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► To cite this version:

Matija Bašić, Željka Milin Šipuš. Students' understanding of the interplay between geometry and algebra in multidimensional analysis: representations of curves and surfaces. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02422575

HAL Id: hal-02422575

<https://hal.science/hal-02422575>

Submitted on 22 Dec 2019

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Students' understanding of the interplay between geometry and algebra in multidimensional analysis: representations of curves and surfaces

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The aim of this study is to investigate mathematics students' understanding of geometrical objects such as curves and surfaces in space, when these objects are presented algebraically, that is, by their equations. Flexible use of different representations, in this case geometric and algebraic, as well as the ability to convert between implicit and parametric types of equations, is often assumed as a prerequisite in multivariable calculus. We have performed an in-depth analysis of students' answers to exam questions to identify types of students' difficulties in these conversions. However, our results also point to the impact of the didactic contract on students' productions which possibly prevented students to fully present their understanding.

Keywords: Transition across university mathematics, multivariable calculus, curve, representation, didactic contract

Introduction

In this paper we investigate learning difficulties that students face in the transition from the first courses in linear algebra and multivariable calculus, to more advanced mathematical courses that involve mathematical analysis of functions of several variables. As described by Kondratieva and Winsløw (2018), requests for efficiency and economy of exposition at university level often lead to narrow and disconnected teaching modules. Our teaching experiences indicate similar disparities in the transition from courses in analytical geometry, linear algebra and calculus on one side, to mathematical analysis of functions of several variables and differential geometry on the other. We noticed misconceptions and knowledge gaps in dealing with basic objects such as curves and surfaces in three-dimensional space, in using their implicit and parametric representations, and in converting from one representation to the other. These notions are essential for calculating curve integrals, applying Green-Stokes theorem, or in introductory lessons in differential geometry, for instance, the introduction of curve and surface curvature.

A curve in three-dimensional space can be presented either by a parametrization, that is, as a trace of a moving particle, or by a system of equations in three variables, that is, as an intersection of surfaces. Transitions from one representation to the other and vice versa are theoretically underpinned in multivariable calculus by implicit and inverse function theorems. However, students' work in basic examples usually happens in the domain of algebra and geometry – solving systems of (not necessarily linear) equations and giving them geometrical meaning. We noticed many students' difficulties in this process, starting from a non-meaningful manipulation of equations in a way that just leads to new equations, understanding what a solution of a system of equations is, especially when a solution is not unique, and, finally, providing a geometrical interpretation of the obtained objects. Students' difficulties in the same field but with linear objects,

in recognition of lines and planes given by their implicit (Cartesian) and parametric equation, and the presentation of these objects as sets (subsets in space satisfying the same conditions or equations) are well evidenced (Artigue, 1999; Alves-Dias; 1998; Nihoul, 2016). However, students' work related to curved objects in space is an under-researched area.

Motivation

Our study was motivated by the following question posed as one of five questions in the midterm exam for the course Introduction to differential geometry in spring semester 2017:

Question. Prove that the space curve given by $x^2 + y^2 = z$, $x^2 + y^2 + 2x + 2y = 1$ is planar and determine its osculating plane.

The question could be answered by following the procedure taught during the exercise classes (for a somewhat more complicated curve), which requires a curve to be parametrized. From the parametrization, one calculates its torsion, which should vanish for a planar curve, or determines its binormal field, which should be constant. However, in the above example the tedious calculation of a torsion (which often results in a calculation error) could have been avoided by using the definition of the planar curve emphasized during lectures. By a simple manipulation of the given equations, it is possible to eliminate the square terms to get $2x + 2y + z = 1$, which could be recognized as an equation of a plane in which the curve lies. Among 136 students who provided any answer to this question (of 172 who participated in the exam), only 7 % (10 students) solved the task using that approach, and 10 % more solved it using both approaches, that means, they obtained the equation $2x + 2y + z = 1$, and then still parametrized the curve and applied the tedious procedure. Some 27 % of students obtained the plane equation without providing an interpretation, and 30 % of students solved the question by immediately parametrizing the curve and applying the procedure only. The remaining 26 % did not obtain the equation nor a correct parametrization. This high number of students who focused only on procedure and failed to apply the definition of a planar curve, encouraged us to implement a more thought-out approach in the next year courses followed by a deeper analysis of students' work.

Theoretical framework and research questions

Mathematical concepts are identified and represented through various representations, which reflect possibly different features of a concept, but simultaneously complement each other. Treating a concept within a certain representation (representational mode or register) and successfully converting between different registers of representation, is considered as a prerequisite for conceptual understanding. Conversion processes entail recognition of the same mathematical objects through representations from different registers and therefore require coordination of registers. Many students' difficulties can be described and explained by the lack of coordination of different registers of representation (Duval, 1993, 2006). Research provides examples (Artigue, 1999; Alves-Dias, 1998; Nihoul, 2016) of students' difficulties in using the algebraic representation of lines and planes in space, and vice-versa, in recognizing straight lines and planes in space from their equations, and in converting between parametric and implicit viewpoints.

In this study we tried to answer the following research question: What are students' learning difficulties in relation to different representations of curves and surfaces in 3-dimensional space?

As observed in the motivational example, it seems that students tend to use lengthy procedures as opposed to conceptual definitions. This indicates that the work of some students does not show their complete mathematical understanding and difficulties, but rather that their answers are governed by reasons of didactical nature. We use the notion of didactic contract from the Theory of Didactic Situations (TDS) by Brousseau (1997), a framework useful to study phenomena at university level (González-Martín, Bloch, Durand-Guerrier, & Maschietto, 2014):

A core TDS conceptual tool is Brousseau's notion of the *didactic contract*, the implicit set of expectations that teacher and students have of each other regarding mathematical knowledge and regarding the distribution of responsibilities during the teaching and learning processes. ... The *didactic contract* is linked to an institution and, in particular at university level, the terms of the *didactic contract* can be quite strong. (pp. 119, 131)

Although our ultimate goal is to study the cognitive and epistemological dimensions for the concept of a curve in space, we first had to analyze the didactical constraints that govern students' answers. Hence, motivated by our students' work we wanted to trace the impact of the didactic contract on students' exam productions and investigate if students' understanding could be improved through some changes in this contract.

Methodology and mathematical context

The participants of our study in 2018 were in total 173 students at a department of mathematics in Croatia, while participating in elective course Introduction to differential geometry (students of undergraduate program in mathematics and in mathematics education). The course has 2 hours of lecture and 2 hours of exercise class per week for 13 weeks. It is naturally divided into two parts, with the first part that focuses on the use of differential calculus to study geometrical properties of curves in 2D and 3D (curvature and torsion). The second part focuses on the study of surfaces and special curves on surfaces. The emphasis of the course is on low dimensions to encourage students to use the geometrical register in interpreting and understanding the results. The second author is the lecturer, while the first author is one of three teaching assistants for the course.

In 2018, during the exercise classes, the teaching assistants led the discussion about the exam question from the year before. The aim of this discussion was to re-negotiate the didactic contract built during previous courses in which teachers assign tasks and students answer in the way they were taught. In this case, it means that students apply the lengthy procedure, not the definition of a planar curve in space, which is intuitive and, although presented during the course, not exclusively a part of differential geometry content. As learning can be accomplished, in many cases, through the ruptures of the didactic contract (Brousseau, 1997), the assistants showed both strategies and emphasized that the aim of the course is to encourage students to use the geometrical interpretation. In other words, it is the conceptual and not the lengthy approach that was expected and aimed for. It was important for the students to hear that all correct answers are acknowledged at the exam, not just those that follow procedures from the exercise classes, since their comment during the discussion on the "non-standard" approach was that "they did not know this was allowed". In

addition, the teaching assistants addressed other students' difficulties observed by the preliminary insight into students' work in 2017, like false generalizations and analogies from plane to space geometry (2D to 3D) in interpretation of equations and non-adequate conversions between implicit and parametric viewpoints of a curve.

The source of data for the in-depth analysis were midterm exam solutions. Questions were formulated in the geometric register, where curves are given as intersections of surfaces. The properties of the intersection curve can be recognized algebraically (from its equation) or geometrically (by some geometrical properties, such as its curvature). In this paper we analyze the questions presented in Figure 1.

Question 1. Determine the set of points in space given by the equations $-x^2 + y^2 + z^2 = 4$, $x - y = 0$. Find its parametrization and determine its curvature. Which object does it represent geometrically?

Question 2. The curve c is given by the equations $x^2 + y^2 + z^2 = \frac{17}{2}$, $y = x + 1$. (a) Explain why c is planar. (b) Determine the curvature of c and decide what curve it is. (c) What curve is obtained by projecting c onto the xz -plane?

Figure 1: Exam questions analyzed in the study

In Question 1, the set of points satisfying both equations, the equation of a hyperboloid of one sheet and a special plane, is to be identified either from a system of transformed implicit equations, or from its obtained parametrization which enables determining the curvature. Simple calculation in the algebraic register gives the system of transformed implicit equations as $x = y$, $z = \pm 2$, therefore the intersection set consists of two parallel lines that are given as intersections of two pairs of non-parallel planes.

In Question 2, the expected solution is to state that $y = x + 1$ is an equation of a plane, so the curve is planar by definition. For the second part it is expected that the equation of the sphere is recognized, and that the curve is a circle since it is the intersection of the sphere and the plane. The radius R of the circle may be determined geometrically, so the curvature can be expressed as $1/R$. The projection onto xz -plane is given by the equation $2(x + \frac{1}{2})^2 + z^2 = 8$, which is the equation of an ellipse. Another correct approach is to parametrize the obtained curve, and then to calculate its curvature by using the formulas from the lecture. Conclusion that the curve is a circle should follow from the fact that the curvature is a non-zero constant, and the projection curve could be recognized from the parametrization with $y = 0$.

Students' answers were processed in an iterative way. First, based on our mathematical analysis of the question, we formulated codes regarding the register in which the task was solved: algebraic (following the algebraic procedure given in the exercise class) or geometric (geometrically interpreting equations), as well as the success of the students (if the calculation was completed or the conclusion reached). Second, the codes were refined to track students' different strategies and

unexpected answers pointing to some difficulties. Next, the observed categories and special cases were discussed between the authors and an agreement was reached to cluster the codes with low frequency pointing to the same type of difficulties. As an example, in Table 1 we present the data for Question 2 (a).

Question 2 (a)			
No answer		57	33%
Recognizing the equation of the plane		46	27%
Using torsion = 0 as criterion	Showing that torsion equals 0	8	5%
	Only stating the criterion	23	13%
Wrong arguments	$y = x + 1$ is a line, and lines lie in planes	15	9%
	$x^2 + y^2 + z^2 = 17/2$ is a plane	7	4%
	Substituting and stating that the curve is an ellipse in yz -plane	17	10%

Table 1: Codes and frequencies for Question 2(a)

Results

In Question 1, approximately half of the students (51 %) noted correctly that the intersection of a doubly-curved surface (an hyperboloid of one sheet) and a plane is a pair of parallel straight lines. Most of the students (76 %) who reasoned correctly relied on parametrizing the obtained set as $c(t) = (t, t, \pm 2)$ and calculating its curvature, which turned out to be 0 in all of its points, and some (29 %) relied on recognizing the lines from the system of transformed equations, whereas 5 % of students used both approaches.

Among students who did not provide a complete solution, approximately half gave no answer at all, further 30 % successfully obtained transformed equations or calculated the curvature, but did not interpret the result geometrically and approximately 20 % misinterpreted the obtained result. Most of the errors in the last group occurred when trying to interpret a transformed implicit equation, which was seen as the requested intersection curve, and not as a new surface on which the intersection curve lies (the intersection set described as a plane, a cone, or a hyperbolic paraboloid); many errors also come from the misinterpretation of a transformed implicit equation without z -coordinate as a curve (a hyperbola instead of a hyperbolic cylinder).

In Question 2, 46 out of 173 students (27 %) noted that the given equation is an equation of a plane and successfully concluded that the curve is planar without further calculation, and a smaller number (18 %) of them went still for calculation of the torsion. More students than in 2017 successfully concluded that a curve is planar by the first approach, but still more than 70 % did not use this approach. Among the students that did not provide a solution, 9 % stated that $y = x + 1$ is an equation of a straight line in space, which is a well-evidenced false generalization in transition from 2D to 3D in linear algebra (Nihoul, 2016). Further students' false argumentation runs as follows "since this is a line, and lines belong to a certain plane, therefore the curve is planar". Furthermore, careless approach to ambient space in which a curve is given seems to generate another case of false transition from 2D to 3D, since 38 students chose the formula for curvature that is the formula for curves in \mathbb{R}^2 , not in \mathbb{R}^3 .

Among 122 students who tried to parametrize the curve, 80 students (65 %) used the local parametrization $c(t) = (t, t+1, \sqrt{17/2 - t^2 - (t+1)^2})$ of the intersection curve, which is not adequate to conclude that the (whole) curve is planar, contrary to the linear case, where the standard procedure provides a global parametrization. To determine the curvature of the curve, students predominantly (70 %) used parametrization for further calculation over geometric approach. Only 15 (9 %) of them used a geometric approach by stating that the plane intersection of a sphere is a circle, with only 4 of them even moving forward to determine geometrically the radius of a circle.

Discussion

In general, we may state that students' difficulties in *converting algebraic equations into geometrical objects* are persisting, even after the first courses in linear algebra and calculus. In the case of a planar curve identification, students predominantly focus on the known procedure, although it is burdened with calculation, over using the conceptual definition of a planar curve which involves recognizing a plane by its implicit equation. We may also say that we observed the influence of the didactic contract, as the students tried to ensure that their answers included newly learned procedures during the exercise classes. However, students' difficulties in recognizing a plane by its equation can be seen as a particular case of a more general lack of coordination between algebraic equations and spatial geometrical objects. When manipulating equations defining a spatial curve, students interpret the new implicit equation as that intersection curve, not as a new surface on which the curve lies.

The above general observations point out to several further remarks. The first remark concerns evidence on students' false generalizations and analogies in their *passage from 2D to 3D*. As already stated, this was particularly the case when they identified the equation of a special plane with a line, a circular cylinder with a circle, or a hyperbolic cylinder with a hyperbola. Similar difficulties are observed in linear algebra, e.g. in the case of a plane in space given by an equation $ax + by = c$ (Alves-Dias, 1998; Nihoul, 2016). It is a clear example of the fact that a piece of existing knowledge (a single equation represents a curve) may undergo a hasty generalization, and therefore standing in the way of full comprehension. It is not irrelevant to notice that the missing insight of different ambient (2D and 3D) is also identified in students' inappropriate use of formulas for the curvature of curves.

Our second remark concerns students' use of and conversion between *the implicit and the parametric representation (viewpoints)* of curves. Again, students' difficulties in this process in linear algebra are well evidenced (Artigue, 1999; Alves-Dias; 1998; Nihoul, 2016). In our study with curved objects, we observed that students did not invoke the need for conversion to a parametric representation, nor is the conversion found as flexible as it is assumed, already discussed in the case of linear algebra (Artigue, 1999, p. 1384):

Flexibility seems to be considered as automatically internalized once one has "understood" the notion, as if it were a simple question of homework that one can leave to the private work of the student.

Another remark is related to students' difficulty coming from *the character of non-linear problems*. Conversion from implicit equations to parametrization and vice-versa, that relies on a procedure of

“eliminating a parameter” or “substituting a variable by a parameter” from linear algebra has a local character in the non-linear case and cannot produce an instant answer as in the linear case. This conversion brings many different transformation approaches, usually non-routine ones (especially when trigonometric and hyperbolic functions are involved). For example, in Question 2, a local parametrization is not adequate to conclude that the curve is planar. A relevant explanation can be provided with the implicit and inverse function theorems, which although do not provide a recipe to change representation, should be discussed during the class to emphasize the subtleties in the non-linear case. Students’ focus on procedures suggests that their knowledge is an “amalgamation of practical blocks” (Brandes & Hardy, 2018, p. 506). Moreover, the observed absence of theoretical insight in students’ knowledge might be seen as a missing brick in Transition of type 1 (Winsløw, Barquero, De Vleeschouwer, & Hardy, 2014).

Conclusion

Different representations of curves and surfaces and their flexible conversions are essential for various mathematical courses that involve the analysis of functions of several variables. We also repeat that our study tries to see some of the students’ bad performances as a result of the impact of the didactic contract. However, students’ productions firstly point out to the existence of difficulties which were also observed in linear algebra in representations of lines and planes (Artigue, 1999; Alves-Dias; Nihoul, 2016). False generalizations from 2D to 3D appear (a line in space is represented by a single implicit linear equation, and vice-versa, single implicit linear equation with, for instance, no z -coordinate is an equation of a line in space; the same student’s reasoning appears for e.g. a circle and a circular cylinder). This observation may also point to the presence of some learning obstacles in the sense introduced by Brousseau (1997), which needs to be further explored. According to Brousseau, obstacles of various origins can stand in the way of an efficient learning of mathematics. Brousseau distinguishes between cognitive (due to the state of mental development), didactical (due to a certain way of teaching) and epistemological obstacles (due to the very nature of mathematical concepts). Clear distinction among them, due to the complex nature of human’s knowledge acquisition, is not always possible. Obstacles become evident in students’ errors which are not related to chance but persist, or in difficulties and problems that students encounter that slow down the learning process. When this becomes evident, it does not point out to the lack of knowledge, but often to the piece of knowledge that was adequate in a previous situation, but now leads to incorrect reasoning. Brousseau suggests that to overcome an obstacle, a sufficient flow of new situations challenging old knowledge is needed. Whether careful task design with different potentials for student learning would be beneficial as well (Gravesen, Grønbæk, & Winsløw, 2017), is another question to be explored in the future.

Acknowledgment

We dedicate this paper to the memory of our dear colleague doc.dr.sc. Stipe Vidak (1980-2018) and to our many fruitful joint discussions.

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