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The use of history of mathematics and, in particular, of historical problems is a very useful tool for teacher training. In this work we present some partial results obtained after the design and implementation of an activity based upon an arithmetic problem from the "Conpusicion de la arte de la arismetica y juntamente de geometria" by the Spanish 16th century Dominican friar Juan de Ortega. This activity was conducted with 48 prospective Secondary education teachers enrolled on an on-line Masters' degree. We focus on the part of the activity related to generalization and, more particularly, on the relation between generalization and use of algebraic language. Our results show that there exists a positive significant relation between generalization and the use of algebraic language. However, we also found out that many students were able to provide correct and satisfactory generalizations using just natural language.

Keywords: History of mathematics, word problems, teacher training, generalization, algebraic language.

Introduction and objectives.

The use of history of mathematics in mathematics education can be approached in many different ways and with many different motivations and objectives (Jankvist, 2009). Regarding the classroom implementation, one interesting possibility is the use of historical problems and, in particular of problems "having clever, alternative, or exemplary solutions" (Tzanakis et al., 2000, p. 224).

In the context of teacher training, Mosvold, Jakobsen and Jankvist (2014) show how all the different domains of teachers' Mathematical Knowledge for Teaching (MKT) might profit from the use of history of mathematics. In particular, the analysis of historical problems and their solutions can be a very interesting task for prospective teachers (Meavilla-Seguí & Oller-Marcén, 2015).

Generalization belongs to the two main domains of the MKT model described by Ball, Thames and Phelps (2008): Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). From the point of view of SMK it is clear that generalizing is one of the main components of mathematical activity (Harel & Tall, 1991). Regarding PCK, Ellis (2007, p. 221–222) points out that "one of the primary aims of educational practice is to help students to develop robust, generalizable knowledge that will support their abilities to create generalizations in the classroom". Generalizing and specializing are two key aspects of one possible approach to algebra (Lee, 1996) and the use of symbolic algebraic language is strongly related to the generalization of arithmetic properties or to the modelization of different situations (Usiskin, 1988).

Even in the absence of symbolic algebraic language, the search for some kind of balance between generalization and specialization can be clearly seen in many ancient mathematical texts. These texts included very long collections of problems possibly hoping that the reader would be able to

infer the general rules used to solve the seemingly different problems. In the *Zhou bi suan jing*, a Chinese text from the 3rd century, this idea was even explicitly stated in a dialogue between a master and a student:

> Illuminating knowledge of categories [is shown] when words are simple but their application is wide-ranging. When you ask about one category and are thus able to comprehend a myriad matters, I call that understanding [my] Way. (Cullen, 1996, p. 15)

This underlying philosophy can still be found in Fibonacci's *Liber Abaci* (Sigler, 2002), for example, about one thousand years later. In fact, many mathematical texts written during the 16th and 17th centuries contained series of problems whose solutions were just the description of the operations that had to be performed with the given data.

In this work, we analyze part of an activity designed for and implemented with prospective Secondary school teachers which is based on an arithmetic problem excerpted from a 16th century Spanish arithmetic textbook. Our main objective is to explore if and how the participants are able to provide generalizations of the statement and the solution of the problem. More particularly, we focus on the interplay between generalization and the use of algebraic language. In other words, we focus on two ways to generalize, according to which semiotic register is used: algebraic language or natural language.

The source.

The only known biographical facts about friar Juan de Ortega is that he was born in Palencia, that he was a Dominican and that he taught mathematics in Spain and Italy both publicly and privately (Madrid, 2016).

As far as we know, he only wrote one book, the *Conpusicion de la arte de la arismetica y juntamente de geometria* (Ortega, 1512). Ortega's book was rather popular and it was reedited several times in different countries during the 16th century (Rey Pastor, 1926). In fact, it was reprinted (sometimes with slightly different titles) and translated at least in Roma and Lyon (in 1515), Messina (in 1522), Sevilla (in 1534, 1537, 1542 and 1552) and Granada (in 1563). Some authors (Carabias, 2012) point out that the 1515 translation published in Lyon was the first text on commercial arithmetic written in French. Marquant (2016) presents an interesting discussion on the authorship of some of these translations.

Regarding its content, the book consists of 204 folios organized according to 36 chapters and it covers the usual topics in a Renaissance commercial arithmetic text: elementary operations with natural and fractional numbers, progressions, square and cubic roots, rule of three and its applications and some elements of geometry. The editions of 1534, 1537 and 1542 included a method to approximate square roots that improved the known methods at his time, on this issue we refer the reader to the classical works by Rey Pastor (1926, pp. 72–81) and Barinaga (1932). Finally, the 1552 edition included a collection of 13 problems solved using algebraic techniques that was inserted by Gonzalo Busto, the editor of this probably posthumous edition. This is an interesting feature of the text, since the first book with a systematic introduction to algebraic

language written in Spanish, the *Libro Primero de Arithmetica Algebratica* (Aurel, 1552) was published that very same year (Puig & Fernández, 2013).

In addition to many classical problems (Métin, 2018), in chapters 14 through 17, Ortega provides a collection of 34 examples of what he calls "extraordinary rules" and that he defines (Ortega, 1512, fol. 60r) as "rules outside the usual way of adding, subtracting, multiplying and dividing and that involve hidden ways to apprise those who know little". They are, in fact, a collection of arithmetic problems and their solutions presented in a merely descriptive fashion. For example, the eleventh example from chapter 14 goes as follows (our translation):

> If you wanted to know, or if it was asked to you, which are those three numbers such that two fifths of the first one is the same as three sevenths of the second one and the same as four ninths of the third one, you will do the following. Put the numbers as you see here: $\frac{2}{5}\ \frac{3}{7}\ \frac{4}{9}$. Then, multiply the 5 below the 2 by the 3 above the seven to get a 15. Multiply this 15 again by the 4 above the 9 and you will get 60, which is the first number. After that, multiply the 7 below the 3 by the 4 above the 9 to get a 28, which you shall multiply by the 2 above the 5 in order to get a 56, which is the second number. After that, multiply again the 9 below the 4 by the 2 above the 5 and you will get an 18 which you shall multiply again by the 3 above the 7 to get 54. This is the third number. If you want to check that it is true, look for the two fifths of 60 and you will see that it is 24. In the same way you will see that three sevenths of 56 is 24 and that four ninths of 54 is 24, as you see in the example. (Ortega, 1512, fols. 63r–63v)

Activity and participants.

The experiment was carried out in the context of the subject "Didactics of arithmetic and algebra" during the academic year 2017-2018 with 48 students of the online Master's degree in Didactic of Mathematics in Secondary education from the International University of La Rioja. The age of the participants ranged from 25 to 56 (with a mean of 37.7 and a standard deviation of 7.9) and most of them (91.7%) had at least six months of prior experience teaching mathematics at secondary level. Even though background of the students was diverse, most of them had prior teaching experience or a degree on a STEM or education discipline and hence a good knowledge of mathematics could be assumed. Table 1 provides further information about the participants.

Gender		**Nationality**				**Degree**		
Male	*Female*	*Spain*	*Colombia*	*Ecuador*	*Mexico*	*STEM*	*Education*	*Other*
52%	48%	17%	44%	37%	2%	61%	35%	4%

Table 1: Some data about the sample

The designed activity was based upon the problem transcribed above. It consisted of three exercises, each of them addressing to a different issue. Being an online Master's degree, the students had to complete this activity at their homes and they were given a two-week deadline in order to obtain as many answers as possible.

- In exercise 1, a short version of the original problem was presented using modern natural language and the students were asked to solve it using whichever method they wanted.

- In exercise 2, a short version of the original solution was presented using modern natural language and the students were asked to:
  - Give their opinion about this solution.
  - Compare this solution with their own solution from exercise 1.
  - Explain which of the two solutions is "better" and in which sense.
- In exercise 3, the students were asked to provide a general version of the statement of the problem from exercise 1 and a general solution based on the solution from exercise 2.

As we see, exercises 1 and 3 mostly deal with purely mathematical aspects so they are more related to SMK. Exercise 2, on the other hand, is more focused on PCK because it involves the analysis of different solutions to the same problem, their comparison and the reflection about the advantages and disadvantages of each of them. In this work, we will present the results of the analysis of exercise 3. In particular, we will focus on two variables: generalization and language. For the first one we will focus on its correctness (correct or incorrect) and on its completeness (partial or full) while for the second one we will consider the attributes natural and algebraic.

Results.

Although in this work we focus only on exercise 3, it is interesting to point out that almost all the students (46 out of 48) were able to solve the problem in exercise 1 at some extent. However, not all the answers were equally complete. For example, very few students mention that the problem has an infinite number of solutions. In any case, we conclude that the students were able to understand the problem

Exercise 3 was answered by 47 out of the 48 participants. Only 28 of them (about 60%) were able to provide some kind of correct generalization for the statement and the solution of the problem. In Table 2 we see that those participants that used algebraic language were more capable to provide a correct generalization of the situation. In fact, performing a χ^2 test of independence, we obtain a statistically significant relationship between both variables (significance level of about 95%).

	Algebraic language	Natural language
Correct generalization	18	10
Incorrect generalization	7	12

Table 2: Correctness versus use of algebraic language

Most of the wrong answers that tried to generalize the problem using only natural language simply provided incomplete statements. One student's answer, for example, was: "Do the following exercise, multiplying one denominator by the remaining numerators", which is a just simple explanation of the method rather than a generalization.

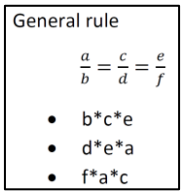

Figure 1: Incorrect generalization

The incorrect generalizations that used algebraic language mostly looked like the example in Figure 1. The statement of the problem (top of the figure) is expressed incorrectly and, at the bottom, we just find the required operations expressed symbolically.

It is noteworthy that some students seem to identify the process of generalization just with the use of symbols. Figure 2 is an interesting example of this phenomenon. There is no generalization in the student's answer, just the use of the letters X, Y and Z to denote the unknown numbers.

Being three quantities X; Y; Z proportional to each other, such that:

2/5 X = 3/7 Y = 4/9 Z

It is possible to find those quantities multiplying the denominator of each fraction by the numerators of the other two fractions

Figure 2: Use of symbols that does not imply generalization

We recall that 28 students provided a correct generalization of the original problem and its solution. However, not all of them provided a full generalization of the situation. In Table 3 we analyze the relation between the use of algebraic language and the degree of generalization of the answer.

	Algebraic language	Natural language
Full generalization	4	7
Partial generalization	14	3

Table 3: Degree of generalization versus use of algebraic language

We obtain again a statistically significant relationship between both variables (significance level of about 95%) showing that the students that were able to correctly generalize the original situation using natural language provided full generalizations more often than their partners. The following transcription is a good example of full generalization that essentially uses only natural language:

> If in a set of *n* numbers that are proportional to each other in a rational way, we multiply the denominator of one of them by the (*n* – 1) remaining numerators, we obtain that number. If we successively proceed in this way, we will obtain the *n* numbers of the solution. Since it is a compatible and indeterminate system with (*n* – 1) equations and *n* unknowns, the system has infinitely many solutions…

a.- General statement.

Given the following n fractions $\frac{a_i}{b_i}$ $i \in \{1,2,\dots,n\}$

find n natural numbers x_i $i \in \{1,2,\dots,n\}$

such that $\frac{a_i}{b_i}x_i = \frac{a_{i+1}}{b_{i+1}}x_{i+1}$ $\forall i \in \{1,2,\dots,n-1\}$

b.- General solution.

$$x_i = b_i \cdot \prod_{\forall j \neq i} a_j \qquad \forall i \in \{1,2,\dots,n\}$$

Figure 3: Full generalization using only algebraic language

Regarding the 4 answers that fully generalized the problem using algebraic language, three of them combined algebraic and natural language and only one (Figure 3) could be described as "completely algebraic". Only one of these 4 answers stated that the problem has infinitely many solutions.

Now, 17 students provided correct generalizations that were only partial generalizations. In particular, they only considered a situation with three unknown numbers as in Figure 4.

To find three numbers such that a/b of the first one is the same as c/d of the second and as e/f of the third

$$\frac{a}{b}x = \frac{c}{d}y = \frac{e}{f}z$$

Figure 4: Partial generalization using algebraic language

Even if most of these 17 answers were very similar to that in Figure 4 (only varying in the degree of use of algebraic language), two of them were particularly incomplete generalizations. For example, in Figure 5, a student considers arbitrary fractions in the statement of the problem but imposes a very particular relation between numerators and denominators inspired by the particular numbers of the original problem.

To find three natural numbers a, b and c such that the x/(x+3) of the first one, a, is the same as the (x+1)/(x+5) of the second, b, and as the (x+2)/(x+7) of the third, c

Figure 5: A very restricted generalization

The other restricted generalization imposed the additional conditions that the numerators (n_i) of the three fractions are consecutive numbers and the denominators are of the form $2n_i + 1$.

Discussion and final remarks.

From the purely mathematical point of view, it was somewhat surprising that about 40% of the students were not able to provide a correct generalization of the given situation. This might be caused in some cases just by some lack of SMK (Figures 1 or 2) while in other cases it might be related to a misconception about generalization that identifies it with the mere use of symbols in any way (see Figure 2). Regarding the completeness of the generalizations we see that, in terms of Ellis (2007, p. 235), many students generalize just by *removing particulars* (what we called partial generalization, like in Figure 4) rather than by *expanding the range of applicability* (what we called full generalization, like in Figure 3). It is also noteworthy that some students show a different behavior between the statement and the solution. For example, in Figure 2, the student does not generalize the statement at all (he only introduces letters for the unknown quantities) but in the solution he refers to "each fraction" instead of to particular values. This fact probably deserves more attention.

Throughout the students' responses there is a varying degree in the use of algebraic language. From those using only natural language to those using only algebraic language. Nevertheless, most of the students lie in between showing in their answers a mixture of algebraic and natural language. The results of our work suggest that the use of algebraic language facilitates the process of generalization of the considered problem, illustrating, as Usiskin (1988) points out, that one of the main uses of variables is as "pattern generalizers". However, and somewhat paradoxically, this use

of algebraic language seems to constitute some kind of obstacle in order to fully generalize the considered situation. There can be several explanations for this phenomenon. One reason can be the context of the exercise, since the answer that was provided to the students only used natural language. Another reason can be related to SMK, because some students might be uncomfortable using algebraic language. Finally, the beliefs, conceptions and expectations of the students about the notion of generalization (Strachota, 2015) might play an important role in their performance.

Zazkis and Liljedahl (2002, p. 400) reached similar conclusions finding out that "there is a gap between students' ability to express generality verbally and their ability to employ algebraic notation comfortably". Nevertheless, it must be noted that they worked with preservice elementary school teachers in a context of pattern generalization, while we have worked in a purely arithmetic setting with Secondary education teachers. Either way, there exist contradictory results in the literature. For example, in (Richardson, Berenson & Staley, 2009) most of the elementary prospective elementary teachers participating in the study expressed their explicit generalizations using algebraic notation.

The designed activity, an example of using history as a tool (Jankvist, 2009), involved other exercises. We have already mentioned that most of the students solved the problem correctly. However, after a preliminary and incomplete analysis, we have observed different solving methods some of which involve the use of other semiotic registers besides algebraic or natural language. For example, a few students solved the problem using diagrams. It would be interesting to analyze the influence of different aspects of the students' solution to the problem over their ulterior process of generalization.

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