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# On tasks that lead to praxeologies' formation: a case in vector calculus

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*A praxeology comprises a type of task, a technique to accomplish it, a discourse (technology) explaining this technique and a theory supporting this discourse. We consider a situation where a praxeology was presented to students in a lecture, but they discovered an alternative one while working on a routine exercise. This exercise was then followed by a more challenging question related to immediate results obtained by the students. We analyzed this student experience in view of Chevallard's six steps of constructing a praxeology. We suggest that observing students' work on standard exercises may give some productive and innovative ideas for the design of tasks that help students deepen their understanding and encourage theoretical justifications of their solutions. We also comment about possible effects of the explicit requirement to search for the most efficient solution on the formation of praxeologies by students.*

*Keywords: Praxis, logos, task design, osculating plane, parametric curve.*

## Introduction

It has been observed that the majority of university mathematics classes are given in the format of lecture and largely employ the pedagogy of “massive scaffolding and direct telling” (Sfard, 2014, p. 202). This teaching approach has been criticized for its limited capacity to empower students in both critical reflection on given information and applying their knowledge in unfamiliar situations. In contrast, within the inquiry-based approach learners start with considering a problem, and then planning, developing and validating their own solutions (Polya, 1945). In this *adidactic* phase (Brousseau, 1997) learners construct their knowledge by working on a problem via thinking about it autonomously, studying related literature and talking to peers. In the didactic phase the teacher discusses the ideas produced by students, corrects them, validates and develops the subject further. However prolific it may sound, this approach is not fully applicable in teaching university mathematics due to the large amount of advanced material to be studied in a relatively short timeframe. Nevertheless, one can ask: to what extent is it possible to embed elements of inquiry in the lecture format courses? Where do we find questions that promote students' “intellectual needs” (Harel, 2012) for critically processing the information given in the lecture? What principles should guide the design of educational tasks that help students to progress from the direct application of given methods to the analysis and justification of them (Gravesen, Grønbaek & Winsløw, 2017)?

Perhaps the central requirement to these tasks is that they should help students to think about and reflect on their actions – what am I doing and why? – so that students can extract new knowledge directly from solving tasks. It is desirable that tasks have *adidactic* potential letting students to “find at least initial and partial answers” (Gravesen et al, 2017, p. 15). The same material can be taught with a different degree of initiative required from students. Typical examples illustrating general concepts and methods may be shown in class and then students can be assigned questions that are just a slight variation of already considered tasks. Such questions, known as *exercises*, contain little challenge but permit verifying students' initial grasp of the material, as students basically mimic

instructor's actions and adjust them according to his or her expectations. In contrast, *problems* require an adaptation or alternation of a given method or a combination of several previously discussed ideas and serve as a form of inquiry. A task that stimulates students to “construct new inferences between results or definitions, reformulate, generalize and instantiate them” is said to have a deepening potential (Gravesen et al, 2017, p. 13). Similarly, research potential of a task measures the possible degree of engagement with activities similar to ones of research mathematicians. However, even if a task is specifically designed, its potential may not be realized in a particular didactic situation. Thus, it is important to consider cases when students gain rich mathematical experience and to notice the circumstances that have a positive effect. This paper presents such a case confirming that even when a lecturer “tends to tell the learner what to do” and “rushes to present their own solutions” (Sfard, 2014, p. 200) it is possible and desirable to delegate to students some work that aims at the development of theoretical knowledge “while working on her own in the quiet of her room” (Sfard, 2014, p. 201).

### **Theoretical framework and research question**

We work under the assumption that “mathematics is a human activity of study of types of problems” (Barbé, Bosch, Espinoza, & Gascon, 2005, p. 236) and mathematical learning (even in a lecture based course) results from solving educational tasks (e.g. problems). Each task is a system of “given-required”, where “given” includes objects and relations between them, while “required” challenges the solver to derive new facts based on the “given” and their existing knowledge. A solution to a mathematical problem can be considered from two angles: “What is done?” and “Why does it work?”. In terms of the Anthropological Theory of Didactics (ADT) proposed by Chevallard (1999), the first question belongs to the domain of praxis, which includes techniques such as formulas, algorithms and methods, while the second - to the domain of logos, which provides corresponding theory, conditions of applicability, derivations and relations to general settings. The praxis consists of the types of tasks and corresponding techniques to solve them. The logos includes technology (explanations of the techniques, proofs of related theorems) and a broader theory within which the theorems are stated. Together praxis and logos components form a praxeology, described by Chevallard (2006, p. 23) as a “basic unit in which one can analyze human activity at large”. Praxeologies can occur at different levels of generality: punctual, local, regional or global. Punctual praxeologies address a unique type of task. Several punctual praxeologies can be integrated in different local mathematical organizations each of which provides a common technological discourse (e.g. algebraic). Local technologies can be further integrated into regional ones.

In the teaching of primary and secondary mathematics there is a tendency to focus on the praxis component with very minimal exposition of the students to the related logos. Even at the university level, there exists an artificial division of courses into e.g. Calculus and Analysis, assuming significant dis-balance towards the praxis in the former and leaving the theory until the latter (see Kondratieva & Winsløw, 2018). The lack of an explanation of why and when a technique works (i.e. lack of technology) leads to incomplete praxeologies; this is a disservice to learners at all levels as it kills natural learners' curiosity and reduces the flexibility of the technique. Even if technology is given by the teacher it may not necessarily lead to formation of a praxeology by students – to the practice when the use of a technique is informed by related technology. In this paper we look at a

case when students were engaged in the formation of a praxeology; the research question being: what are the instructional conditions making this possible at the undergraduate university level?

Our discussion will refer to the following six moments of creating a praxeology (Chevallard, 1999).

- 1) The moment of *first encounter* of a certain type of problem related to a praxeology;
- 2) The *exploratory* moment of finding and elaborating techniques suitable for the problem;
- 3) The *technical* moment of using and improving the technique;
- 4) The *technological-theoretical* moment in which alternative techniques are assessed;
- 5) The *institutionalization* moment, when one is aiming to identify the elaborated praxeology;
- 6) The *evaluation* moment, when one examines the value of the constructed praxeology.

Here the *adidactic* situation leads students to propose some techniques that need to be assessed and theorized. Students react on the situation in accordance with an existing *didactic contract* (Brousseau, 1997), that is, a set of responsibilities, rules and norms of schooling behavior. In our instructional setting we encountered a slightly modified sequence of events presented below.

## **Instructional settings and observations**

This section describes several episodes that occurred in a vector calculus course taught by the author of this paper in the Spring 2018 term at Memorial University (MUN) in Canada. The episodes had been selected in order to present the story of success. They describe a didactic contract, an error and its correction, made by students, leading to formation of a new praxeology.

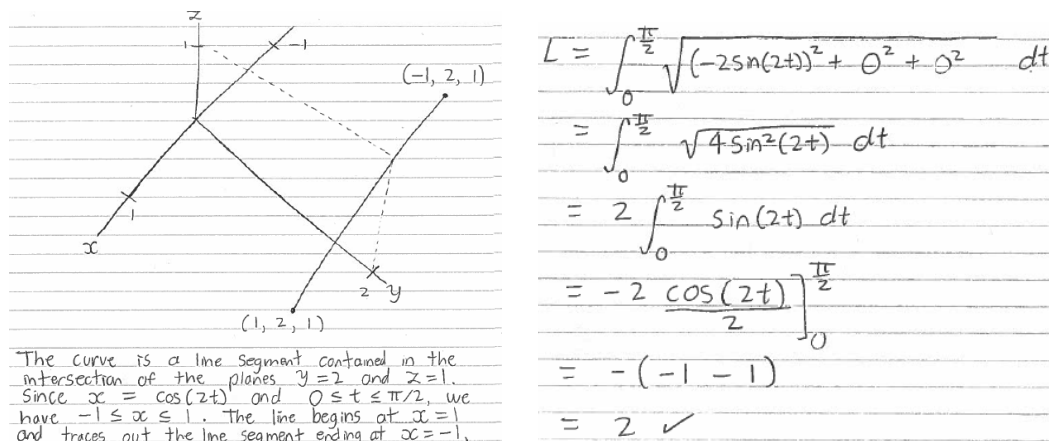
### **The settings**

The vector calculus course is the last course in the sequence of calculus courses offered by the department of Mathematics and Statistics at MUN. One-hour lectures are given three times a week for a twelve-week period (with regular homework assignments and a midterm exam) followed by a 3-hour written final exam. The course discusses such topics as parametric curves and surfaces, multiple integrals, Green's, Stokes' and the Divergence theorems with some motivations from physics. The course is usually taken by students majoring in either mathematics, physics, computer science or engineering. Similarly to preceding calculus courses it focuses mostly on computational techniques and obtaining numerical answers. Some sketches of proofs are discussed in lectures but they are not tested in exams. While the syllabus of the course is prescribed, the instructor has some freedom in composing questions for assignments and tests. In the considered case the homework assignments were marked by the instructor and consisted of the following types of questions: (1) short summary describing definitions, formulas, and conditions of their applicability as discussed in lectures; (2) exercises in direct application of formulas listed in (1); (3) problems that require minor alternations of formulas listed in (1); (4) bonus problems that may require additional reading and creative thought. Note that questions of types (1) and (4) were not typical in students' previous experience. Twenty students completed the course in Spring 2018.

### **The observations and interventions**

*Episode 1: Efficiency of solutions as a part of didactic contract.* This observation took place in the very beginning of the course. The students just learned how to find the curve length given a parametric description of a curve  $\mathbf{r}(t) = \{(x(t), y(t), z(t)), t \in [a, b]\}$ . Using the integration

formula  $\int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2} dt$  they were able to find the length of some exotic curves such as a part of a helix. However the assignment also included examples of parts of a straight line and a circle. Many students still used the general method to find the length of these curves even when they had recognized them correctly. It was then pointed out to students that they should aim at the most efficient way of finding their answers or at least mention that their calculations were consistent with the obvious answer. For example, the length of a straight line  $r(t) = \{(\cos(2t), 2, 1), t \in [0, \frac{\pi}{2}]\}$  connecting points  $(-1, 2, 1)$  and  $(1, 2, 1)$  (Figure 1, left) is obviously 2, which makes the calculation in (Figure 1, right) unnecessary. Similar situations appeared in other assignments. Some hints referred to efficiency in numerical calculations, such as



**Figure 1: Student's work on curve  $r(t) = \{(\cos(2t), 2, 1), t \in [0, \frac{\pi}{2}]\}$ . Description and length**

a suggestion to use the property  $\sqrt{(ka)^2 + (kb)^2} = k\sqrt{a^2 + b^2}$  ( $k \geq 0$ ), or warned that the simplification  $\sqrt{A+B} = \sqrt{A} + \sqrt{B}$  is not valid unless  $AB=0$ . All in all, such systematic instructional feedback shaped the didactic contract for this group of students.

*Episode 2: Accidental efficiency.* Another assignment included a problem of finding an osculating plane for the curve  $r(t) = (\cos t, \sin t, \cos t)$  at  $t = \pi/2$ . The instructor explained a method, which consisted of the following steps:

1. Find the unit tangent vector  $T = \frac{r'}{\|r'\|}$  to the curve at a given point.
2. Find the unit normal vector  $N = \frac{T'}{\|T'\|}$  to the curve at a given point.
3. Find the unit binormal vector  $B = T \times N$  to the curve at a given point.
4. An osculating plane by definition includes vectors T and N and therefore is orthogonal to vector B. Thus the equation of the plane is  $B \cdot (r - r_0) = 0$ , where  $r = (x, y, z)$  and  $r_0$  is the point on the curve where the plane is required.

An example considered in class in order to illustrate this method was helix of radius  $\alpha$ :  $r(t) = (\alpha \cos t, \alpha \sin t, t)$ . Note that in this case  $r'(t) = (-\alpha \sin t, \alpha \cos t, 1)$  and so  $\|r'\| = \sqrt{1 + \alpha^2}$  is  $t$ -independent. Consequently, the normal unit vector  $N = \frac{r''}{\|r''\|}$ . However, if  $\|r'\|$  is a function of  $t$ , this must be taken into consideration when calculating  $N$ . When working on the assignment described above, some students ignored this circumstance and used the formula  $\frac{r''}{\|r''\|}$  for  $N$  in step 2.

Nevertheless, they still got a correct answer for the proposed problem, the plane  $z = x$ . The mistake was pointed out by the instructor during a lecture, but the fact of accidental efficiency was surprising for students and caught their attention leading to an in-class discussion. Their first thought was that maybe the value of the parameter  $t = \pi/2$  was special, but the “wrong method” still gave a correct equation of the osculating plane for  $t = \pi/4$  and some other variations they tried. So, a bonus problem naturally emerged: to explain why and when the “wrong method” works.

*Episode 3: A praxeology given in the lecture.* The method of finding the osculating plane was not completely unexplained in the lecture. Prior to Episode 2 the students were presented with the statement that if a vector-function  $f(t)$  has a  $t$ -independent norm  $\|f\| \equiv \sqrt{f \cdot f} = \text{const}$  then the vector function  $f'(t)$  is orthogonal to  $f(t)$  for every  $t$ . Indeed, if we differentiate both sides of the equation  $f \cdot f = \text{const}$  we obtain  $f' \cdot f = 0$ . Now, the unit tangent vector has norm  $\|T\| = 1$ , therefore  $T'(t)$  is orthogonal to  $T(t)$  by the above statement. After normalization it gives the unit normal vector  $N(t)$ . The cross product of two unit vectors gives a unit vector orthogonal to both initial vectors. This way the binormal unit vector  $B(t)$  is obtained. These explanations presented a technology related to the praxis of finding an osculating plane. However, as it could be inferred from the error made by the students who were somehow misled by the example of helix in Episode 2, this praxeology, given by the instructor, remained external to them and the application of the method was not guided by the related theory. This revelation prompted the instructor to encourage students to develop their own explanations of the method that they had accidentally discovered.

*Episode 4: Towards a new praxeology construction by students.* Proving was not a well developed skill for many students, so their initial attempts naturally were flawed. We will follow the progress of a student called Sam (a pseudonym). Firstly, Sam arrived at the correct statement, but it was not very useful in the situation he needed: if  $r'(t) \perp r''(t)$  for  $t \in [a, b]$  then  $N = \frac{r''}{\|r''\|}$ . Indeed, the conclusion was desirable but the condition was not met in the problem they had. After some time Sam managed to prove algebraically that if  $r'(t_0) \perp r''(t_0)$  then  $N(t_0) = \frac{r''}{\|r''\|}$  (Figure 2, left). When this result was shared with others in class, it was noted by students that the statement is obvious from the geometrical point of view: if both vectors  $N$  and  $r''$  are orthogonal to  $r'$ , they must be proportional. In this way the case  $t = \pi/2$  was successfully explained. However, it was still unclear why the “wrong method” gave the correct equations of the osculating plane even in the case when  $N$  was not proportional to  $r''$  (see e.g. calculations for case  $t = \pi/4$  in Figure 2, right).

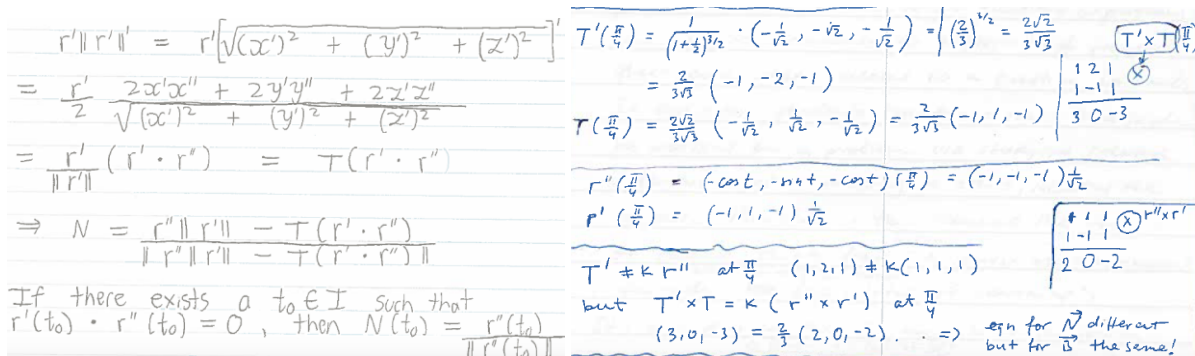


Figure 2: Student’s work on a proof for the alternative technique: cases  $t = \frac{\pi}{2}$  (left) and  $t = \frac{\pi}{4}$  (right)

*Episode 5: The new method explained.* After looking at several examples that produced a correct osculating plane by the “wrong method”, Sam started to realize that he should not try to prove that  $N$  is proportional to  $r''$  (which in general is not so) but instead he should try to prove that the span on vectors  $r'$  and  $r''$  forms exactly the same plane that the span of vectors  $T$  and  $N$  does. This shift of attention made Sam perceive a new property, which according to Mason (2008, p. 38) is feasible “when you are aware of a possible relationship and you are looking for elements to fit in”. Once he knew what he was looking for, a simple line sufficed: since  $r' = ||r'||T$ , by differentiation we obtain  $r'' = v'T + vT'$ , where  $v = ||r'||$  and  $v'$  are scalars. Thus, vector  $r''$  is a linear combination of vectors  $T$  and  $T'$ , or equivalently of vectors  $T$  and  $N$ . Therefore, whenever  $v \neq 0$ , vectors  $r'$  and  $r''$  can be used for finding the osculating plane initially defined by vectors  $T$  and  $N$ .

## **Concluding discussion**

In this paper we explored the possibility of imbedding an inquiry task, allowing students to construct technique-technological links and eventually to develop praxeologies by themselves, within a praxis-dominated lecture-based calculus course. Here we refer to the six moments of constructing a praxeology (listed in the Theoretical Framework section) in order to comment on two conditions that in our view had a positive effect on the successful outcome reported in Episode 5.

The first condition is the systematic attention paid by the instructor to the issue of the efficiency of students' work. This requirement for a solution of a problem is entwined with the inquiry-based methodology. In a situation when more than one approach is applicable to a problem in hand, one would naturally ask for the simplest answer, which is based on the insight about the nature of the situation and avoids unnecessary lengthy calculations. Indeed, “as mathematical thinking develops, it should become not only more powerful, but more simple” (Tall, 2013, p. 19) because “not only mathematics, but science as a whole progresses only if we understand things ... and explain [ideas] in simple terms” (interview with Sir M. Atiyah, as quoted in Tall, 2013). Episode 1 confirmed the observation that students in North America spend little time on planning their solution and jump into calculations right away (Schoenfeld, 1985). While the cognitive mechanisms for finding simple solutions in general are not clearly understood (see discussion in Koichu, 2010), it is important to establish the work habit of looking for legitimate simplifications whenever possible. Viewed as a part of didactic contract, the higher efficiency of the alternative method of finding the osculating plane prompted the students to keep trying to justify it. This efficiency gave the value to the praxeology being constructed (moment 6: evaluation).

The second condition is the fact that an additional new task emerged from the instructor's feedback on a mistake made by students while solving an exercise. Let us look at it a bit closer. The initial task in Episode 2 was to find an osculating plane for a curve at a point, which was defined in terms of the unit tangent and normal vectors -- the objects that were introduced to the students at the same time as was the osculating plane. The definitions prescribed the way of constructing all listed above objects. Thus, in the scenario of Episode 2 the first two moments (first encounter, exploration) were missing as such, and in this sense the task was an exercise in which a given technique was supposed to be used (a modified moment 3). On the other hand, the curve was chosen in such a way that it allowed a more intuitive approach in line with examples discussed in Episode 1. A “simple”,

intuitive solution was to notice that the curve belonged to the intersection of the cylinder  $x^2 + y^2 = 1$  and the plane  $z = x$ . Therefore the entire curve lied in the plane  $z = x$ , which thus was the osculating plane for any value of the parameter  $t$ . Finding this solution relied on explorations and noticing specific relations between components of the vector-function  $r(t)$  describing the curve. Indeed, several students had commented on this but only after they found the answer by a different method. Erroneously, they altered the method given to them attempting to (illegitimately) reduce the amount of calculations. The fact that the altered method provided an answer consistent with the “simple” intuitive approach led to a new task: to explain why it happened. It was the instructor’s role to formulate this task, but because the method appeared from the students’ own omission they claimed ownership of this task. Here the *adidactic* phase occurred later in the process and the development of praxeology by students basically started from moment 4, that is, from the assessment of the alternative technique. The successful outcome in Episode 5 was warranted by students’ reflection on the ideas related to the original praxeology with technology presented by the instructor in Episode 3, and modifying them accordingly. The institutionalization moment (5) was completed after Episode 5, when related physical motivation was also discussed in the lecture. Indeed, the observation that played the key role in the justification of the alternative method could be interpreted as follows: no matter how the object moves through the space the acceleration vector  $a = r''$  always belongs to the osculating plane:  $a = v'T + \kappa v^2N$ , where  $\kappa = ||T'||/v$  is the curvature of the trajectory and  $v = ||r' ||$  is the speed of motion. Vector  $T$  gives the direction of the motion and vector  $N$  points in the direction the object is turning. So  $T$  and  $N$  describe the (osculating) plane to which the trajectory of motion locally belongs. The expressions for the tangential and normal components of the acceleration vector make sense if we think of a passenger in a car making a turn (Stewart, 2003, p. 875).

Note that the praxeology developed by students in Episodes 2, 4 and 5 is punctual as it addresses just one particular problem of constructing an osculating plane. It is a refinement of the initial praxeology given in Episodes 2 and 3, and we conjecture that due to students’ involvement in its justification it will be more readily integrated in the future in local and regional types.

In sum, our case involved a routine exercise and a lot of “direct telling” as in a typical calculus course. This setup presumes that the two steps (first encounter, exploration) in the construction of a praxeology could be missing. Instead, through working on exercises using given information and techniques, the groundwork for developing alternative techniques and raising new questions is established. This situation gives us a clue about possible sources of questions that appeal to students and are perceived as important or meaningful, questions that help them to critically process given information and eventually develop praxeologies. We conclude that innovative ideas for task design can come from observing and analyzing students’ mistaken work with practice exercises. Sometimes the instructor can predict or even provoke those mistakes. Reshaping standard exercises so that they have multiple ways to solve them, like in our case above, may enrich the possibilities for the tasks that follow. If these tasks “aim directly to develop and refine knowledge in progress” (Gravesen *et al.*, 2017, p. 28), or explain certain puzzling phenomena, they will have the deepening potential and perhaps the research potential, helping students to work with new material in a meaningful way and gain richer experience of mathematical work. Because of their origin such



tasks (compare to *a priori* designed ones) may have a stronger and more personalized effect on learners when used in a didactic setting.

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