics majors, and one student was an economics major. We note that this is a required course for mathematics, engineering, and physics majors, and that students outside the aforementioned disciplines rarely take this course as an elective. Eight students were freshmen, and the other three students were sophomores. Data for the study consisted of transcriptions of audio-recordings of the task-based interviews and work written by the 11 students during each task-based interview session.

Data analysis

Data analysis was done in two stages. In the first stage, we used a priori codes that consisted of the themes on students' difficulties when solving UOPs discussed earlier. Specifically, we carefully

read through each interview transcript and coded: (1) instances where students reasoned about the objective function, (2) instances where students reasoned about finding and interpreting critical points and/or extrema, and (3) instances where students reasoned about justifying/verifying extrema. In the second stage of the analysis, we looked for patterns in each of the codes identified in the first stage of the analysis. These patterns included trends in the students' understandings, or difficulties they had in connection with each of the a priori codes identified in the first stage. The common understandings or difficulties in students' reasoning found in the second stage of our analysis provided answers to our research question.

Results

There are three main findings from this study. First, setting up the objective function was easy for nearly all the students. Second, most of the students successfully determined and correctly interpreted critical points and extrema. Third, verifying extrema was problematic for a majority of the students.

Setting up the objective function

Ten students were successful in setting up the objective function (i.e., the volume function) for the problem posed in the task. These students performed a quantitative operation by conceptualizing a measure of the quantity of volume as a quantitative product, i.e., volume as the product of the length, width, and height of the bag. The following excerpt illustrates how Sally, whose reasoning is representative of these students, thought about the objective function and constraint function.

Researcher: What was the first thing you did when solving the problem?

Sally: I wrote down the equations for the volume [$V = lwh$], which is what we are trying to solve for and maximize, and our constraint [$l + w + h = 45$] that we have, in terms of the variables we have, which is length, width, and height of the carry-on bag.

Researcher: What are the units of the l , w , and h ?

Sally: They will be in inches.

Researcher: How about the units of V ?

Sally: Cubic inches.

After reading the problem statement, Sally recognized that the volume of the bag is the product of the length, width, and height of the bag, and that to maximize the volume of the bag, the sum of the dimensions of the bag will have to be 45 inches. She quickly turned these recognitions to algebraic equations for the objective function and constraint function. Sally also recognized the volume, length, width, and height of the bag as quantities (and not as unit-less variables) when she assigned units of measure to the variables l , w , h , and V . Only one student, Andy, had difficulty setting up the objective function. This student conflated the objective function with the constraint on the dimensions of the bag given in the task. Consequently, he incorrectly claimed that the volume function would be $f(x) = xyz = 45$, where $f(x)$ represents "the volume of the bag" and x , y , and z represents the length, width, and height of the bag respectively. Andy's notation for the volume

function suggest that this student either did not know how to write the correct function notation for a real-valued function of three variables or that he treated the variables y and z as constants.

Determining and interpreting critical points and/or extrema

Eight students correctly determined the critical point of the objective function using algebraic methods. The following excerpt illustrates how Ronda, who is representative of how six of these students used the Lagrange multiplier method (hereafter, LMM) to find the critical point of the objective function.

Researcher: What did you do next [after finding the objective function and constraint function]?

Ronda: I found the partial derivatives for the volume equation [$f(x, y, z) = xyz$ in Figure 1], and I did the same thing for the other equation [$g(x, y, z) = x + y + z$ in Figure 1]. From there, I tried solving. I used the Lagrange multipliers [method] but I don't know if that is what I was supposed to do but that's what I ended up doing. I ended up getting that the length, the width, and the height will equal each other, and I got $45/3$ or 15 inches for all of those [length, width, and height].

Handwritten work showing the Lagrange multiplier method for finding the critical point of the volume function $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + y + z = 45$. The student sets up the partial derivatives, equates them to each other and to the constraint, and solves for x , y , and z , finding $x = y = z = 15$. The final volume is calculated as $V = 15^3 = 3,375$.

Figure 1: Ronda's solution

Ronda correctly used the LMM as presented during classroom instruction and in the course textbook to find the critical point of the objective function. When asked about the meaning and units of the quantities she used in her solution, Ronda stated that the variables V , x , y , and z represent the volume, length, width, and height of the bag respectively and that the units of the variable V are “inches cubed” while those of x , y , and z are “inches.” In the process of using the LMM to find the critical point of the objective function, Ronda created new quantities (i.e., performed quantitative operations). Three of the quantities she created are the partial derivatives $f_x = yz$, $f_y = xz$, and $f_z = xy$. When asked about the meaning of the partial derivative $f_x = yz$, Ronda said, “that would be saying like the width and the height would both be constant, so they wouldn't be changing at all, x is the only variable.” While Ronda's interpretation of the partial derivative f_x shows that she clearly understood that only x is allowed to vary while y and z are held constant, it does not show that she understood the partial derivative to be a “rate” quantity. She stated that the units of the partial derivative $f_x = yz$ would be “inches cubed,” suggesting that she conflated a “rate” quantity with an “amount” quantity, i.e., volume. Ronda gave similar

interpretations and units for the other two partial derivatives, namely f_y and f_z . Ronda and three other students (Sally, Austin, and Tim) went on to engage in quantification by evaluating the volume function at the critical point they found (i.e., 15 inches for the length, width, and height respectively) to determine the volume of the bag.

Two other students (Noel and Paul) used another algebraic method to find the critical point of the objective function. Because of space limitations, we will not show student work that illustrates the method used by these students. Two other students (Tom and Austin) drew on their prior experiences with calculating volumes of rectangular prisms and correctly determined the critical point of the objective function. Another student (Andy) incorrectly assumed that the length, width, and height of the bag will each be $\sqrt[3]{45}$ inches long.

Verifying extrema

Eight students had difficulty verifying that the dimensions of the bag they found would result in a bag of largest volume. For example, when asked how they can convince someone that the dimensions of the bag they found are the dimensions of the largest volume bag a passenger can carry on an American Airlines flight, Ronda commented, “honestly, I am not sure” while Garry said “I would probably google Lagrange multipliers to refresh myself. Lagrange multipliers would sort of be my proof.” In response to the same question, one of the students who gave an acceptable response, stated that he could use a numerical approach that involves randomly trying other numeric values for the quantities of length, width, and volume besides 15 inches and showing that these values would result in a smaller volume than when each side of the bag is 15 inches long. He also stated that the Second Derivative Test, discussed during classroom instruction and in the course textbook, could be used. The student correctly explained how this test could be used, in addition to using it correctly to verify extrema in another task not reported in this paper.

Discussion and conclusions

Although previous research (e.g., LaRue & Infante, 2015) has reported on students’ difficulties with setting up the objective function when solving UOPs, this was not the case for most of the students in this study. To some extent, findings of this study suggest that students’ difficulties when solving MOPs are similar to students’ difficulties when solving UOPs. In particular, verifying that the dimensions of the bag the students found (correct or incorrect) were the dimensions of the largest volume bag a passenger can carry on an American Airlines flight was problematic for eight of the 11 students who participated in this study. Similar results have been reported among students when solving UOPs (Borgen & Manu, 2002; Mkhathshwa, 2018). One student (Ronda) interpreted a “rate” quantity represented by a partial derivative as an “amount” quantity with units of volume (i.e., inches cubed). Similar results have been reported in other studies that have examined students’ reasoning about “rate” and “amount” quantities in real-world contexts (e.g., Mkhathshwa & Doerr, 2018). As noted earlier, the students in this study had limited exposure to solving MOPs that have real-world contexts. The presentation of optimization problems during course lectures closely followed the presentation of optimization problems in the course textbook. In particular, verifying extrema using informal and formal techniques such as the Second Derivative Test received minimal attention during course lectures. Hence, we argue that the students’ difficulties with verifying that

the dimensions they found would result in a bag of largest volume may be directly related to the limited opportunities they had to verify extrema via examples that were given during course lectures. Calculus instructors may have to supplement examples given in course textbooks in order to maximize students' opportunity to learn about MOPs. Future research might examine whether or not there are gaps in students' prior knowledge that impact them in understanding MOPs and the use of technology (especially graphs and spreadsheets) during classroom instruction to help students understand MOPs.

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