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Justifying a calculation technique in years 3 and 6

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In this progress report, we present the outline and the first results of a research project which aims to study the argumentative capacity of year 3 (age 8) to year 6 (age 11) students, in a numerical context involving natural and decimal numbers. This project centers on a technique for dividing natural numbers by two, a technique which appears in Al-Khwarizmi's Kitāb al-ḥisāb al-Hind.

Keywords: Al-Khwarizmi, primary school, operations, place value, proof.

Preliminary remark: This paper is a progress report on an ongoing project. In Chorlay (forthcoming), which was written before the first classroom experiments were carried out, we presented in some detail the historical background, the general motivation, and elements on the *a priori* analysis; these will only be sketched here. Our main goal is to present some of the empirical data gathered, and discuss the theoretical tools which can help us analyze them.

Rationale.

Historical background.

Although Al-Khwarizmi is usually remembered for his treatise on algebra and quadratic equations, he also wrote several other books for which the oldest manuscripts available now are Latin translations. In particular, in his *Book on Indian Reckoning (Kitāb al-ḥisāb al-Hind)*, he introduced into the Arabic-speaking scientific community a technique to write whole numbers (the decimal place value system), and written calculating techniques relying on this conveniently uniform coding system; techniques for addition, subtraction (with the borrowing method), *mediation* and duplication (i.e. division and multiplication by 2), multiplication, division, and square root extraction (with mediation as one of its steps). Although he found these in the Indian mathematical and astronomical traditions, it should be noted that they were also in use in Chinese mathematics.

Both the book on algebra and that on Indian reckoning were first translated into Latin in the 12th and 13th centuries, more often than not by scientists working in Andalusia with a prime interest in astronomy. For a recent study on these treatises, we refer the reader to Allard's edition and French translation of the three oldest extant Latin translations/adaptations of the lost Arabic text (Al-Khwarizmi, 1992).

General didactical motivation.

For reasons which we will spell out below, we thought the algorithm for *mediation* (i.e. division by 2) could be used in mathematics education research, in order to study phenomena which have no direct connections to the use of history of mathematics in the classroom. In a general report on the future of mathematics education published in 2003, a French study-group headed by Pr. Kahane stressed that:

(...) Most of the reckoning algorithms whose teaching and learning took so much school-time are now available in the most basic calculator. By contrast, new questions emerge, in particular about the digital representation of the mathematical objects on which they bear (such as numbers), and about the efficiency of algorithms (beyond their correctness) ... question which were not issues in teaching until now. The calculating power of the new tools (...) calls for renewed investigations into the connections between calculation and reasoning, and fosters exploration, simulation and experimentation. (quoted in (Charnay, 2007, p. 204). Our free trans.)

On this basis, Roland Charnay suggested that the main interest of studying written calculating techniques was that they provide

opportunities to work on the properties of numbers and operations by focusing the greater part of the effort on understanding and justifying these algorithms (...). This requires that we do not content ourselves with a *technicist* teaching (bearing on the *how*), but endeavour to justify the various steps and their connections (the *why*) ... and actually *do* mathematics! (Charnay, 2007, p. 206. Our free trans.).

Of course, these statements call for qualification: on the one hand, it would not be fair to say that the justification of numerical algorithms played no part in teaching, even in elementary school (for instance, see Clivaz (2016) and Constantin (2017) for recent investigations into teacher practices and mathematical-knowledge-for-teaching with regard to the justification of reckoning techniques). On the other hand, it is likely that some familiarity with calculation routines is a prerequisite to entrust students with *meta*-tasks – “*meta*” in so far as do not bear on specific numbers or numerical problems, but on mathematical procedures; procedures which are to be described, reformulated, compared with others, assessed, or justified (Chorlay, 2016). In this context, the procedures change status: from epistemic tools (i.e. answer-generating tools, which can, in themselves, be black-boxes) to epistemic objects (i.e. question-generating objects). It should be pointed out that, as stressed in Chorlay (2016), this change of status allows for the insertion of justification tasks in the larger set of meta-tasks. One of the underlying macro-hypothesis is that enculturation of the students into the class of meta-task could be a viable pathway to the teaching of the specific and highly demanding meta-task “prove”.

Al-Khwarizmi’s technique.

The three oldest Latin texts expound variants of the same technique. The *Dixit Algorismi* reads¹:

To divide any number by two, begin with the first place and divide it by two. If its number is odd, divide the pairs [or: the even one] in two halves, and there will remain one to be divided by two – that is, to be divided in two halves – and you shall set one half, which is a fraction of thirty

¹ *Cum volueris mediare aliquem numerum, accipe a prima differentia et media eam. In qua si fuerit numerus impar, media pares, et remanebit unum quod mediabis, idest divides in duas medietates, constituesque medietatem unam triginta partem ex sexaginta (...). Deinde mediabis sequentem differentiam, si fuerit numerum eius par. Et si fueris impar, accipe medietatem paris et pone eam in loco eius, et constitue medietatem unius residui quinque, et pone eos in differentia que est ante ipsam. (...) et similiter operare in universis differentiis.* (Al-Khwarizmi, 1992)

out of sixty (...). Then, you shall divide by two the next place, if its number is even. If it's odd, take half of the even one and put it in its place; set 5 as half of the remaining unit and put it in the place before. (...) Operate the same way for all places. (Al-Kwarizmi, 1992. Our trans. based on Allard's translation into French).

No justification for this specific technique is given, but the three manuscripts provide justifications for most of the others and probably consider this one to be self-explanatory. Even though the meaning of "first place" is ambiguous in this excerpt, it is not in the treatises: it denotes the units; so, if we write the number in today's standard form, the algorithm works from right to left, from units to tens, then to hundreds etc. When applying the technique to 92, one can display the successive states of the slate as the shown in the boxes below:

92 [2 → 1] 91 [9 = 8+1; 8 → 4] 41 [carry 5 to the right, 1 → 6] 46

In square brackets, we described the operations to be carried out either on the slate (→ for "erase and replace") or mentally (decompose², carry).

From a purely mathematical viewpoint, the validity of this algorithm rests of the fact that halving a sum is the same as summing the halves (which can be seen as a special case of the distributivity of \times over $+$, if we agree to count "one half" as a number), on the multiplicative linearity of the "halving" and, eventually, on the canonical decomposition of whole numbers in a place value system. The fact that we are working in base ten is reflected in the "carry 5 to the right" rule: a "1" digit in the second place stands for "ten", so its half is 5 units; a "1" digit in the third place represents one hundred, which is 10 tens, so its half is 5 tens etc. If numbers were written in base 4, the technique would be the same except for a "carry 2 to the right" step. Resting on the same mathematical bases, this algorithm is structurally similar to the other "Indian" algorithms: you don't need to consider the number as a whole; rather, you iteratively apply the same procedure to every digit, which means that you can process any whole number as long as you can process 1-digit numbers. A more detailed analysis can be found in Chorlay (forthcoming).

First report on the experiments.

Outline of the experiments and theoretical background.

The fact that this technique is not known to French students, but rests only on the basic principles of the decimal place-value system (namely: place value, and the "exchange rate" of 10 between two adjacent places³) led us to assume that it was fit to be used to investigate whether or not *meta*-tasks – including "justify" – could be entrusted to young students. In school year 2017-2018, we designed teaching experiments based on this material and tried it out in a year 6 class (1st year of middle-

² Or "subtract one", or "consider the predecessor", or "take the greatest even number less than (or contained in) 9".

³ The validity of the algorithm also rests on the distributive property, a property which is not taught explicitly before year 7 in France. However, the distributive property is used *in act* at very early stages in the study of multiplication, either in mental calculation, or in the standard written techniques. We assumed that the distributive property of "halving" would "go without saying" for students, that is, would neither be mentioned nor questioned explicitly.

school in France, students aged 11) and in a year 3 class (students aged 8). A key difference lies in the fact that year 6 students know decimal numbers, whereas year 3 students do not. Indeed, in the experiments, year 3 students typically said that halving 3 is impossible – or cannot be carried out completely accurately – whereas all year 6 students said that half 3 is 1.5.

For the researchers to gather data, students were asked to produce written traces of their individual reflection. These worksheets were collected, and the collective phases were video-recorded. The experiment took place in ordinary classes of about 25 students each. The class was taught by the regular teacher (or maths-teacher in year 6); the teachers had been associated to the design of the sessions from the outset. The researcher was present but played no active part in the sessions.

We have no room here to present the specifics of the teaching sessions, which differed for year 3 and year 6 students. To put it in a nutshell, the starting point was a silent performance – on the blackboard, by the teacher – of a few instances of mediations along Al-Khwarizmi's technique, with even whole numbers, some with even digits only, some with odd digits as well. Over the course of the sessions, the various tasks entrusted to students were:

1. Make hypotheses as to the function performed by the algorithm (that is: division by 2).
2. Emulate the technique with new even numbers (with or without odd digits).
3. Write (for year 6 students) or describe orally (for year 3 students) the method shown.
4. Justify its correctness.

Before we focus on points 3 and 4 (which, as we will see, were not treated independently by most students), let us mention a few empirical results regarding points 1 and 2. Regarding point 1, even year 3 students guess very quickly that it is a halving technique. When asked how they could test this conjecture, they suggested the following three correct techniques: add the output with itself and compare with the input; multiply the output by 2 and compare with the input; subtract the output from the input and compare with the output; they had not yet studied Euclidean division. By contrast, few students managed to identify the procedure for the treatment of odd digits (that is: subtract one, halve the result⁴, and add a carry of five to the digit of the output immediately to the right). Thus, tasks 2 turned out to be significantly more difficult than task 1.

As far as points 3 and 4 are concerned, we wanted to investigate:

- A. What – if any – knowledge about the decimal place value system was used by students to describe, then make sense of or justify the algorithm.
- B. What forms of justification – if any – did students supply.

In our investigation into A, we relied on the work of Frédéric Tempier (2016) who, after Christine Chambris, regards “tens”, “hundreds” etc. as a system of (non-independent) units. In his analysis of the work of students at the end of primary school he identified four interpretations of numbers written in standard form: 1. Juxtaposition of identical units (584 seen as 5 units, 8 units and 5 units); 2. Place-value juxtaposition (the student is able to read in terms of “hundreds”, “tens”, and “units” but with no functional connection between these terms); 3. Simple units (584 seen as 500 units plus

⁴ Or: halve the greatest even number contained in (less than) the odd number; halve the number before/the predecessor.

80 units plus 5 units); 4. System of units (“units”, “tens”, “hundreds” are seen as units, and relationships such as 1 hundred = 10 tens = 100 units are available). Of course, when dealing with numbers (either in rewriting tasks, as studied by Tempier, or to justify mediation), interpretations 1 and 2 are not functional, whereas interpretations 3 and 4 are. A key aspect is that many students tend to regard written calculation as a black-box operating *only* on number *signs*, and indeed, the strength of place value systems lies in the fact that the algorithms can be described (and applied) as series of graphic operations bearing on written signs; of course, applying these techniques rests on properties of one-digit numbers, properties which are usually to be learnt by heart for an efficient and routine use of the technique. In the following, we will not consider the use of memorized numerical facts bearing on one-digit numbers to be indicative of an engagement in justification.

As far as argumentation is concerned, our starting point is the seminal work of Nicolas Balacheff (1987). He distinguished between two forms of “proof”: pragmatic proof (by actually performing and checking) *vs* intellectual proof (through a discourse bearing on object, thus dependent on means of representation of the objects). He also considered a hierarchy of forms of argumentation: naïve empiricism (check on a few cases), *experimentum crucis* (the generality issue is considered, and addressed by checking a specially difficult or large case), generic proof (where the actions are performed on a specific object which is used as a representative for a class of objects, and one claims that arguments – or moves – hold – or can be performed – for all other cases “just as well”; see (Yopp & Ely, 2016) for a recent survey), the thought experiment (the action is described without being performed on a specific case, and the general properties are indicated in some other way than by the result of their use), and formal proof. This framework has since proved inspirational for many at an international level, also in works bearing on primary school mathematics (Stylianides, 2007). In recent years, Balacheff also suggested that we should distinguish between *explanation* (which strives for coherence) and *demonstration* (striving for a type of proof).

A sample of results.

We will focus on the results for year 6, with occasional asides about year 3. After silently carrying out several mediations on the blackboard, the teacher staged a first collective discussion on the nature of the results. Once a consensus had been reached on the fact that it worked out halves (or divided by two) the teacher asked students to work on individual worksheets for 5 minutes. The two questions were: 1. “A schoolmate of yours has not seen the teacher carry out this operation and would like to divide by two as he did. Write down the method for him.” 2. “Justify each step of the method”.

One student – we will call him K – gave a remarkable answer; an answer which is actually quite close to that of the medieval treatises:

1. First write down the number that you want to divide by 2. Consider the number of units⁵, if it’s even write down its half, if it’s odd take away 1, write down its half. You add “.5”. Move on

⁵ Strictly speaking this is incorrect, at least for numbers greater than 9. The correct version would read: the unit digit.

to the number of the tens, if it's even write down its half under it⁶, if it's odd we take away 1 from the number; write down its half under it, and add five to the number that came before. Do the same for the hundreds, the thousands, the millions.

2. It works because even numbers are divisible by 2, and odd numbers are even numbers +1, so we work out the half + 0.5. $1:2 = 0.5$

Beyond its thoroughness and clarity, this answer differs from all the others in the sample in three significant ways. First, K is the only one who gave separate and clearly distinct answers to questions 1 and 2, thus distinguishing between the algorithmic-imperative and the justificatory genres of text. As we will see below, all the other students gave a single answer, and inserted elements of “justification” as they described the technique. Second, he is the only student who anticipated the adaptation of the technique to odd numbers. Third, we claim that the numerical facts he relied on put him in Tempier’s category 4. In the written worksheet, he resorted to decimals to make sense of the role of “5”, but in the second session, while commenting on a mediation of 256 on the occasion of which another student claimed that “half of 5 is 2.5” but was unable to explain why a “5” should be added to the 3, he explained that “duh! In fact it’s simple, because, for real, it’s not 2.5 but 2.5 *tens*, so5 is for units”. The recordings show that, when the teacher asked the class if they agreed that “2.5 tens” was equal to 25 (which had been suggested by another student to clarify K’s remark), he stirred puzzlement in many, even disbelief in some students.

By contrast, the written work of other students shows no distinction between the expression of the algorithm and attempts at accounting for / explaining / justifying the steps; on the basis of the written traces, we hypothesize that the approach of most students was: 1/ endeavour to devise a technique to divide by two (without using Euclidean division), 2/ check whether or not it gives the correct results, and 3/ compare it with the teacher’s technique. In year 6, for lack of a preliminary phase enabling us to assess whether or not students were able to emulate the technique for new even inputs, we cannot say whether they regarded looking for their own technique as the only possibility, or as something which they took to address the “justify” question. The low level of identification of the technique in year 3 (where it was tested independently of its written description), and the absence of a divide between the “describe” and the “justify” answers suggest that the first alternative holds for a large majority of students. In year 6, besides K, only two students clearly stated the “add five to the right” step in their description, without any justification though.

Some students tried to devise a written technique for mediation by adapting steps or moves they were familiar with in other calculating techniques. In year 6, 3 out of 24 students tried to use the “bridge-trick” from their technique for Euclidean division; for instance, in order to divide say 345 by 4, you start from the left and try to divide 3 by 4; since you can’t, you try to divide 34 by 4 etc. Unfortunately, in mediation, this trick works in cases such as 834 (one half of 4 is 2, one half of 3 – stop, one half of 34 is 17 [probably from mental calculation], one half of 8 is 4, so half of 834 is 417), but not so well with several consecutive odd digits. This can be seen in some students’ worksheets, for example when dealing with 334. In year 3, student G suggested a mediation

⁶ On the blackboard, the output of the algorithm was written under the input.

technique which used the “borrowing” trick from the subtraction technique he knew: to divide 9014 by two, starting from the left, write 4 (half of 8), and in 9014 write 10 instead of 0 (“because we took away 1 from 9”), so the half is 5 etc. This is a correct and efficient techniques, but not Al-Khwarizmi’s. We did not investigate whether G was able to justify this “trick” on the basis of numerical properties, or just performed a graphic operation which he considered legitimate because 1. It was already in use in class (in another context, however), 2. It worked.

Let us finally compare two other suggestions. To do this, we will use the notion of *unfolding* of a numerical algorithm. We will call an unfolding of algorithm X, the description of another algorithm X’ along with: 1. A (possibly crude) description of how its steps are related to that of X, and 2. Indications that its connection to X’ *accounts* for X; as we will see below (analysis of C’s unfolding), we consider that accounts *can* but *need not* be justificatory. This notion emerged from our analysis of Liu Hui’s first justification of the correctness of the technique for fraction multiplication in the *Nine Chapters* (Chorlay, 2017): he described a longer algorithm than the one actually performed, but which necessarily gives the same results (the extra-steps it involves neutralize one another) and whose correctness rests on explicit general properties of multiplication.

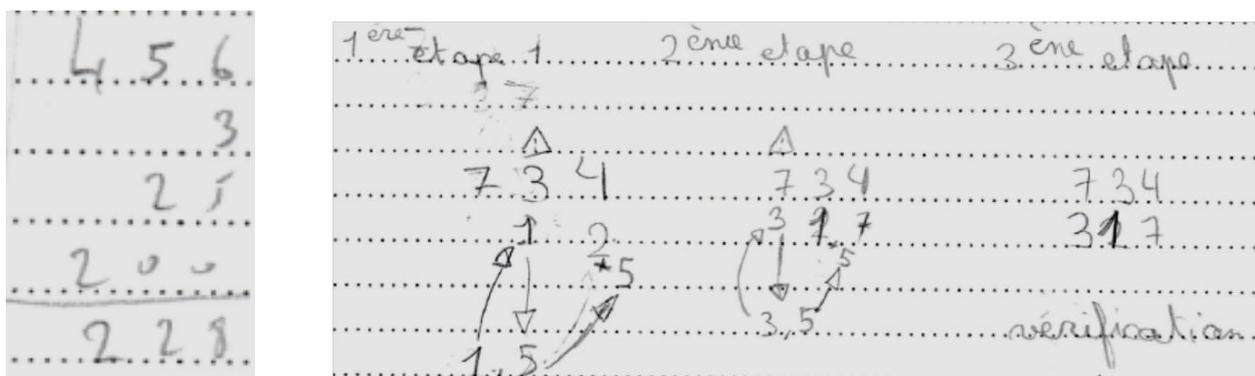


Figure 1: Worksheets of students L (left) and C (right)

L’s method (fig. 1 left) rests on a type 3 decomposition (in Tempier’s framework), as his oral explanations in session 2 showed: “3 is the half of 6, 25 is the half of 50, and 200 is the half of 400”. So he devised a correct technique for mediation and was able to justify it by mentioning valid and relevant properties of the numbers involved, which we take to be indicative of a proof by generic example. We have no way to say if he only looked for a mediation technique (which he managed to justify), or actually considered this to be an unfolding of the teacher’s technique.

The technique used by C (fig.1 right) is correct in so far as it leads to correct halves, and is clearly an unfolding of Al-Khwarizmi’s technique, embedding it in a larger algorithm involving calculations (with decimal numbers) and a “carry five to the right” step. She described her diagram as something which shows what she “did in her head” when applying Al-Khwarizmi’s technique. However, we do not take C’s correct unfolding to be of a justificatory nature, since, when asked why the .5 in 1.5 and 3.5 could be used as a “+5 to the right”, she mentioned no numerical properties. She justified her moves by saying that writing 1.5 under 3 was “impossible” (meaning: this is not a proper standard form), so she just wrote 1; and the fact that the output had extra “5”s (although in the “.5” form) could account for the move she had identified in the technique: “carry

five to the right”. So she unfolded the technique as a graphic algorithm dealing with number signs, and her unfolding achieves *coherence* (between what she knows about halves of one-digit numbers, and the steps of Al-Khwarizmi’s technique), without providing proof.

Perspectives.

Let us mention three alterations considered for a new round of classroom experiments in 2018-2019:

- Since a large share of students (even in year 6), fail to identify the “carry 5 to the right”-step, it should be made explicit before any attempts to entrust students with description or justification tasks.
- Asking for written traces helps the researcher gather data, but has several drawbacks: 1. Many students are not willing or able to engage in this writing tasks, which results from a combination of cognitive (in particular when comparing students of different ages) and social factors (Bautier & Goigoux, 2004). 2. As the cases of L and C show, the extent to which written traces testify to an actual engagement in justification is difficult to assess without a follow-up interview.
- We would like to study how students assess the various “arguments”.

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