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Creation and annihilation of point-potentials using Moutard-type transform in spectral variable*

P.G. Grinevich [†] R.G. Novikov [‡]

Abstract

We continue to develop the method for creation and annihilation of contour singularities in the $\bar{\partial}$ -spectral data for the two-dimensional Schrödinger equation at fixed energy. Our method is based on the Moutard-type transforms for generalized analytic functions. In this note we show that this approach successfully works for point potentials.

Key words: two-dimensional Schrödinger equation, Faddeev eigenfunctions, fixed-energy spectral problem, spectral singularities, generalized analytic functions, Moutard-type transforms, point potentials.

MSC: 30G20, 37K35, 35J10, 47A40, 81Q05, 81U15.

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1 Introduction

We continue studies of the two dimensional Schrödinger equation at fixed energy E

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad (1)$$

where $v(x)$ is a real-valued function on \mathbb{R}^2 with sufficient decay at infinity.

More precisely, for equation (1) we continue studies of the direct and inverse scattering transform. We are focused on the simplest case of real negative energy E .

In order to define generalized scattering data $B = B_E$ for equation (1) at a negative energy E , one uses the Faddeev's eigenfunctions $\psi = \psi(x, k)$ (see [5, 6, 20, 7, 13]), specified by the asymptotic condition:

$$\psi = e^{ikx} (1 + o(1)) \quad \text{as } |x| \rightarrow \infty, \quad k \in \Sigma_E, \quad (2)$$

where

$$\Sigma_E = \{k \in \mathbb{C}^2 : k^2 = E\}, \quad E < 0, \quad E \in \mathbb{R}. \quad (3)$$

We say that Σ_E is the spectral variety for fixed E . We recall that $\Sigma_E \approx \mathbb{C} \setminus 0$ and has the following parametrization:

$$\Sigma_E = \left\{ k_E(\lambda) = \left(\left(\lambda + \frac{1}{\lambda} \right) \frac{i\sqrt{|E|}}{2}, \left(\lambda - \frac{1}{\lambda} \right) \frac{\sqrt{|E|}}{2} \right) : \lambda \in \mathbb{C} \setminus 0 \right\}. \quad (4)$$

We recall that the restriction of the Faddeev eigenfunction $\psi(x, k_E(\lambda))$ to Σ_E for each fixed x is a generalized analytic function with respect to the spectral variable λ ; see [8, 7]:

$$\frac{\partial}{\partial \bar{\lambda}} \psi(x, k_E(\lambda)) = B_E(\lambda) \overline{\psi(x, k_E(\lambda))}, \quad E < 0, \quad \lambda \in \mathbb{C} \setminus 0, \quad (5)$$

and

$$\psi = e^{ik_E(\lambda)x} (1 + o(1)) \quad \text{as } |\lambda| \rightarrow \infty \quad \text{and as } |\lambda| \rightarrow 0. \quad (6)$$

The coefficient $B_E(\lambda)$ in (5) does not depend on x , and plays the role of generalized scattering data ($\bar{\partial}$ -spectral data) for equation (1). Here and below, the notation $f = f(\lambda)$ does not mean that f is holomorphic in λ .

For an introduction to the classical theory of generalized analytic functions (pseudoanalytic functions), see [25, 2].

Note that for real $v(x)$ the data B_E has the following symmetries (see [8]):

$$B\left(\frac{1}{\lambda}\right) = -\lambda^2 \overline{B(\lambda)}, \quad B\left(-\frac{1}{\lambda}\right) = \lambda \bar{\lambda} B(\lambda), \quad \lambda \in \mathbb{C} \setminus 0. \quad (7)$$

We consider the following inverse scattering problem.

Problem 1. Given $B_E(\lambda)$ at fixed $E < 0$, find $v(x)$.

Note that Problem 1 has direct applications to integrating the Novikov-Veselov hierarchy; see, for example, [21, 7, 8]. Note also that Problem 1 admits direct applications to reconstructing potential v , supported in a bounded domain, from the Dirichlet-to-Neuman boundary map; see, for example, [14, 15].

The standard approach to Problem 1 uses the following scheme:

$$B_E \xrightarrow{(5),(6)} \psi \xrightarrow{(1)} v. \quad (8)$$

It is well-known that Problem 1 is uniquely solvable and scheme (8) works perfectly if B satisfies (7), and $B \in L_p$, $p > 2$, at the unit disc $|\lambda| \leq 1$ (see [8, 20]). These conditions are fulfilled if $v(x)$ is sufficiently regular and $|E|$ is sufficiently large. However, if $E_1 < E < 0$, where E_1 denotes the smallest discrete eigenvalue (ground state energy), then B and ψ have strong singularities, which are typically contour poles, and the classical methods of the generalized analytic functions theory do not work. In [8] hypothetical solvability conditions on the singularities at the contours were formulated. In [10, 11, 12] we have shown that singularities with these constraints can be locally removed by applying the Moutard-type transformations for generalized analytic functions. In turn, the construction of [10, 11, 12] was stimulated by articles [23, 24] by I.A. Taimanov dedicated to the Moutard-type transforms for the Dirac operators. For extra information about Motard-type transforms, see also [18, 22, 17, 26, 19, 4].

In this note for the first time we show that, at least in some interesting cases, our approach to creation and annihilation of spectral contour singularities works globally. In particular, we demonstrate global creation and annihilation of spectral contour singularities for the case of two-dimensional analogs of the Bethe-Peierls-Fermi-Zel'dovich-Berezin-Faddeev point potentials; see Sections 3, 5, 6.

As about other approaches to fixed-energy inverse problems for the Schrödinger equation (1) in the case when Faddeev's spectral singularities may be present, see, for example, [20, 3, 16].

2 “Scattering” functions

We recall that for sufficiently regular $v(x)$ the Faddeev eigenfunctions $\psi(x, k)$ satisfy the following generalized Lippmann-Schwinger integral equation (see, for example, [7, 20]):

$$\psi(x, k) = e^{ikx} + \iint_{\mathbb{R}^2} G(x - y, k)v(y)\psi(y, k)dy, \quad (9)$$

where

$$G(x, k) = e^{ikx}g(x, k), \quad (10)$$

$$g(x, k) = -\left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi, \quad k \in \Sigma_E, \quad x \in \mathbb{R}^2, \quad E < 0. \quad (11)$$

Here Σ_E is defined by (4), $G(x, k)$ is the Faddeev’s Green function for $\Delta + k^2$. Note also that

$$|\Re k_E(\lambda)| + |\Im k_E(\lambda)| = \begin{cases} \sqrt{|E|}|\lambda|, & |\lambda| \geq 1, \\ \sqrt{|E|}|\lambda|^{-1}, & |\lambda| < 1, \end{cases} \quad E < 0, \quad (12)$$

where k_E is defined in (4).

In addition, the following formulas hold (see, for example, [7, 20]):

$$B_E(\lambda) = \frac{\pi \operatorname{sign}(\lambda\bar{\lambda} - 1)}{\lambda} b(k_E(\lambda)), \quad (13)$$

where

$$b(k) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i\bar{k}x} v(x)\psi(x, k)dx. \quad (14)$$

Here B_E is the $\bar{\partial}$ -spectral data arising in (5), $b(\lambda) = b_E(\lambda) = b(k_E(\lambda))$ is a part of the Faddeev generalized scattering data. Actually, $b(\lambda)$ can be treated as a non-linear analog of the Fourier transform of the potential $v(x)$. For real $v(x)$ the function $b(\lambda)$ has the following symmetries [8]:

$$b\left(\frac{1}{\bar{\lambda}}\right) = \overline{b(\lambda)}, \quad b\left(-\frac{1}{\bar{\lambda}}\right) = b(\lambda), \quad \lambda \in \mathbb{C} \setminus 0. \quad (15)$$

Symmetries (15) are equivalent to symmetries (7).

If the potential $v(x)$ depends only of the distance from the origin: $v(x) = v(|x|)$, then

$$b(\lambda) = b(|\lambda|), \quad b(\lambda) = \overline{b(\bar{\lambda})}, \quad \lambda \in \mathbb{C} \setminus 0. \quad (16)$$

3 Point potentials

For the two-dimensional analogs of the Bethe-Peierls-Fermi-Zel'dovich-Berezin-Faddeev point potentials $v_{0,\alpha}(x)$ with support at the point $x = \{0\}$, the following formulas hold (see [9]):

$$\psi = \psi_{0,\alpha}(x, k) = e^{ikx} \mu_{0,\alpha}(x, k) = e^{ikx} \left[1 + \frac{\alpha}{1 - \frac{\alpha}{2\pi} \ln(|\Re k| + |\Im k|)} \cdot g(x, k) \right], \quad (17)$$

$$b = b_{0,\alpha}(k) = \left(\frac{1}{2\pi} \right)^2 \frac{\alpha}{1 - \frac{\alpha}{2\pi} \ln(|\Re k| + |\Im k|)}, \quad (18)$$

where $g(x, k)$ is defined in (11), $\alpha \in \mathbb{R}$, $k \in \Sigma_E$, $x \in \mathbb{R}^2$, $E < 0$.

We recall that the function $\psi_{0,\alpha}(x, k)$ satisfies the Schrödinger equation (1) with the point-type potential $v(x) = v_{0,\alpha}(x)$ and condition (2) at infinity (the exact meaning of the Schrödinger equation with the point-type potential is discussed, in particular, in [1], [9]).

In addition, the $\bar{\partial}$ -equation (5) holds, where ψ , b are given by (17), (18) (see [9]). Note also that

$$\frac{\partial}{\partial \bar{\lambda}} a_{0,\alpha}(k_E(\lambda)) = B(\lambda) \overline{a_{0,\alpha}(k_E(\lambda))}, \quad E < 0, \quad \lambda \in \mathbb{C} \setminus 0, \quad (19)$$

where $B(\lambda) = B_E(\lambda)$ is given by (13), (18) and

$$a_{0,\alpha}(k) = \lim_{|x| \rightarrow 0} \frac{1}{2\pi} (\ln(|x|))^{-1} \psi_{0,\alpha}(x, k), \quad (20)$$

$$a_{0,\alpha}(k) = \overline{a_{0,\alpha}(\bar{k})} = b_{0,\alpha}(k), \quad k \in \Sigma_E, \quad E < 0. \quad (21)$$

Finally, we recall that the spectral singularities of $\psi = \psi_{0,\alpha}$ and $b = b_{0,\alpha}$ are described as follows.

Let $E_1 = -\exp(4\pi/\alpha)$, $\alpha \neq 0$, $x \neq 0$. Then (see [9]):

1. if $E < E_1$, then $\mu_{0,\alpha}(x, k_E(\lambda))$ and $b_{0,\alpha}(k_E(\lambda))$ are continuous in $\lambda \in (\mathbb{C} \cup \infty)$ (i.e. in this case there are no spectral singularities);
2. if $E = E_1$, then $\mu_{0,\alpha}(x, k_E(\lambda))$ and $b_{0,\alpha}(k_E(\lambda))$ are singular in λ on the contour

$$C_{E_1,\alpha} = T = \{\lambda \in \mathbb{C} : |\lambda| = 1\},$$

and are continuous in $\lambda \in (\mathbb{C} \cup \infty) \setminus T$;

3. if $E_1 < E < 0$, then $\mu_{0,\alpha}(x, k_E(\lambda))$ and $b_{0,\alpha}(k_E(\lambda))$ are singular in λ on the contours

$$C_{E,\alpha} = \{\lambda \in \mathbb{C} : |\lambda| = \sqrt{E/E_1}\} \cup \{\lambda \in \mathbb{C} : |\lambda| = \sqrt{E_1/E}\},$$

and are continuous in $\lambda \in (\mathbb{C} \cup \infty) \setminus C_{E,\alpha}$.

4 Moutart-type transforms for generalized analytic functions

Consider the basic pair of conjugate equations of the generalized analytic function theory:

$$\partial_{\bar{\lambda}}\psi = B\bar{\psi} \quad \text{in } D, \quad (22)$$

$$\partial_{\bar{\lambda}}\psi^* = -\bar{B}\bar{\psi}^* \quad \text{in } D, \quad (23)$$

where D is an open domain in \mathbb{C} , $B = B(\lambda)$ is a given function in D , $\partial_{\bar{\lambda}} = \partial/\partial\bar{\lambda}$; see [25].

Let $f(\lambda)$, $f^*(\lambda)$ denote a pair of fixed solutions of equations (22) and (23), respectively. A simple Moutard-type transform $\mathcal{M} = \mathcal{M}_{B,f,f^*}$ for the pair of conjugate equations (22), (23) is given (formally) by the formulas (see [9]):

$$\tilde{B} = \mathcal{M}B = B + \frac{f\bar{f}^*}{\omega_{f,f^*}}, \quad (24)$$

$$\tilde{\psi} = \mathcal{M}\psi = \psi - \frac{\omega_{\psi,f^*}}{\omega_{f,f^*}} f, \quad \tilde{\psi}^* = \mathcal{M}\psi^* = \psi^* - \frac{\omega_{f,\psi^*}}{\omega_{f,f^*}} f^*, \quad (25)$$

where ψ and ψ^* are arbitrary solutions of (22) and (23), respectively, and $\omega_{\psi,\psi^*} = \omega_{\psi,\psi^*}(\lambda)$ denotes imaginary-valued function defined by:

$$\partial_{\lambda}\omega_{\psi,\psi^*} = \psi\psi^*, \quad \partial_{\bar{\lambda}}\omega_{\psi,\psi^*} = -\overline{\psi\psi^*} \quad \text{in } D, \quad (26)$$

where this definition is self-consistent, at least, for simply connected D , whereas a pure imaginary integration constant may depend on concrete situation. The point is that the functions $\tilde{\psi}$, $\tilde{\psi}^*$ defined in (25) satisfy the conjugate pair of Moutard-transformed equations:

$$\partial_{\bar{\lambda}}\tilde{\psi} = \tilde{B}\bar{\tilde{\psi}} \quad \text{in } D, \quad (27)$$

$$\partial_{\bar{\lambda}}\tilde{\psi}^* = -\bar{\tilde{B}}\bar{\tilde{\psi}}^* \quad \text{in } D. \quad (28)$$

5 Creation of point potentials by Moutard-type transforms

Let us start with the Schrödinger equation (1) with the zero potential $v(x) \equiv 0$. In this case for $E < 0$ one has:

$$\psi(x, k) = e^{ikx} = e^{-\frac{\sqrt{|E|}}{2}(\lambda\bar{z} + \frac{z}{\lambda})}, \quad \lambda \in \mathbb{C} \setminus 0, \quad (29)$$

$$B(\lambda) = B_E(\lambda) \equiv 0, \quad (30)$$

where $B(\lambda)$ is defined in (13),

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad \partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad (31)$$

and λ is the same as in (4).

Let D_+ and D_- denote the set of points inside and outside the unit circle respectively:

$$\begin{aligned} D_+ &= \{\lambda \in \mathbb{C} : 0 < |\lambda| \leq 1\} \\ D_- &= \{\lambda \in \mathbb{C} : |\lambda| \geq 1\} \end{aligned} \quad (32)$$

Equations (22), (23) with $B \equiv 0$ have the following pair of solutions

$$f(\lambda) = 1, \quad f^*(\lambda) = \frac{i \operatorname{sign}(\lambda\bar{\lambda} - 1)}{\lambda}, \quad \text{for } \lambda \in D_+ \text{ or } \lambda \in D_-. \quad (33)$$

For the conjugate equation (23) with $B \equiv 0$ we also consider the solution

$$\psi^*(z, \lambda, E) = \frac{i}{\lambda} \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda\bar{z} + \frac{z}{\lambda} \right) \right], \quad E < 0, \quad \lambda \in \mathbb{C} \setminus 0. \quad (34)$$

Consider Moutard-type transforms $\mathcal{M}^\pm = \mathcal{M}_{B,f,f^*}^\pm$ for equations (22), (23) in the domains D_\pm , respectively, constructed according to formulas (24)-(26).

Theorem 1. *Let $\tilde{B}_\pm = \mathcal{M}_{B,f,f^*}^\pm B$, $\tilde{\psi}_\pm = \mathcal{M}_{B,f,f^*}^\pm \psi$, $\tilde{\psi}_\pm^* = \mathcal{M}_{B,f,f^*}^\pm \psi^*$ in D_\pm , where B , ψ , ψ^* are given by (30), (29), (34) (and correspond to the Schrödinger equation (1), with $v(x) \equiv 0$, $E < 0$), $f(\lambda)$, $f^*(\lambda)$ are given by (33).*

Then for a proper choice of integration constants in formulas (26) we obtain:

$$\tilde{B}_\pm(\lambda) = \tilde{B}(\lambda) = \frac{\pi \operatorname{sign}(\lambda\bar{\lambda} - 1)}{\bar{\lambda}} b_{0,\alpha}(k_E(\lambda)), \quad (35)$$

$$\tilde{\psi}_\pm(z, \lambda) = \tilde{\psi}(z, \lambda) = \psi_{0,\alpha}(x, k_E(\lambda)), \quad (36)$$

$$\tilde{\psi}_\pm^*(z, \lambda) = \tilde{\psi}^*(z, \lambda) = \frac{i}{\lambda} \psi_{0,\alpha}(x, k_E(-\lambda)), \quad (37)$$

where $\lambda \in \mathbb{C} \setminus 0$, $b_{0,\alpha}$, $\psi_{0,\alpha}$ are given by (18), (17).

The point is that the Moutard-transformed data \tilde{B} and eigenfunctions $\tilde{\psi}$, $\tilde{\psi}^*$ correspond to the Schrödinger equation (1) with the point-type potential $v(x) = v_{0,\alpha}(x)$, and, in particular, have spectral singularities if $E_1 \leq E < 0$.

Proof of Theorem 1. Under our assumptions, for potentials $\omega_{f,f^*}(\lambda)$, $\omega_{\psi,\psi^*}(\lambda)$ in (24)-(26) the following formulas are valid:

$$\omega_{f,f^*}(\lambda) = i[|\ln(\lambda\bar{\lambda})| + \ln(|E|)] - i \begin{cases} c_{f,f^*}^+, & \lambda \in D_+ \\ c_{f,f^*}^-, & \lambda \in D_-, \end{cases} \quad (38)$$

$$\omega_{\psi,f^*}(\lambda) = 4\pi i G(z, \lambda, E) + i \begin{cases} c_{\psi,f^*}^+(z), & \lambda \in D_+ \\ c_{\psi,f^*}^-(z), & \lambda \in D_-, \end{cases} \quad (39)$$

$$\omega_{f,\psi^*} = 4\pi i \operatorname{sign}(\lambda\bar{\lambda} - 1) G(z, -\lambda, E) + i \begin{cases} c_{f,\psi^*}^+(z), & \lambda \in D_+ \\ c_{f,\psi^*}^-(z), & \lambda \in D_-, \end{cases} \quad (40)$$

where c_{f,f^*}^+ , c_{f,f^*}^- , $c_{\psi,f^*}^+(z)$, $c_{\psi,f^*}^-(z)$, $c_{f,\psi^*}^+(z)$, $c_{f,\psi^*}^-(z)$ are some real constants with respect to λ , $G(z, \lambda, E) = G(x, k_E(\lambda))$ is defined by (10), (11). Formula (38) can be checked directly. Formulas (39), (40) can be also checked directly using the relations

$$\frac{\partial}{\partial \bar{\lambda}} G(z, \lambda, E) = \frac{1}{4\pi} \frac{\operatorname{sign}(\lambda\bar{\lambda} - 1)}{\bar{\lambda}} \exp \left[-\frac{\sqrt{|E|}}{2} \left(\bar{\lambda} z + \frac{\bar{z}}{\bar{\lambda}} \right) \right], \quad (41)$$

$$\overline{G(z, \lambda, E)} = G(z, \lambda, E), \quad \text{where } \lambda \in \mathbb{C} \setminus 0, \quad (42)$$

(see [20], page 423).

Using formulas (24), (30), (33) and (38) with $c_{f,f^*}^+ = c_{f,f^*}^- = \frac{4\pi}{\alpha}$, we obtain formula (35). Using formulas (25), (29), (33), (34) and (39), (40)

with $c_{\psi, f^*}^+(z) = c_{\psi, f^*}^-(z) \equiv 0$, and $c_{f, \psi^*}^+(z) = c_{f, \psi^*}^-(z) \equiv 0$ we obtain the formulas:

$$\tilde{\psi}_{\pm}(z, \lambda) = \tilde{\psi}(z, \lambda) = \exp \left[-\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}, \quad (43)$$

$$\tilde{\psi}_{\pm}^*(z, \lambda) = \tilde{\psi}^*(z, \lambda) = \frac{i}{\lambda} \left\{ \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, -\lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \right\}, \quad (44)$$

where $\lambda \in \mathbb{C} \setminus 0$. Formulas (43), (44) can be rewritten as (36), (37). \square

6 Annihilation of point potentials by Moutard-type transforms

Now we start with the Schrödinger equation (1) with the point-type potential $v_{0,\alpha}(x)$ mentioned in the Section 3. In this case for $E < 0$ one has:

$$B(\lambda) = B_E(\lambda) = -\frac{\text{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} \frac{1}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}, \quad (45)$$

$$\psi(z, \lambda) = \psi(z, \lambda, E) = \exp \left[-\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}, \quad (46)$$

$$\psi^*(z, \lambda) = \psi^*(z, \lambda, E) = \frac{i}{\lambda} \left\{ \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, -\lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \right\}, \quad (47)$$

and ψ, ψ^* satisfy the Schrödinger equation (1) with $v(x) = v_{0,\alpha}(x)$ and the $\bar{\partial}$ -equations (22), (23), where B is given by (45), $D = D_{\pm}$.

In addition, equations (22), (23), where B is given by (45), $D = D_{\pm}$, have the following particular pair of conjugate solutions:

$$f = a(\lambda) = -\frac{1}{\pi} \frac{1}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}, \quad (48)$$

$$f^* = \frac{i \operatorname{sign}(\lambda\bar{\lambda} - 1)}{\lambda} a(\lambda) = -\frac{i \operatorname{sign}(\lambda\bar{\lambda} - 1)}{\pi\lambda} \frac{1}{|\ln(\lambda\bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}. \quad (49)$$

Here $a(\lambda) = a_E(\lambda) = a_{0,\alpha}(\lambda)$ is given by (20), (21).

Theorem 2. Let $\tilde{B}_\pm = \mathcal{M}_{B,f,f^*}^\pm B$, $\tilde{\psi}_\pm = \mathcal{M}_{B,f,f^*}^\pm \psi$, $\tilde{\psi}_\pm^* = \mathcal{M}_{B,f,f^*}^\pm \psi^*$ in D_\pm , where B , ψ , ψ^* are given by (45), (46), (47) (and correspond to the Schrödinger equation (1), with $v(x) = v_{0,\alpha}(x)$, $E < 0$), $f(\lambda)$, $f^*(\lambda)$ are given by (48), (49).

Then for a proper choice of integration constants in formulas (26) we obtain:

$$\tilde{B}(\lambda) \equiv 0, \quad (50)$$

$$\tilde{\psi}_\pm(z, \lambda, E) = \tilde{\psi}(z, \lambda, E) = \exp\left[-\frac{\sqrt{|E|}}{2} \left(\lambda\bar{z} + \frac{z}{\lambda}\right)\right], \quad (51)$$

$$\tilde{\psi}_\pm^*(z, \lambda, E) = \tilde{\psi}^*(z, \lambda, E) = \frac{i}{\lambda} \exp\left[\frac{\sqrt{|E|}}{2} \left(\lambda\bar{z} + \frac{z}{\lambda}\right)\right], \quad (52)$$

where $\lambda \in \mathbb{C} \setminus 0$.

The point is that the Moutard-transformed data \tilde{B} and eigenfunctions $\tilde{\psi}$, $\tilde{\psi}^*$ correspond to the Schrödinger equation (1) with zero potential $v(x) \equiv 0$ at fixed energy $E < 0$, whereas, in particular, the initial data B and eigenfunctions ψ , ψ^* have spectral singularities if $E_1 \leq E < 0$.

Proof of Theorem 2. Under our assumptions, for potentials $\omega_{f,f^*}(\lambda)$, $\omega_{\psi,f^*}(\lambda)$ in (24)-(26) the following formulas are valid:

$$\omega_{f,f^*}(\lambda) = \frac{i}{\pi} a(\lambda) + \begin{cases} c_{f,f^*}^+, & \lambda \in D_+ \\ c_{f,f^*}^-, & \lambda \in D_-, \end{cases} \quad (53)$$

$$\omega_{\psi,f^*} = -\frac{4i G(z, \lambda, E)}{|\ln(\lambda\bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} + i \begin{cases} c_{\psi,f^*}^+(z), & \lambda \in D_+ \\ c_{\psi,f^*}^-(z), & \lambda \in D_-, \end{cases} \quad (54)$$

$$\omega_{f,\psi^*} = -\frac{4i \operatorname{sign}(\lambda\bar{\lambda} - 1) G(z, -\lambda, E)}{|\ln(\lambda\bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} + i \begin{cases} c_{f,\psi^*}^+(z), & \lambda \in D_+ \\ c_{f,\psi^*}^-(z), & \lambda \in D_-, \end{cases} \quad (55)$$

where c_{f,f^*}^+ , c_{f,f^*}^- , $c_{\psi,f^*}^+(z)$, $c_{\psi,f^*}^-(z)$, $c_{f,\psi^*}^+(z)$, $c_{f,\psi^*}^-(z)$ are some real constants with respect to λ , $G(z, \lambda, E) = G(x, k_E(\lambda))$ is defined by (10), (11).

Formula (53) follows from the formulas

$$\frac{\partial}{\partial \bar{\lambda}} a(\lambda) = \frac{\pi \operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} (a(\lambda))^2, \quad (56)$$

and

$$\frac{\operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} (a(\lambda))^2 = \overline{if(\lambda)f^*(\lambda)}. \quad (57)$$

Formula (56) follows from the formulas (19), (21). Finally, formula (57) follows from (48), (49).

To prove (54), we use the formulas:

$$\psi(z, \lambda) - \exp \left[-\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] = -\frac{4\pi G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}, \quad (58)$$

$$\frac{\partial}{\partial \bar{\lambda}} \psi(z, \lambda) = B(\lambda) \overline{\psi(z, \lambda)} = i\pi \overline{\psi(z, \lambda) f^*(\lambda)}, \quad (59)$$

and formula (42).

To prove (55), we use the formulas :

$$\psi^*(z, \lambda) - \frac{i}{\lambda} \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] = -\frac{i}{\lambda} \frac{4\pi G(z, -\lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha}, \quad (60)$$

$$\begin{aligned} \frac{\partial}{\partial \bar{\lambda}} \frac{\lambda}{\pi} \psi^*(z, \lambda) &= \frac{\lambda}{\pi} \overline{B(\lambda) \psi^*(z, \lambda)} = \operatorname{sign}(\lambda \bar{\lambda} - 1) a(\lambda) \overline{\psi^*(z, \lambda)} = \\ &= \operatorname{sign}(\lambda \bar{\lambda} - 1) \overline{f(\lambda) \psi^*(z, \lambda)}, \end{aligned} \quad (61)$$

and formula (42).

Next, we choose

$$c_{f, f^*}^+ = c_{f, f^*}^- = 0, \quad c_{\psi, f^*}^+(z) = c_{\psi, f^*}^-(z) = c_{f, \psi^*}^+(z) = c_{f, \psi^*}^-(z) \equiv 0. \quad (62)$$

Formula (50) follows from the following computation:

$$\begin{aligned} \tilde{B}(\lambda) &= \frac{\pi \operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} a(\lambda) + \frac{f \bar{f}^*}{\omega_{f, f^*}} = \\ &= \frac{\pi \operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} a(\lambda) + \frac{a(\lambda) \frac{-i \operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} a(\lambda)}{\frac{i}{\pi} a(\lambda)} = \\ &= \frac{\pi \operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} a(\lambda) + a(\lambda) \frac{-\pi \operatorname{sign}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} = 0. \end{aligned} \quad (63)$$

Formula (51) follows from the following computation:

$$\begin{aligned}
\tilde{\psi}(z, \lambda) &= \psi(z, \lambda) - \frac{\omega_{\psi, f^*}}{\omega_{f, f^*}} f = \\
&= \exp \left[-\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} + \\
&+ \frac{4i G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \frac{a(\lambda)}{\frac{i}{\pi} a(\lambda)} = \\
&= \exp \left[-\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} + \\
&+ \frac{4\pi G(z, \lambda, E)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \frac{a(\lambda)}{a(\lambda)} = \\
&= \exp \left[-\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right].
\end{aligned} \tag{64}$$

Formula (52) follows from the following computation:

$$\begin{aligned}
\tilde{\psi}^*(z, \lambda) &= \psi^*(z, \lambda) - \frac{\omega_{f, \psi^*}}{\omega_{f, f^*}} f^* = \\
&= \frac{i}{\lambda} \left\{ \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{4\pi G(z, -\lambda)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \right\} + \\
&+ \frac{4i G(z, -\lambda)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \text{sign}(\lambda \bar{\lambda} - 1) \frac{\frac{i \text{sign}(\lambda \bar{\lambda} - 1)}{\lambda} a(\lambda)}{\frac{i}{\pi} a(\lambda)} = \\
&= \frac{i}{\lambda} \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right] - \frac{i}{\lambda} \frac{4\pi G(z, -\lambda)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} + \\
&+ \frac{i}{\lambda} \frac{4\pi G(z, -\lambda)}{|\ln(\lambda \bar{\lambda})| + \ln(|E|) - 4\pi/\alpha} \frac{a(\lambda)}{a(\lambda)} = \frac{i}{\lambda} \exp \left[\frac{\sqrt{|E|}}{2} \left(\lambda \bar{z} + \frac{z}{\lambda} \right) \right].
\end{aligned} \tag{65}$$

This completes the proof of Theorem 2. \square

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