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► **To cite this version:**

Mika Gabel, Tommy Dreyfus. To whom do we speak when we teach proofs?. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02422627

**HAL Id: hal-02422627**

**<https://hal.science/hal-02422627>**

Submitted on 22 Dec 2019

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# To whom do we speak when we teach proofs?

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*We assess rhetorical aspects of the flow of proof, a notion that encapsulates various aspects of classroom presentation of proof, proposing Perelman's New Rhetoric (PNR) as a theoretical framework. We present findings from semi-structured interviews conducted with experienced mathematics lecturers, who were asked to reflect about general features and pedagogical considerations of mathematical proofs by relating to two proofs of the 'Two Pancakes theorem'. We focus on PNR's concept of audience and argue that lecturers generally address two types of audience, particular and universal, when teaching proofs and need to balance between the requirements of each audience. We conclude that PNR is suitable to capture this inherent tension in teaching proofs.*

*Keywords: Proof teaching, flow of proof, Perelman's New Rhetoric, mathematical argumentation*

## **Reconnecting mathematics and rhetoric**

The notion 'flow of a proof' has been used by Gabel and Dreyfus (2017) to examine different aspects of classroom presentation of proof: the way that the lecturer chooses to present the logical structure of the proof, to incorporate informal features into the presentation and to account for various mathematical and instructional contextual factors. The analysis of the informal and contextual features requires a theoretical lens that can account for rhetorical features of the presentation. However, traditionally, scholars perceive rhetoric and mathematics as antithetical disciplines (Reyes, 2014), two banks of a river. On one bank - mathematics, dealing with the establishment of unequivocal truths based on the rigorous laws of formal logic, and on the other bank - rhetoric, mostly related to the study of style, expression and ornamental aspects of discourse.

Yet, scholars have been gradually building bridges between the two banks by using rhetorical concepts to gain better understanding of mathematics and mathematical education. Davis and Hersh (1987), for example, challenged the then prevailing opinion that "mathematical truth is established by a unique mode of argumentation, which consists of passing from hypothesis to conclusion by means of a sequence of small logical steps, each of which is in principle mechanizable..." (pp. 59-60) and claimed that "mathematical proof has its rhetorical moments and its rhetorical elements" (p. 60). Ernest (1999) related to other scholars who share that approach and stated that in education this approach is less controversial than in mathematical and philosophical circles. Reyes (2014) adds that mathematics, as a practice of writing, thinking, and arguing, should be studied by rhetorical scholars, and that rhetorical scholars should explore mathematics discourse as its importance is constantly increasing. In this work, we analyse rhetorical features of mathematical proof presentation through the lens of an argumentation theory called 'The New Rhetoric' (PNR, Perelman and Olbrechts-Tyteca, 1969). PNR is a comprehensive theory that allows the considerations of a varied range of argument aspects within a single unifying theory and ties these different aspects to the adaptability to the audience for whom the argumentation is intended.

Therefore, it enabled us to relate to different aspects of the flow as well as to the ways these aspects are intertwined.

### **Perelman's new rhetoric (PNR)**

PNR assumes a speaker addressing an audience and studies techniques that aim to increase audience adherence to the theses presented by the speaker. It concerns the effective use of informal reasoning, i.e. reasoning that promotes audience adherence. PNR was initially designed to complement formal logic and to show how choices, decisions, and actions can be justified on rational grounds; thus, it relates to dialectical, rhetorical and contextual features. It asserts that form is subordinated to content and to the effort to persuade, and that reducing an argument to its formal features undermines the rhetoric features that support its meaning. PNR offers a description of various aspects of argumentation: (i) *scope and organization*, which is a result of the lecturer's need to take into account a complicated, at times contradictory, set of considerations in the proof presentation; (ii) the constitution of a shared *basis of agreement* between the speaker and the audience (iii) the manner by which the lecturer uses rhetorical figures to endow elements with *presence* and to focus audience's attention on them; and (iv) different *argumentation techniques*. Moreover, PNR stresses that an effective argument must be adapted to the audience. In Gabel and Dreyfus (2017) we explained our adaptation of PNR for analysing mathematical proof teaching and how we use PNR to account for different aspects of the flow of a proof; we also discussed aspects (i) and (iii). In the current paper, we focus on PNR's notion of *audience* and examine its relation to inherent tensions in the teaching of mathematical proofs.

### **The universal audience and the particular audience in Perelman's new rhetoric (PNR)**

The audience plays a pivotal role in PNR. Perelman and Olbrechts-Tyteca (1969) make a distinction between argumentation aimed to persuade the *particular audience* addressed by the speaker, which is the actual, physical audience (and therefore the argumentation needs to be adjusted to particular knowledge, experiences, expectations, opinions and norms), and argumentation that transcends particularity and is aimed to convince what is called a *universal audience*. The universal audience is a mental construct of the speaker, composed of all normally reasonable and competent people, where competence is specific to a discipline or culture, and consists of a series of beliefs, agreements and language that are typical for this discipline, whether it is of scientific, juridical or other nature. Such agreements may be the result of certain conventions that characterize audiences, usually distinguishable by their use of a technical language of their own (van Eemeren et al., 2013). While every argument is directed to a specific individual or group, the speaker decides what information and approaches will be convincing according to the universal audience that s/he has in mind. Thus, some arguments appeal only to particular groups in a particular context and some arguments attempt to have a broader appeal. By addressing differences between universal and particular audiences, Perelman believes he can better distinguish between a merely effective argumentation that persuades a particular audience, and a genuinely valid argumentation that convinces the universal audience. In that sense, a universal audience may be used as a standard of relevance (Crosswhite, 1989).

## Rationale and goals

This work is situated within the growing research field concerned with different styles that mathematical lecturers employ (e.g., Hemmi; 2010) and the various pedagogical considerations they apply while teaching proof (e.g., Dawkins & Weber, 2017; Lai & Weber, 2014).

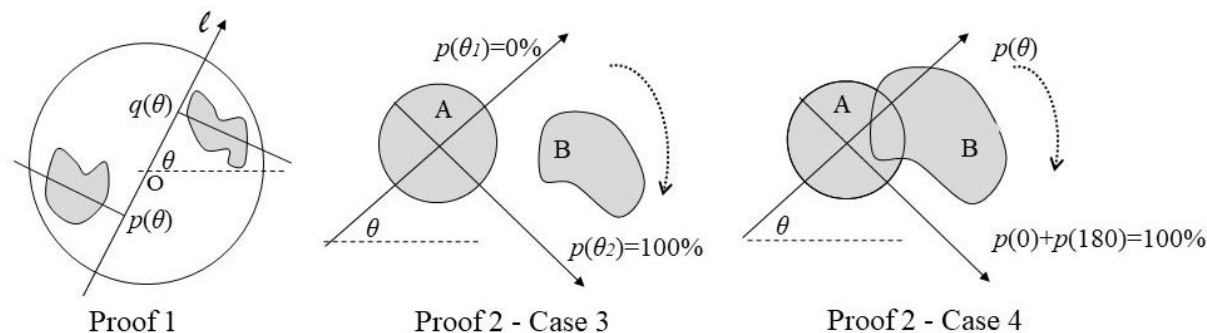
The goals of this paper are firstly to present different pedagogical dilemmas that mathematicians have when teaching proof at the undergraduate level and secondly to demonstrate how these dilemmas can be interpreted by using PNR's notion of two audiences. We show how the tension between the universal and particular audiences can explain decisions taken by mathematics lecturers and focus on conflicts that emerge during proof teaching between the lecturers' own views of mathematics and the characteristics of their students.

## Method

We present findings from interviews conducted with five experienced mathematics lecturers (10-40 years of experience), teaching a variety of tertiary level mathematics courses to diverse student populations (engineering students in college or university, mathematics students, prospective teachers, computer science students, high-school students). They volunteered to be interviewed and were asked about features of mathematical proofs and considerations for proof teaching by relating to the 'Two Pancakes theorem' and its two proofs (Davis and Hersh, 1983) outlined below:

Theorem: Given two arbitrary closed and bounded areas in the plane, A and B, there exists a line that simultaneously bisects the two areas.

*Proof 1* uses an arbitrary point O and a directed line  $\ell$  rotating through O (Figure 1); one defines functions  $p(\theta)$ ,  $q(\theta)$ : the coordinates on  $\ell$  of the lines perpendicular to  $\ell$  that bisect areas A, B. If  $r(\theta)=p(\theta)-q(\theta)$  is positive for some  $\theta$ , then it is negative for  $\theta+180^\circ$ . Hence, there exists  $\theta_0$  for which  $r(\theta_0)=0$ , i.e.  $p(\theta_0)=q(\theta_0)$ , according to the intermediate value theorem (IVT).



**Figure 1: Drawings for Proof 1 and Proof 2 (Case 3, Case 4)**

*Proof 2* is based on five successively more general cases. Cases 1-2 are trivial: If A and B are circles (concentric in Case 1, non-concentric in Case 2) then the line through their centres bisects both. Case 3 (Figure 1): A is a circle, B does not overlap A; when rotating the diameter of A, the part of the area of B that lies on one side of the diameter,  $p(\theta)$ , changes from 0 to 1; the existence of  $\theta_0$  for which  $p(\theta_0)=0.5$  follows from the IVT. Case 4 (Figure 1): A is a circle and B partially overlaps A; now  $p(0^\circ)+p(180^\circ)=1$ , and the claim follows from the IVT on  $0^\circ \leq \theta \leq 180^\circ$ . Case 5:

Two arbitrary areas; the same argument as in Case 4 applies with the diameter of A replaced by a line that halves the area of A.

The interviews were semi-structured. Most questions related to features of mathematical proofs in general and pedagogical considerations for teaching proof. The interview began with questions relating to the proofs of the above theorem and proceeded to more general questions. The interviews were audio-recorded, transcribed and analysed using principles of verbal analysis (Chi, 1997). The analysis was carried out in five steps: (1) Segmenting and reducing: Segmenting the transcriptions into modules that contain an answer to a specific question. Each module was reduced by selecting significant utterances; (2) Coding and organizing: Most categories were created naturally by the interview questions and some were created when reading the reduced data. The relevant utterances were summarized and placed in the appropriate category; (3) Operationalizing evidence: A collection of utterances that constitute evidence for each category was created; (4) Seeking for patterns in the organized data. When such patterns emerged, they were validated by looking for further evidence; (5) Repeating the procedure in order to verify the coding and alter it if necessary.

In this paper, we focus on the lecturer's answers to the following three questions:

1. Which of the two proofs would you use for teaching this theorem, and why?
2. What aspects of the proof do you emphasize when teaching a proof?
3. What language (formal/informal) do you use when teaching a proof?

## Findings

We start by presenting the interviewees' answers to the first question: which of the two proofs they would teach and why. Proof 1 was chosen by three interviewees: Sally, Dana and Max. Max claimed that Proof 1 is more correct and properly built: "[it] is built as a proof: you start at the beginning and reach the end. [In Proof 2] I am told: take a case and prove it... What would happen if there were 79 cases?! ... It's more elegant to find one proof that does not require the separation into cases"; he also stated that "students should learn the proper way to prove claims" and that the many cases of Proof 2 might "drive the student crazy". Dana stated that the division into cases makes "Proof 2 hard to remember" and Sally stated that it worries her: "...I am always concerned that a case is missing, it's stressful...". Proof 2 was chosen by Tara and Anne who stated that it is more intuitive and less 'tricky'. Tara said: "I relate better to the idea of halving the first domain and then the second one, than to the idea of the perpendicular lines". Anne said that "it enables to teach students how to 'play'" and that the gradual increase in complexity allows students to fully understand each case before proceeding.

The lecturers also suggested how to improve the proofs. Sally, Dana and Max preferred teaching Proof 1 but were aware of its difficulties, mainly the 'Deus ex Machina' nature of the function  $r(\theta)$ . Sally suggested to "divide the board, write down the stages and finally ... fill in the details. First the 'how' then ... the 'what' ... define a-priori where [you are] heading", because "...completing the details is [only] half of the work...". Max suggested using Cases 1-2 of Proof 2 to increase students' intuition regarding the meaning of the claim. Dana suggested emphasizing one central idea: the construction of the distances. These lecturers chose Proof 1 because it agrees with their beliefs about what a "good, proper mathematical proof" is, but contemplated how to improve its

communication to their students. Tara and Anne preferred teaching Proof 2; they felt its intuitive nature is better adapted to their students. The division into cases, a disadvantage for others, is an advantage for Anne, who believes that the gradual increase in complexity improves students' learning experience. Tara said that Proof 2 is more intuitive to her than the perpendicular lines of Proof 1; She said: "...I don't like it when proofs are based on an idea that I would not spontaneously think about... if it can be done... without 'tricks', I prefer it". Yet she suggested shortening the proof by omitting Cases 1-2.

Although these findings indicate different lecturer choices, there were noticeable commonalities in their answers. Firstly, Tara and Max chose different proofs, but were both bothered with the imprecise way of using the IVT and stated that one should explicitly define functions and justify their continuity before applying the IVT. For them, some standard of mathematical rigor is important. Secondly, several lecturers refer to meta-proof issues. For example, Sally refers to a proof plan, Anne to pre-proof activities ('play') and Max to the pedagogical value of using precise language. Thirdly, the lecturers address affective aspects, and subjectivity is demonstrated by the opposite attributes that different lecturers relate to the same proof feature. Whereas Sally and Max feel that the division into cases might disturb the students, Anne sees the gradual increase in complexity as giving students a sense of understanding and stability. Thus, the lecturers' choice of proof is highly influenced by their perception of the students attending the lesson. Moreover, earlier in the interview, when asked about their proof preference some lecturers stated that the choice mostly depends on the intended audience. Sally, for example, stated that Proof 1 is clearer in her opinion but that Proof 2 is better for her students. Then, when asked to choose which proof to teach she chose Proof 1 and described pedagogical ways to improve its presentation. The choice of proof appears to be an amalgam of mathematical and pedagogical considerations, personal preferences and convictions.

In the second question, the interviewees were asked what aspects they emphasize while teaching proofs. They all stated that their answer is population dependent. Table 1 summarizes their answers and reveals that lecturers invest a lot of effort in communicating the proof, and raising students' awareness of meta-proof and rhetorical aspects (e.g., aesthetics, and significance). Tara tries "...to look for... the significance, why this conclusion... [and] the way [it] was derived is important...". Anne focuses on pre-proof activities and states that she partly "... wants to teach them how to transform a 'game' into a proof...how to produce... the simplest example that still maintains the features of a problem, ... [how] to try to prove or refute when you still don't have a clue if the claim is true or false...". Max adds that his verbal explanations are what "really explains it to students".

Highlighted aspects	Suppressed aspects
1. Ideas that are: (i) repeated in future proofs or (ii) related to the current content {A, S, D} 2. The thoughts and intuitions that led to the proof {M} 3. What was proved and what "skipped" {T, D} 4. Beauty, aesthetics {T} 5. Intermediate summaries {S} 6. Proof structure, proof type (e.g., proof by induction)	1. Repetitive stages that do not contain new ideas or new calculations {D, S} 2. Details that are not directly relevant to the work of the students {S} 3. Stages that are too difficult to the current level of the students

{A}; dividing the proof into stages/ parts {M} 7. Flow; one thing arises from another, like a chain{A} 8. Completely justifying each passage {A} 9. Useful “pre-proof” activities: how to formulate an effective example; how to approach the proof when it is still not known if the claim is true or false {A} 10. Difficult points in the proof {M} 11. Significance and relevance of the theorem {T, D}	{S} 4. Sometimes – formality, if the proofs are trivial enough and the concepts have been thoroughly exercised {A}
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**Table 1: Proof features that are highlighted or suppressed while teaching**

In the third question, four were explicitly asked about their use of formal language (the fifth interviewee referred to the use of formal language throughout the interview). Sally used an analogy to explain the importance of using formal language: “... lawyers use words that have meaning only in the context of the law discipline. It's not the same meaning as their dictionary definition...” She says that similarly, students should master the proper way of writing proofs, using the necessary symbols and language. Tara and Anne demonstrate flexibility regarding the use of formality in places where it might obstruct student’s learning. For example, Tara states: “in one classroom I might describe the concept of a limit very informally... as a sequence of values getting nearer some value; in a different classroom, I have to use the epsilon-delta definition with extreme formality”. Anne admits that “...somewhat reducing the formality” might harm students’ understanding but supports her choice by stating that otherwise students may “... lose their grasp of ‘the whole’... they need to deal with parts... [and] sub-parts that are not well constructed yet...” Dana’s approach marks the other end of the range, for she speaks of ‘gluing’ things in a non-formal way and clearly expresses her preference to emphasize central ideas rather than to dwell on technical steps. Dana advises students to concentrate on the main, less technical, proof ideas. Thus, the lecturers’ answers to the third question reflect a range of attitudes: from a strict use of formal language, via compromises between formality and audience, to an entirely context dependent attitude. They balance between helping the students to be sufficiently acquainted with the customary mathematical language on one hand, and not getting lost in the rigidity and details of extreme formality on the other hand. At the same time, the proof presented to the students should reasonably satisfy some standard of the mathematical community, the same way a legal document should accept approval of law experts.

To summarize, the interviewees discussed a rich collection of proof features, constantly referring to pedagogical considerations. Two themes, subjectivity and context-dependent value of proof features, particularly the type of student population, were prominent throughout the interview.

### **Discussion of the findings within PNR framework**

The findings show that the interviewed lecturers have a clear vision of different mathematical ideas and features of the two proofs, combined with a rich collection of pedagogical considerations. These findings are consistent with the literature (e.g., Hemmi, 2010; Lai and Weber, 2014). However, we wish to focus on the manner in which the lecturers consider their students, i.e. the audience, while teaching proofs. Lai and Weber (2014) explored factors that mathematicians claim to consider when preparing a pedagogical proof and found that audience diversity influenced

mathematicians in four aspects: (a) the assumed previous knowledge; (b) the actions that need to be taken to avoid potential student difficulties; (c) using techniques students find familiar or comfortable; (d) what mathematical ideas are emphasized by the proof. Indeed, most of these issues were also raised by our interviewees.

Our data show that when mathematicians teach proofs a tension between two poles arises: the first pole is the wish to present the proofs in a rigorous manner acceptable to professional mathematicians (as Max said “the proper way to prove”); the second pole is the wish to accommodate students’ cognitive and affective needs so that they will gain adequate understanding and a rewarding learning experience. This tension is widely found in the literature (e.g., Dawkins and Weber, 2017; Lai and Weber, 2014). Dawkins and Weber (2017), for example, investigate values and norms of mathematicians regarding proof and acknowledge that sometimes the needs of the mathematical community differ from those of the classroom community, which cannot perfectly mimic practices of professional mathematicians. They claim that expecting students to adopt mathematicians’ proof norms without perceiving the underlining values might cause a dissonance between the mathematicians’ and the students’ communication culture. They also state that one outcome is that “researchers... have sensibly advocated loosening various norms for the purpose of encouraging students’ genuine insights” (p. 133). Within PNR, this dilemma is an inherent feature of simultaneously addressing the universal and particular audiences. It is not about “compromising” but about adjusting the argument and creating a shared basis of agreement with the students while maintaining a proof presentation that would still be acceptable by professional mathematicians (possibly imagined as sitting at the back of the class). Moreover, PNR not only relates to the difference between the lecturer’s and the students’ norms and values but also to other rhetorical and informal features of the proof classroom presentation (Gabel and Dreyfus, 2017), as well as to other types of lecturer-student gaps in the argumentation and their sources.

This may also explain why during the interview, the lecturers frequently answered interview questions focused on mathematical proofs in general by raising pedagogical considerations related to the particular audience, constantly shifting between two perspectives: mathematician and teacher. We give three examples. Sally stated that as a mathematician, she prefers Proof 1, but Proof 2 better fits her students. Nevertheless, when asked to choose a proof to teach she chose Proof 1, suggesting pedagogical ways to improve its presentation. Sally also stressed the importance of formal language in class, but admitted skipping repetitive stages of proofs, and suppressing details that might currently be too difficult for her students. A similar tension can be found in the suggestion Max made regarding the improvement of Proof 1. Max declared the importance of proof formality and preciseness and chose Proof 1 both as his personal preference and as the better proof to teach. However, he suggested improving the presentation of Proof 1 by adding Cases 1–2 of Proof 2, in order to enhance students’ intuition regarding the claim’s meaning. A similar tension appeared when Anne, who chose to teach Proof 2 for the feeling of understanding it provides referred to formal language by saying that “...one can skip formality if one has the ability to reconstruct it ... but [students] don't know how to do that, so it's a kind of ‘pretence’ understanding”.



Mathematics lecturers address two audiences when teaching proofs: particular and universal. The particular audience is the actual group of students attending the lesson, who bring with them their previous knowledge, attitudes, learning habits, beliefs and cultural conventions. The lecturer needs to address the needs of the particular audience, to maintain a shared basis of agreement with the students and persuade them that the claim was truly proved. In parallel, the lecturer needs to convince the universal audience, which is a mental construct of the lecturer; it may include professional mathematicians, experts of the taught material or admired lecturers. Convincing the universal audience requires different standards of formality, preciseness and rigor. Whenever mathematics lecturers teach proofs, they need to balance between the different requirements of both audiences. The need for this balance is demonstrated, for example, by Dana, who admits that in class she prefers to emphasize important ideas rather than to dwell on technical steps “in an attempt not to give up on classroom proving altogether because then understanding drops to a very low level”.

Thus, PNR is a unifying and comprehensive theoretical framework that can be used to interpret a major challenge of proof teaching. Within PNR this challenge is considered an inherent tension that always exists between the need to present a proof that is convincing to the universal audience, in our context a complete, flawless proof accepted by expert mathematicians, and between the need to persuade the particular audience, in our context the students actually attending class. PNR also relates to other rhetorical and informal features of classroom presentation of proof, such as means to endow proof elements with presence (Gabel and Dreyfus, 2017) as well as to lecturer-student gaps in the argumentation and their sources. Therefore, PNR may be used not only to describe mathematicians’ proof presentations but also to evaluate the presentations and to provide practical suggestions of how to improve them.

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