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THE IMPACT OF A LOWER ORDER TERM IN A DIRICHLET PROBLEM WITH A SINGULAR NONLINEARITY

LUCIO BOCCARDO AND GISELLA CROCE

ABSTRACT. In this paper we study the existence and regularity of solutions to the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-1}\nabla u) + u|u|^{r-1} = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

proving that the lower order term $u|u|^{r-1}$ has some regularizing effects on the solutions.

1. INTRODUCTION

In this paper we study the existence and regularity of weak solutions to the following nonlinear problem

$$(1.1) \quad \begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-1}\nabla u) + u|u|^{r-1} = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded open subset of \mathbb{R}^N , $N \geq 2$, f is a positive (that is $f(x) \geq 0$ and not zero a.e.) function in $L^1(\Omega)$ and $0 < \theta < 1$. Due to the singularity in the right hand side, a solution of this problem is a function u in $W_0^{1,1}(\Omega)$ such that for all $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u \geq c_\omega > 0$ in ω , and satisfying

$$(1.2) \quad \int_{\Omega} a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi + \int_{\Omega} |u|^{r-1}u\varphi = \int_{\Omega} \frac{f\varphi}{u^\theta} \quad \forall \varphi \in C_0^1(\Omega).$$

This paper is motivated by the results of [6], where the authors studied the existence, regularity and uniqueness of weak solutions to the

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singular semilinear problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

M was supposed to be an elliptic bounded matrix, $0 \leq f \in L^m(\Omega)$, $m \geq 1$, or f a Radon measure and $\theta > 0$.

In [11] a nonlinear version of the above problem was studied, considering an operator as the p -laplacian instead of $-\operatorname{div}(M(x)\nabla u)$ (see also Remark 4.2). In [13] the author added a lower order term growing as $|u|^{r-1}u$, and studied existence, regularity and uniqueness of solutions to problem (1.1). Although the right hand side is singular at u , the lower order term in the left hand side has a regularizing effect. In the right hand side, f is assumed to belong to $L^m(\Omega)$ with $m > 1$ in the case $\theta < 1$. In this paper we will analyse the limit case $f \in L^1(\Omega)$ showing that the lower order term $|u|^{r-1}u$ has still a regularizing effect on the solutions.

We observe that the regularizing effect of the lower order term $|u|^{r-1}u$ is well known for similar problems. It was pointed out in [7, 5, 8] for elliptic problems. Later it was showed for elliptic problems with degenerate coercivity in [9, 2, 3].

In this paper we will show the following result.

Theorem 1.1. Let $f \in L^1(\Omega)$ be a positive function ($f(x) \geq 0$ and not zero a.e.). Assume that $r > \frac{(1-\theta)(p-1)N}{N-p}$, $p \geq 1 + \frac{1-\theta}{r}$, $0 < \theta < 1$. Then there exists a function $u \in W_0^{1,q}(\Omega)$, with $q = \frac{p}{1+\frac{1-\theta}{r}}$ which is a solution to problem (1.1) in the sense of (1.2). Moreover $u^{r+\theta}$ belongs to $L^1(\Omega)$.

In order to prove this result, we will work by approximation, “truncating” the singular term $\frac{1}{u^\theta}$ so that it becomes not singular at the origin. We will get some a priori estimates on the solutions u_n of the approximating problems, which will allow us to pass to the limit and find a solution to problem (1.1). In order to obtain a solution in the sense described above, we will apply a suitable form of the Strong Maximum Principle.

We observe that the regularity of the above theorem implies that q can be equal to 1, that is, u can belong to the non-reflexive space $W_0^{1,1}(\Omega)$. This will demand the proof of the equi-integrability of $|\nabla u_n|^q$.

We will then analyse the regularity obtained and point out the regularizing effects of the term $|u|^{r-1}u$.

2. APPROXIMATING PROBLEMS

In this section we prove the existence of a solution to the following approximating problems:

$$(2.1) \quad \begin{cases} -\operatorname{div}(a(x)|\nabla u_n|^{p-1}\nabla u_n) + u_n|u_n|^{r-1} = \frac{f_n}{(u_n + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Due to the nature of the approximation, the sequence u_n will be increasing with n , so that the (strict) positivity of the limit will be derived from the (strict) positivity of any of the u_n (which in turn will follow by the standard maximum principle for elliptic equations).

Let f be a nonnegative measurable function (not identically zero), let $n \in \mathbb{N}$, let $f_n(x) = \min\{f(x), n\}$.

Lemma 2.1. *Problem (2.1) has a nonnegative solution u_n in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let n in \mathbb{N} be fixed, let v be a function in $L^2(\Omega)$, and define $w = S(v)$ to be the unique solution of

$$(2.2) \quad \begin{cases} -\operatorname{div}(a(x)|\nabla w|^{p-2}\nabla w) + |w|^{r-1}w = \frac{f_n}{(|v| + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The existence of a solution $w \in W_0^{1,p}(\Omega)$ follows from the classical results of [12].

Taking w as test function, we have, using the ellipticity of a ,

$$\alpha \int_{\Omega} |\nabla w|^p \leq \int_{\Omega} a(x)|\nabla w|^{p-2}\nabla w \cdot \nabla w = \int_{\Omega} \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} \leq n^{\gamma+1} \int_{\Omega} |w|.$$

By the Sobolev inequality on the left hand side and the Hölder inequality on the right hand side one has

$$\left[\int_{\Omega} |w|^{p^*} \right]^{p/p^*} \leq C n^{\gamma+1} \left(\int_{\Omega} |w|^{p^*} \right)^{\frac{1}{p^*}},$$

for some constant C independent on v . This implies

$$\|w\|_{L^{p^*}(\Omega)} \leq C n^{\gamma+1},$$

so that the ball of $L^{p^*}(\Omega)$ of radius $C n^{\gamma+1}$ is invariant for S . It is easy to prove, using the Sobolev embedding, that S is both continuous and compact on $L^{p^*}(\Omega)$, so that by Schauder's fixed point theorem there

exists u_n in $W_0^{1,p}(\Omega)$ such that $u_n = S(u_n)$, i.e., u_n solves

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) + |u_n|^{r-1}u_n = \frac{f_n}{(|u_n| + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By using as a test function $-u_n^-$, one has $u_n \geq 0$. Since the right hand side of (2.1) belongs to $L^\infty(\Omega)$, the result of [14], Théorème 4.2, imply that u_n belongs to $L^\infty(\Omega)$ (although its norm in $L^\infty(\Omega)$ may depend on n). \square

Lemma 2.2. *The sequence u_n is increasing with respect to n , $u_n > 0$ in Ω , and for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ (independent on n) such that*

$$(2.3) \quad u_n(x) \geq c_\omega > 0 \quad \text{for every } x \text{ in } \omega, \text{ for every } n \text{ in } \mathbb{N}.$$

Moreover there exists the pointwise limit $u \geq c_\omega$ of the sequence u_n .

Proof. Since $0 \leq f_n \leq f_{n+1}$ and $\gamma > 0$, one has (distributionally)

$$-\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) + |u_n|^{r-1}u_n = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} \leq \frac{f_{n+1}}{(u_n + \frac{1}{n+1})^\gamma},$$

so that

$$\begin{aligned} & -\operatorname{div}(a(x)(|\nabla(u_n)|^{p-2}\nabla(u_n) - |\nabla(u_{n+1})|^{p-2}\nabla(u_{n+1}))) + \\ & + |u_n|^{r-1}u_n - |u_{n+1}|^{r-1}u_{n+1} \leq f_{n+1} \frac{(u_{n+1} + \frac{1}{n+1})^\gamma - (u_n + \frac{1}{n+1})^\gamma}{(u_n + \frac{1}{n+1})^\gamma (u_{n+1} + \frac{1}{n+1})^\gamma}. \end{aligned}$$

We now choose $(u_n - u_{n+1})^+$ as test function. In the left hand side we use the monotonicity of the p -laplacian operator as well as the monotonicity of the function $t \rightarrow |t|^{r-1}t$. For the right hand side we observe that

$$\left[\left(u_{n+1} + \frac{1}{n+1} \right)^\gamma - \left(u_n + \frac{1}{n+1} \right)^\gamma \right] (u_n - u_{n+1})^+ \leq 0;$$

recalling that $f_{n+1} \geq 0$, we thus have

$$0 \leq \alpha \int_{\Omega} |\nabla(u_n - u_{n+1})^+|^p \leq 0.$$

Therefore $(u_n - u_{n+1})^+ = 0$ almost everywhere in Ω , which implies $u_n \leq u_{n+1}$.

Since u_1 belongs to $L^\infty(\Omega)$ (see Lemma 2.1), and there exists a constant (only depending on Ω and N) such that

$$\|u_1\|_{L^\infty(\Omega)} \leq C \|f_1\|_{L^\infty(\Omega)} \leq C,$$

one has

$$\begin{aligned} & -\operatorname{div}(a(x)|\nabla u_1|^{p-2}\nabla u_1) + |u_1|^{r-1}u_1 \\ &= \frac{f_1}{(u_1 + 1)^\gamma} \geq \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)} + 1)^\gamma} \geq \frac{f_1}{(C + 1)^\gamma}. \end{aligned}$$

Since $\frac{f_1}{(C+1)^\gamma}$ is not identically zero, the strong maximum principle implies that $u_1 > 0$ in Ω (see [15]), and that (2.3) holds for u_1 (with c_ω only depending on ω , N , f_1 and γ). Since $u_n \geq u_1$ for every n in \mathbb{N} , (2.3) holds for u_n (with the same constant c_ω which is then independent on n). \square

3. A PRIORI ESTIMATES

Let $k > 0$. We denote by $T_k(x)$ the function $\max\{-k, \min\{s, k\}\}$.

Lemma 3.1. *Let $k > 0$ be fixed. The sequence $\{T_k(u_n)\}$, where u_n is a solution to (2.1), is bounded in $W_0^{1,p}(\Omega)$.*

Proof. It is sufficient to take $T_k(u_n)$ as a test function in problems (2.1). \square

Theorem 3.2. Assume $r > \frac{(1-\theta)(p-1)N}{N-p}$, $p \geq 1 + \frac{1-\theta}{r}$. The sequence of solutions $\{u_n\}$ to (2.1) is bounded in $W_0^{1,q}(\Omega)$, with $q = \frac{p}{1+\frac{1-\theta}{r}}$. Moreover the sequence $\{u_n^{r+\theta}\}$ is bounded in $L^1(\Omega)$.

Proof. We use $(u_n + h)^\theta - h^\theta$ as a test function in (2.1): we get

$$\alpha \int_{\Omega} \frac{|\nabla u_n|^p}{(u_n + h)^{1-\theta}} + \int_{\Omega} u_n^r [(u_n + h)^\theta - h^\theta] \leq \int_{\Omega} f.$$

At the limit as $h \rightarrow 0$ we get

$$\alpha \int_{\Omega} \frac{|\nabla u_n|^p}{u_n^{1-\theta}} + \int_{\Omega} u_n^{r+\theta} \leq \int_{\Omega} f.$$

Then

(3.1)

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^q &= \int_{\Omega} \frac{|\nabla u_n|^q}{u_n^{\frac{q}{p}}} u_n^{(1-\theta)\frac{q}{p}} \leq \left[\int_{\Omega} \frac{|\nabla u_n|^p}{u_n^{(1-\theta)}} \right]^{\frac{q}{p}} \left[\int_{\Omega} u_n^{(1-\theta)\frac{q}{p} \frac{p}{p-q}} \right]^{\frac{p-q}{p}} \\ &\leq \left[\frac{1}{\alpha} \int_{\Omega} f \right]^{\frac{q}{p}} \left[\int_{\Omega} u_n^{(1-\theta)\frac{q}{p-q}} \right]^{\frac{p-q}{p}}. \end{aligned}$$

Observe that q is such that $(1-\theta)\frac{q}{p-q} = r$. Thus

$$\int_{\Omega} |\nabla u_n|^q \leq \left[\frac{1}{\alpha} \int_{\Omega} f \right]^{\frac{q}{p}} \left[\int_{\Omega} f \right]^{\frac{p-q}{p}} = \left[\frac{1}{\alpha} \right]^{\frac{q}{p}} \int_{\Omega} f.$$

□

Lemma 3.3. *Let u_n be a solution to problem (2.1). Then*

$$\int_{\{k < u_n\}} u_n^r \leq \frac{1}{k^\theta} \int_{\{k < u_n\}} f$$

and $\lim_{|E| \rightarrow 0} \int_E u_n^r = 0$ uniformly with respect to n .

Proof. Let $k > 0$ and ψ_i be a sequence of increasing, positive, uniformly bounded $C^\infty(\Omega)$ functions, such that

$$\psi_i(s) \rightarrow \begin{cases} 1, & s \geq k \\ 0, & 0 \leq s < k. \end{cases}$$

Choosing $\psi_i(u_n)$ in (2.1), we get

$$\int_{\Omega} u_n^{r-1} u_n \psi_i(u_n) \leq \int_{\Omega} \frac{f_n}{(u_n + \frac{1}{n})^\theta} \psi_i(u_n).$$

The limit on i gives

$$\int_{\{k < u_n\}} u_n^r \leq \int_{\{k < u_n\}} \frac{f}{(u_n + \frac{1}{n})^\theta}.$$

Therefore we have

$$\int_{\{k < u_n\}} u_n^r \leq \frac{1}{(k + \frac{1}{n})^\theta} \int_{\{k < u_n\}} f.$$

This implies that

$$\int_E u_n^r \leq k^r |E| + \int_{E \cap \{u_n > k\}} u_n^r \leq k^r |E| + \frac{1}{k^\theta} \int_{\{u_n > k\}} f.$$

Since $f \in L^1(\Omega)$ for any given $\varepsilon > 0$, there exists k_ε such that $\int_{\{|u_n| > k_\varepsilon\}} |f| \leq \varepsilon$. Therefore

$$\int_E u_n^r \leq k_\varepsilon^r |E| + \frac{\varepsilon}{k^\theta}$$

and the statement of this lemma is thus proved. □

Proposition 3.4. *Let u_n be a solution to problem (2.1). Let $q = \frac{p}{1+\frac{1-\theta}{r}}$.*

Then $\lim_{|E| \rightarrow 0} \int_E |\nabla u_n|^q = 0$ uniformly with respect to n .

Proof. From (3.1) and Lemma 3.3 we infer

$$\begin{aligned} \int_{\{k < u_n\}} |\nabla u_n|^q &= \int_{\{k < u_n\}} \frac{|\nabla u_n|^q}{u_n^{\frac{(1-\theta)q}{p}}} u_n^{(1-\theta)\frac{q}{p}} \leq \left[\int_{\Omega} \frac{|\nabla u_n|^p}{u_n^{(1-\theta)}} \right]^{\frac{p}{q}} \left[\int_{\{k < u_n\}} u_n^{(1-\theta)\frac{q}{p} \frac{p}{p-q}} \right]^{\frac{p-q}{p}} \\ &\leq \left[\frac{1}{\alpha} \int_{\Omega} f(x) \right]^{\frac{p}{q}} \left[\int_{\{k < u_n\}} u_n^{(1-\theta)\frac{q}{p-q}} \right]^{\frac{p-q}{p}} \leq \left[\frac{1}{\alpha} \int_{\Omega} f(x) \right]^{\frac{p}{q}} \left[\int_{\{k < u_n\}} f(x) \right]^{\frac{p-q}{p}} \end{aligned}$$

which gives the equiintegrability of $|\nabla u_n|^q$, with the same technique as in the previous result. \square

4. PROOF OF THE MAIN THEOREM

To prove Theorem 1.1 we are going to pass to the limit in problems (2.1). By Lemma 3.1 $T_k(u_n) \rightarrow T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, where u is the pointwise limit of u_n (see Lemma 2.2). We will use the a.e. convergence of ∇u_n to ∇u , stated in theorem 2.3 of [11], that we recall:

Proposition 4.1. *Let u_n be a sequence of functions such that $T_k(u_n) \rightarrow T_k(u_n)$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \rightarrow u \leq u_n$ a.e. in Ω . Assume that $-\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) \geq 0$. Then $T_k(u_n) \rightarrow T_k(u)$ in $W_0^{1,p}(\Omega)$. In particular $\nabla u_n \rightarrow \nabla u$ a.e. in Ω .*

We are now in position to prove Theorem 1.1.

Proof. It is easy to pass to the limit in the right hand side of problems (2.1): indeed $u_n \rightarrow u$ a.e. and $u_n \leq u \in W_0^{1,1}(\Omega)$. Therefore one can apply the Lebesgue theorem.

By the same argument, the second term of the left hand side converges to $\int_{\Omega} u^r \varphi$.

For the first term, we have that $a(x)|\nabla u_n|^{p-2}\nabla u_n$ converges almost everywhere in Ω to $a(x)|\nabla u|^{p-2}\nabla u$ by Theorem 4.1. Furthermore, this term is majorised by $\beta|\nabla u_n|^{p-1}$. Observe that $p-1 \leq q$. The limit follows by Vitali's theorem, which can be applied thanks to Proposition 3.4. \square

Remark 4.2. We now compare the regularity found in [11] (that is the same problem without lower order term) with q , as defined in Theorem 1.1. The solution found in [11] belongs to $W_0^{1,\tilde{p}}(\Omega)$, $\tilde{p} = \frac{N(p+\theta-1)}{N+\theta-1}$, for $2-\theta + \frac{\theta-1}{N} \leq p < N$. One has $\tilde{p} \leq q$ if $r > \frac{N(p+\theta-1)}{N-p}$ which is larger than the lower bound $\frac{(1-\theta)(p-1)N}{N-p}, p \geq 1 + \frac{1-\theta}{r}$ for r of Theorem 1.1.

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