



**HAL**  
open science

# On the didactical function of some items from the history of calculus

Stephan Berendonk

► **To cite this version:**

Stephan Berendonk. On the didactical function of some items from the history of calculus. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02421842

**HAL Id: hal-02421842**

**<https://hal.science/hal-02421842>**

Submitted on 20 Dec 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the didactical function of some items from the history of calculus

Stephan Berendonk<sup>1</sup>

<sup>1</sup>Universität Duisburg-Essen, Germany; [stephan.berendonk@uni-due.de](mailto:stephan.berendonk@uni-due.de)

*The article draws attention to some individual results from the history of calculus and explains their appearance in a course on the didactics of analysis for future high school mathematics teachers. Specifically it is shown how these results may serve as a vehicle to address and reflect on three major didactical challenges: determining the meaning of a result, concept or topic to be taught, teaching mathematics as a coherent subject and understanding the source of misconceptions.*

Keywords: *Didactics of analysis, meaning, coherence, misconceptions.*

Why do we have to learn this *historical* stuff, although this is a course on *didactics*? This is a question frequently posed by future high school mathematics teachers taking my third semester course on the didactics of analysis at the university of Duisburg-Essen. In this course we consider and compare several different approaches to differentiation and integration, uncover some of the relationships of analysis with other mathematical subdomains and try to settle the role and meaning of analysis within the high school curriculum. Although the students have mastered a typical first semester university course on real analysis before taking my course on the didactics, I feel obliged to incorporate several mathematical digressions into this course. These digressions especially include some episodes from the history of calculus that weren't discussed in the course on real analysis. I teach these episodes, however, not as an end in itself or as an introduction to the history of mathematics, but as a means in order to be able to seriously address the didactical issues at stake. In the following article I will mention some of those episodes and I will try to point out their relevance and value from a didactical point of view.

## **Three didactical challenges – meaning, coherence and misconceptions**

Let us consider a typical first semester university course on real analysis. With such a course we may associate the following keywords: axioms for the real numbers, convergence of sequences and series, Heine-Borel, Bolzano-Weierstrass, differentiability, monotony, continuity, integrability, intermediate value theorem, mean value theorem, fundamental theorem of calculus and epsilon-delta proofs. This course has a lot to offer to its students. It introduces them to the mathematical quest for rigor, generality and structural results. It shows that the knowledge about differentiation and integration learned at school can be embedded into a beautiful large-scale deductive system, and thereby exemplifies the architectural nature of mathematics.

One might believe that this course would form a sufficient background regarding the mathematical content knowledge with respect to analysis for a future high school mathematics teacher. However, teaching analysis at school may serve quite different aims than the university course we considered. For instance one might want to teach analysis as a tool to formulate and understand some Newtonian physics or more generally speaking, one might want to teach analysis as an example showing the applicability of mathematics and its role for the other sciences. In any case, in contrast to the university course, analysis nowadays usually isn't taught architecturally at school and

according to Freudenthal (1973) and his legacy it wouldn't be a good idea to do so either. One should be acquainted with a subject, with local orderings of its results and concepts before one should work on or be confronted with a global ordering of that subject. From this situation, namely teaching analysis in a different way, a didactical challenge arises: Find the role and meaning of the concepts and results to be taught with respect to the overall approach chosen, e.g. genetic, applicational or problem-oriented. In the axiomatic approach the concepts and results receive their meaning from their place within the system. The mean value theorem for instance occupies a central position within the system. It can be seen as the linchpin of the theory. Continuity is treated as a central concept. In a different approach, however, the roles and meanings of the concepts and results might be completely different.

A central concept developed and used by the German-speaking educational sciences in order to direct the discussion of school curricula is called 'Allgemeinbildung'. The concept elaborates on the functions of school and thereby it serves as a framework and a benchmark for the development and analysis of curricula. Each school subject is then requested to determine its individual contribution to the Allgemeinbildung. In Heymann's (1997, p.2) version of the concept, which is especially prominent in mathematics education, one major task of school is to teach the subjects in such a way as to endow cultural coherence, meaning not only the relationship between past, presence and future, which Heymann would rather have called 'cultural continuity', but also the interrelations of different subcultures. Analysis seems the perfect choice when trying to reveal the interrelations of mathematics and physics. However, even in mathematics itself there are subcultures or rather subdomains, whose relationship one should uncover. The didactical challenge, therefore, consists in discussing the unity of mathematics and asks the following questions: How is analysis related to geometry, how is it related to algebra? The first semester course on real analysis imagined hardly addresses this issue, since it rather focuses on presenting analysis as a self-contained subject.

Teacher students are usually keen to learn about pupils' misconceptions and the best recipes on how to treat them. It is important for them to notice, however, that the instruction itself may cause some of those misconceptions and that a different instruction, using for instance a different approach, may again cause quite different misconceptions. Thus, some misconceptions may be 'typical' in general, whereas some of them may only be 'typical' relative to a given instruction. It is an important didactical challenge to determine the misconceptions belonging to a certain instruction. Since every mathematics teacher student shares the experience of roughly the same first semester course on real analysis, it appears to be a good exercise to think about misconceptions that this course might favor.

In the preceding passages I have formulated three didactical challenges ('meaning', 'coherence' and 'misconceptions') and I have tried to indicate their importance for mathematics teacher students. These challenges form an integral part of my course on the didactics of analysis. My students, when confronted with these issues in the beginning of the course, have a hard time dealing with them. This is understandable, considering that their analytical background consists solely of a first semester course on real analysis. However, teaching a few episodes from the history of calculus, already helps them to handle the challenges in a much more appropriate and satisfying way. I will now, for each challenge separately, mention the results from the history of calculus that I teach and I will explain how they may contribute to the discussion on the respective challenge.

## Meaning – the integral integrates, the calculus calculates

What is the meaning of the integral? What is the meaning of the calculus in general? What is the meaning of the fundamental theorem of calculus? A mathematics teacher needs to be able to answer these questions in a satisfying way, since ‘how’ he teaches a certain subject or course should of course highly depend on ‘why’ he regards the content as being worth to be taught in the first place. In my view, the best way to approach these questions is to look at the prehistory of the calculus. Let us consider Roberval’s determination of the area (see Whitman, 1943) under the cycloid (Figure 1) and his comparison of the length (see Pedersen, 1971) of an Archimedean spiral with that of a certain parabola (Figure 2).

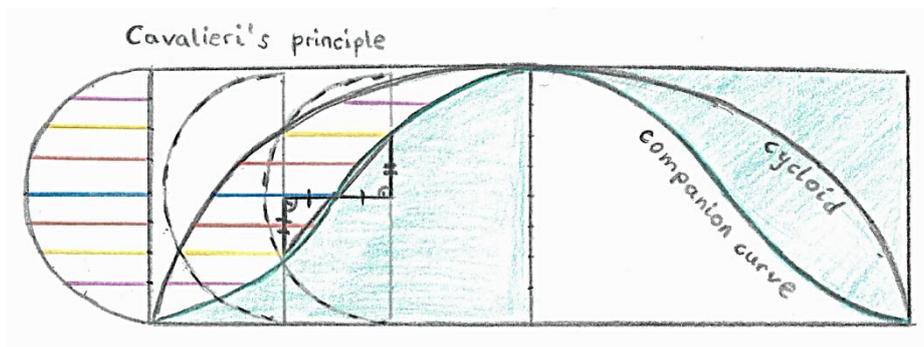


Figure 1: Roberval – The area of the cycloid

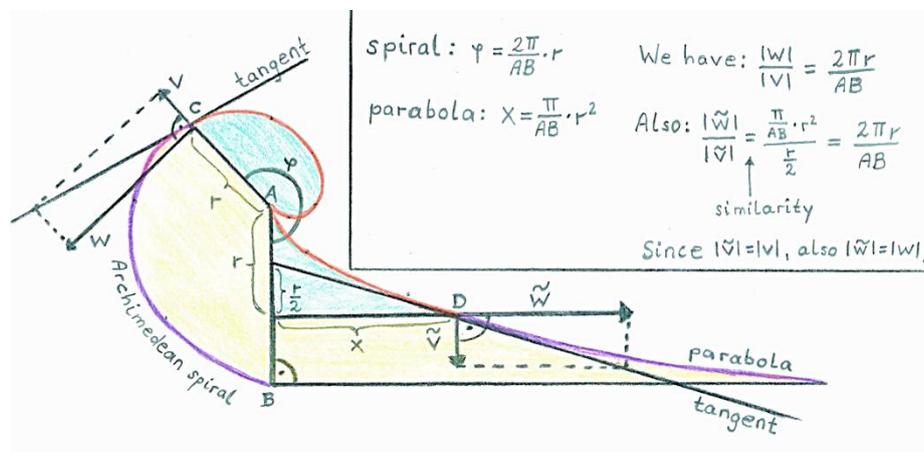


Figure 2: Roberval – The length of the Archimedean spiral

To solve the ‘area’ problem Roberval uses Cavalieri’s principle. To solve the ‘length’ problem he employs his kinematic approach to tangents (see Pedersen, 1969). Thus, the two solutions are based on two completely different methods. This is interesting, since in the world of calculus both, the length problem and the area problem, have become one and the same, in the sense that both ask you to compute some integral. Apparently the integral does what it promises. It integrates the two basic problems associated with a curve. Determine its length and its area. Of course the concept of the integral incorporates ‘volumes’, ‘centers of gravity’ and many other things as well. Wouldn’t this be one possible meaning of the integral that we might wish to convey to learners of mathematics? After all Poincaré once said (see Verhulst, 2012):

Mathematics is the art of giving the same name to different things. (Poincaré, 1914, p.23)

Let us now think about the meaning of calculus and its fundamental theorem. To this end, let us in addition to Roberval's two results consider Archimedes' quadrature of the parabola (see Dörrie, 1965, p.239) and Huygens' determination of the area (see Maanen van, 2003) between the cissoid of Diocles and its asymptote (Figure 3). After looking at these four problems and their solutions one might get the impression that before calculus every curve was treated individually, that at that time people were extremely well acquainted with the geometrical properties of these curves and that one had to come up with a new ingenious idea for every other curve. Today, on the other hand, it needs no ingenuity to solve these questions. With the calculus at hand, these problems have become more or less straightforward routine exercises. The routine goes like this: Formulate the problem in terms of an integral and apply the fundamental theorem, that is, look for an antiderivative and plug in the limits of integration. Finding an antiderivative might be a problem, but that is where the calculus with its powerful tools, integration by parts or by substitution, sets in. We could therefore view the calculus as a facilitation of a major part of mathematics. With calculus everyone has the chance to learn how to handle areas under curves. It seems to be exactly this meaning that Leibniz himself saw in his calculus:

For what I love most in this calculus is that it gives us the same advantage over the ancients in the geometry of Archimedes as Viète and Descartes gave us in the geometry of Euclid and Apollonius; by relieving us of working with the imagination. (as cited in Blåsjö, 2016, p.17)

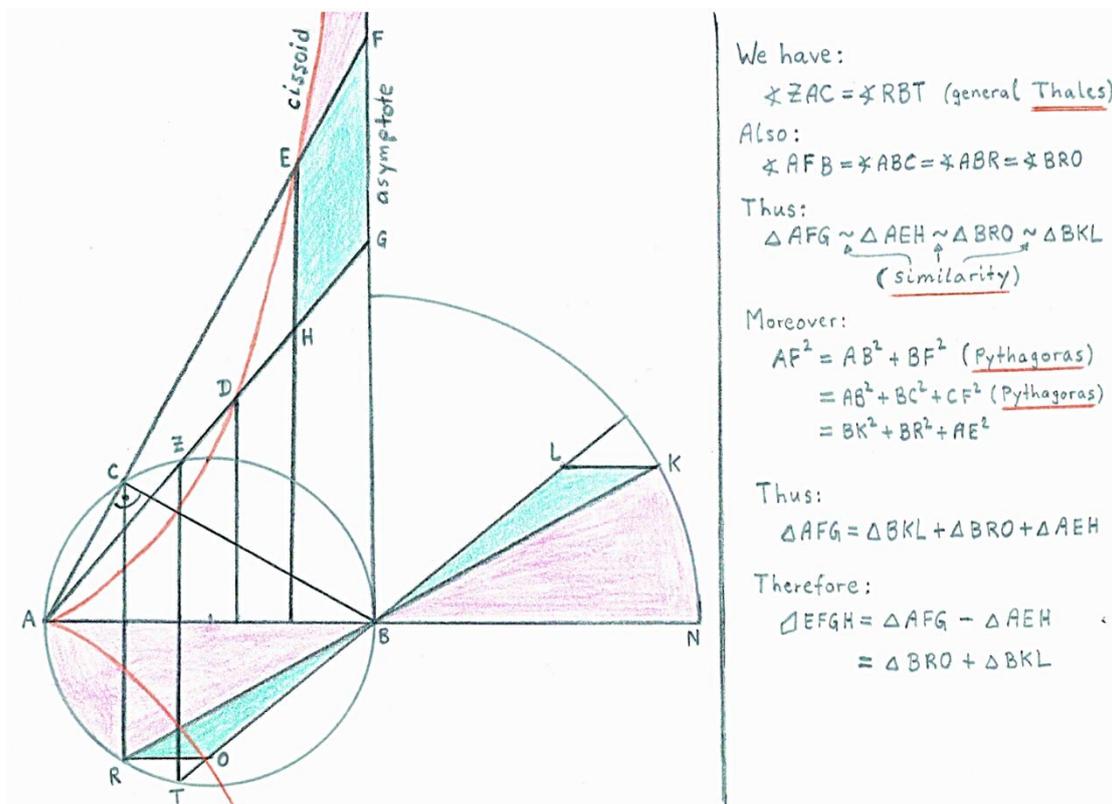


Figure 3: Huygens – The area of Diocles' cissoid

## Coherence – calculus seen as the algebra for transcendental curves

Which of the following terms does not fit with the others?

‘Similarity of triangles’, ‘Thales’ theorem’, ‘Pythagoras’ theorem’, ‘differential equation’.

Of course the last term does not fit, since the first three terms belong to geometry, whereas the last term belongs to calculus. Huygens’ determination of the area of the cissoid, however, may cast some doubts about this decision. To the modern eye his problem might appear as a ‘calculus problem’. His method, on the other hand, is purely geometrical. Thales, Pythagoras and similarity (compare Figure 3), these are exactly the tools that Huygens employs. Calculus and geometry, this is the morale of Huygens’ result, are not as divorced from each other as the typical courses on these subjects might let one think. By looking for instance at Johann Bernoulli’s lectures on the calculus (Bernoulli, 1924) one realizes that differentials and their equations were just an additional tool incorporated into the old game of geometry. Thales, Pythagoras, similarity and a differential equation, in Bernoulli’s determination of the tangents to a cycloid (Figure 4) we have all this at one place.

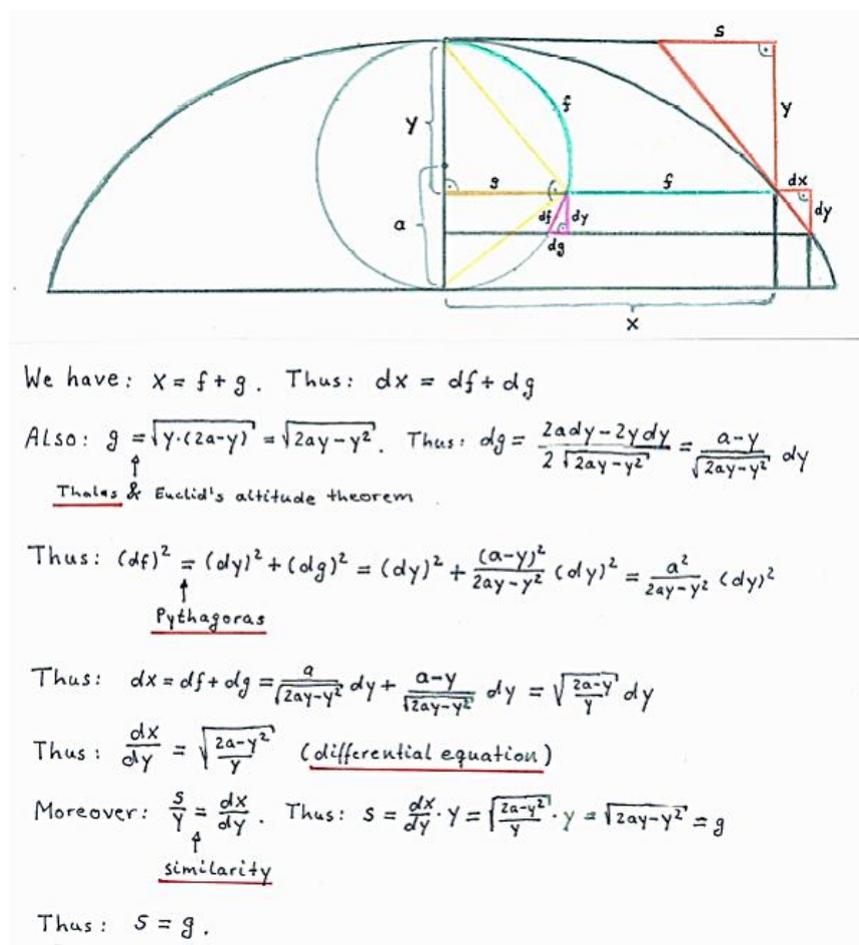


Figure 4: Bernoulli – The tangent of the cycloid

We now turn to the relationship between calculus and algebra. A major challenge in teaching these two subjects coherently is to show where the algebra ends and where the calculus begins, that is, to

reveal their line of demarcation. At least according to Leibniz the line runs precisely between the algebraic curves and the transcendental curves:

As far as the differential calculus is concerned, I admit that there is much in common between it and the things which were explored by both you [Wallis] and Fermat and others, indeed already by Archimedes himself. Yet now the matter is perhaps carried much further, so that now those things can be accomplished which in the past seemed closed even to the greatest geometers as Huygens himself recognized. The matter is almost the same in the analytical calculus applied to conical curves or higher: Who does not consider Apollonius and other ancients to have had theorems which present material for the equations by which Descartes later preferred to designate curves. In the meantime the matter has been reduced to calculation by the method of Descartes, so that now conveniently and without trouble that can be done which formerly required much effort of contemplation and imagination. In the same way, by our differential calculus, transcendentals too, which Descartes himself excluded in the past, are subjected to analytical operations. (as cited in Blåsjö, 2016, pp.16-17)

In this picture the relationship between geometry, algebra and calculus is beautifully simple. At first we have the Euclidean geometry with compass and ruler only. Then Descartes allows some other kinds of construction tools, which on the side of the equations correspond exactly with the algebraic curves. This world can be studied by means of algebra. Leibniz, finally, goes even further and includes modes of constructions that allow also transcendental curves to enter geometry. Algebra falls short of these curves and calculus is needed. A good way to experience this situation is by starting with Descartes' double-root method for finding normals and tangents and by trying to get as far as possible with this method. This is performed in detail in (Range, 2011). One first of all shows that there is a unique line through each point of the graph which intersects the graph at that point with multiplicity at least 2. Then one generalizes this fact to rational functions and finally to algebraic functions. When it comes to transcendental functions, of course, one gets stuck. It is at that point, that one has to generalize one's definition of tangent. It is at that point, where the 'limits' come into play.

### **Misconceptions – tangents require limits, areas require rectangles**

Instead of Descartes' double-root method we might also consider the notion of tangent that Euclid employed in his third book when dealing with the circle:

A straight line is said to *touch* a circle which, meeting the circle and being produced, does not cut the circle. (Euclid, 1956, p.1)

How far can we get with this notion? If we take the notion literally, it already fails when confronted with a parabola, since through each point of a parabola there are two different lines that have only one point in common with the parabola, one of them being a line parallel to the axis of symmetry of the parabola. We can, however, rule out these parallels by requiring the tangents to stay on one side of the parabola. This leads to the following modification of Euclid's definition: A line, which has a common point with a curve, is called *tangent* to that curve, if the curve lies on one side of the line and if it is the only line through the common point with that property. Huygens, for instance,

worked with this notion, as one can see in his work on the cycloid (see Aarts, 2015, pp.3-4). For convex curves this notion works fine, but already in the case of such a simple non-convex curve as the cubical parabola ( $f(x) = x^3$ ) not one single point would have a tangent according to this notion. However, for all but one point on the cubical parabola, the problem is easily resolved by only requiring the curve to stay locally on one side of the line. The point not covered by this extended notion is the inflection point. It seems that we have to give up on Euclid's notion of tangents ultimately when it comes to inflection points. Arnold Kirsch, however, has shown in (Kirsch, 1960a) how to overcome this obstacle. After defining *smoothness* of convex functions by means of the modified version of Euclid's notion of tangent, he goes on to define smoothness of functions in general essentially as follows: A function is *smooth* in some point, if the function lies locally between two smooth convex functions that both go through that point and that share a common tangent at that point. In (Kirsch, 1960b), Arnold Kirsch shows that this notion of *smoothness* is equivalent to our usual notion of differentiability. This is certainly an interesting result, since it clashes with the widespread belief that one needs the concept of *limit* in order to define differentiability and tangents in general.

Another major misconception I come across among teacher students more often than one might imagine is the belief that one can determine areas associated with curves only by means of the Riemann integral or rather by exhausting the relevant area with rectangles. This conception seems to consolidate itself in the course of the received instruction, since children, which have not yet been 'enlightened' by a course on calculus, apparently still use all kinds of figures, triangles, trapezoids, squares or whatever suits, in order to cover the area in question. Against this background it seems important to elaborately discuss examples such as Archimedes' quadrature of the parabola (see Dörrie, 1965, p.239), which uses triangles, or Huygens' determination of the area of the cissoid (see Maanen van, 2003), which uses trapezoids and triangles instead of the notorious rectangles.

## Summary and discussion

In this article I have shared my approach on how to address three major challenges ('meaning', 'coherence' and 'misconceptions') in a course on the didactics of calculus for future high school mathematics teachers. In order to discuss the meaning of the calculus I provided the students with results from Roberval and Huygens, which show firstly, what can be done even without calculus and secondly, how much insight is needed for this. In this way the students may see and appreciate the calculus, just as Leibniz apparently does, as a 'mechanization' of geometry. It should be critically noted that the historical episodes chosen play within geometry and due to this we have left the meaning of calculus for other branches of mathematics and especially its role in Newtonian physics completely untouched. In order to show the coherence between calculus and geometry I offered some examples from Bernoulli's lectures on the differential calculus. To this end it is also worthwhile to consider two different rather recent geometrical approaches (Apostol & Mnatsakanian, 2002 and Kaenders & Kirfel, 2017) to the integration of the basic functions, treated in high school. Both of them, the sweeping-tangent approach by Apostol and Mnatsakanian and the approach using geometrical transformations by Kaenders and Kirfel link calculus and geometry each in their own way. To show the coherence between calculus and algebra I focused on the

difference between algebraic and transcendental curves, especially by maxing out Descartes' double-root method. Finally, regarding the misconceptions, I tried to point out that some of them might be 'instruction-generated'. For example, I showed that generalizing Euclid's notion of tangent can lead to a 'limit-free' general notion of tangent, a fact that seems to clash with the calculus related beliefs of many students. By way of the three didactical challenges I tried to show that and to indicate how courses on the didactics of mathematics may benefit from a selective and targeted recourse to the history of mathematics.

## References

- Aarts, J. (2015). *Christian Huygens: Het Slingeruurwerk – Een studie*. Amsterdam, Epsilon Uitgaven.
- Apostol, T. M. & Mnatsakanian, M. A. (2002), Tangents and Subtangents Used to Calculate Areas, *The American Mathematical Monthly*, 109(10), 900-908.
- Bernoulli, J. (1924). *Die Differentialrechnung*. Ostwald's Klassiker der exakten Wissenschaften Nr. 211. Leipzig, Akademische Verlagsgesellschaft M.B.H.
- Bläsjö, V. (2016). *Transcendental Curves in the Leibnizian Calculus* (Doctoral Dissertation, University of Utrecht, Netherlands), Academic Press.
- Dörrie, H. (1965). *100 Great Problems of Elementary Mathematics*. New York, Dover Publications.
- Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, Springer Netherlands.
- Euclid (1956). *The Thirteen Books of the Elements*. Vol. 2 Book: 3-9 (trans. T. Heath). New York: Dover Publications.
- Heymann, H. W. (1997). Allgemeinbildung als Aufgabe der Schule und als Maßstab für Fachunterricht. *Pädagogik*, 49(2), 42-45.
- Kaenders, R. & Kirfel, C. (2017), Flächenbestimmung bei Basisfunktionen der Schule mit Elementargeometrie, *Mathematische Semesterberichte*, 64(2), 199-220.
- Kirsch, A. (1960a). Eine geometrische Charakterisierung der "Differenzierbarkeit" einer Funktion. *Mathematisch-Physikalische Semesterberichte*, 7, 96-100.
- Kirsch, A. (1960b). Ein geometrischer Zugang zu den Grundbegriffen der Differenzialrechnung. *Der Mathematikunterricht*, 8(2), 5-21.
- Maanen van, J. (2003). Precursors of Differentiation and Integration. In Jahnke, H.N. (Ed.), *A History of Analysis*, 41-72.
- Pedersen, K. M. (1969). Roberval's Method of Tangents. *Centaurus*, 13(2), 151-182.
- Pedersen, K. M. (1971). Roberval's Comparison of the Arclength of a Spiral and a Parabola. *Centaurus*, 15(1), 26-43.
- Poincaré, H. (1914). *Wissenschaft und Methode*. Leipzig, B.G. Teubner.
- Range, M. (2011). Where are Limits needed in Calculus? *The American Mathematical Monthly*, 118(5), 404-417.
- Verhulst, F. (2012). An interview with Henri Poincaré - Mathematics is the art of giving the same name to different things. *Nieuw Archief voor Wiskunde*, 5(13), 154-158.
- Whitman, E. A. (1943). Some Historical Notes on the Cycloid. *The American Mathematical Monthly*, 50(5), 309-315.