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# Mean field flow equations and asymptotically free scalar fields

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The flow equations of the renormalisation group permit to analyse the perturbative  $n$ -point functions of renormalisable quantum field theories. Rigorous bounds implying renormalisability allow to control large momentum behaviour, infrared singularities and large order behaviour in the number of loops and the number of arguments  $n$ . Gauge symmetry which is broken by the flow in momentum or position space, can be shown to be restored in the renormalised theory.

In this paper we want to do a first but important step towards a rigorous non-perturbative analysis of the flow equations (FEs). We restrict to massive scalar fields and analyse the *mean field limit* where the Schwinger or 1PI functions are considered to be momentum independent or, otherwise stated, are replaced by their zero momentum values. We regard smooth solutions of the system of FEs for the  $n$ -point functions for different sets of boundary conditions. We will realise that allowing for nonvanishing irrelevant terms permits to construct *asymptotically free* and thus nontrivial *scalar field theories* in the mean field approximation. We will also analyse the so-called trivial solution so far generally believed to exhaust four-dimensional scalar field theory. The method paves the way to a study of the system of FEs beyond the mean field limit.

## 1 Introduction

Quantum field theory, originally developed to implement the principles of quantum mechanics in relativistic systems, has become the general theoretical framework to study physical systems with an infinite (or large) number of degrees of freedom. Relativistic quantum systems are

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described by relativistic quantum field theory, Euclidean field theory gives access to critical systems in statistical mechanics, systems from solid state physics can be modeled by field theories at finite density and temperature. These systems have different kinematics which is reflected in particular by the form of the (free) propagator or two-point function. Interactions are introduced via the path integral formalism.

Aiming at mathematical rigour one is faced with the problem that path integrals describing interacting systems in field theory are generally not defined a priori. Whereas there is a complete theory of Gaussian measures applying to the noninteracting case, a mathematically oriented study of interacting field theories generally starts from regularised versions of the theory, where the number of degrees of freedom in space and momentum has been made (essentially) finite by hand, through the introduction of regulators like finite volume and large momentum cutoffs. One then studies correlation functions and proves that these have uniform limits in the cutoffs. For a general introduction to these methods see [7], euclidean scalar field theories are analysed in [23, 25].

The functional flow equation is a differential equation for the effective action functional of the field theory considered. When expanded in moments it becomes an infinite system of differential equations for the connected amputated Schwinger functions of the theory. In a seminal paper [22] Polchinski observed that when expanding these functions order by order in the number of loops, there is an airtight inductive scheme which permits to sufficiently control the perturbative functions such that renormalisability follows. In a subsequent paper [10], see also [11], and [12, 20] for reviews, it was shown how to impose physical renormalisation conditions, and the induction hypothesis was sharpened so that cutoff independence became immediate. As a result ultraviolet renormalisability could be largely reduced to power counting once the optimal induction hypothesis had been found. The complicated combinatoric aspects of the problem, which had found their deep solution in Zimmermann's forest formula [31], thus turned out not to be intrinsic to the renormalisation problem, but rather to stem from the fact that the perturbative contributions had been split up in too fine a way, namely into Feynman diagram amplitudes<sup>1</sup>. In contrast, methods originally stemming from statistical physics like cluster and Mayer expansions [18, 2, 7, 23, 3], permit to analyse regularised path integrals non-perturbatively, but are relatively straightforward to apply only in theories which do not have to be renormalised in an essential way like  $\phi_2^4$  and  $\phi_3^4$  or other superrenormalisable models. They are technically very hard to apply in strictly renormalisable theories. The review [3] shows the state of the art and reveals important progress made in this respect over the last decades. Still in quantum field theory with hindsight to particle physics the relevance of the constructive path integral method is also limited by the fact that the physically interesting theories are either plagued by the triviality statement or by infrared problems which are presently beyond scope in mathematical physics, as is the case for quantum chromodynamics. As a consequence the work performed in constructive field theory has not entered text books on quantum field theory outside the realm of mathematical physics, in spite of the fact that the nonperturbative analysis of field theory is generally recognized to be an important problem.

Regarding on the other hand the flow equations, one realises that they have not been used with much success in the rigorous analysis of quantum field theory beyond perturbation the-

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<sup>1</sup>This remark does of course not put into question the value of Feynman diagrams. It only says that they are not optimally adapted for a mathematical analysis of the UV divergences (and related problems).

ory. Whereas the renormalisation problem becomes transparent and easy in this framework, by being related immediately to power counting, there are problems of combinatoric or algebraic origin, which hinder a nonperturbative analysis, even in the absence of renormalisation. To some degree these problems are already present in  $\phi_1^4$  or even  $\phi_0^4$  theory as will be explained more precisely in the following. In this context we cite a beautiful paper by Rivasseau [24] on a construction of planar “wrong sign”  $\phi_4^4$ -theory which is intermediate between constructive and flow equation methods.

In this paper we shall consider scalar field theories in four dimensions in the mean field limit. These have the same power counting and scaling behaviour as the full four dimensional theory and seem to capture well the basic physical properties of four dimensional scalar field theories.

In perturbative quantum field theory, one typically starts from a bare action which contains only a few local monomials of low degree in the fields. In a theory like quantum electrodynamics this leads, after perturbative renormalisation, to results which are in extremely good agreement with experiment. The simple form of the bare action is also justified by the fact that higher order monomials in the fields lead to nonrenormalisable interactions. Theories containing such interactions are generally not predictive in the perturbative framework. From the point of view of the Wilson renormalisation group these low order monomial lagrangians correspond to a fine tuning procedure. Staying with the example of quantum electrodynamics, this means that even if we start at the UV cutoff scale with a bare lagrangian containing only the monomials

$$\bar{\psi}\psi, \quad \bar{\psi}\partial^\mu\psi, \quad \bar{\psi}A^\mu\psi, \quad F_{\mu\nu}F^{\mu\nu}, \quad (\partial_\mu A^\mu)^2,$$

after integrating out degrees of freedom (in whatever small a momentum range) we immediately obtain a nonpolynomial effective lagrangian containing monomials of any degree in the fields and their derivatives (as far as they are allowed by the symmetries of the theory). Otherwise stated, this fine tuning consists in arranging things such that the infinite number of trajectories of all higher n-point functions are forced to pass through 0 each, and for all momentum arguments, at the same value of the renormalisation scale, leaving only the few local terms appearing in the bare lagrangian we wrote above.

The infinite number of terms generated by the renormalisation group evolution can be shown in perturbation theory to contain each an inverse power of the renormalisation group scale, corresponding to its mass dimension, times a suitable function bounded uniformly in the scale up to logarithms. All these terms are uniformly bounded in the UV cutoff [16, 20]. Their contributions do not produce any new ultraviolet divergences, as compared to those stemming from the initial bare lagrangian. Thus the argument of nonrenormalisability disappears in this case due to the aforementioned inverse powers. Nevertheless one also tends to start from those low order monomial bare lagrangians in constructive field theory which is based on the Wilson renormalisation group. There are two reasons for this: the starting point of the construction is easier to control mathematically, and perturbative calculations in physics are based on monomial bare lagrangians. Nor does it seem evident to characterize generic classes of nonpolynomial starting lagrangians.

In this paper we want to insist on the fact that different, generally nonpolynomial, bare actions, scaling with the cutoff as indicated before, may lead to essentially different theories. As we said we restrict to four dimensional scalar field theories in the mean field limit. We will show that, depending on the choice of the bare lagrangian, one may in particular obtain

asymptotically free scalar field theories which escape the so-called triviality statement [5, 1]. Our results are not in contradiction with this statement since we will verify that for the fine tuned bare Lagrangian containing only local terms of the type  $\varphi^4$  and  $\varphi^2$ , the trivial solution emerges indeed. We will characterize this solution quite explicitly. It will turn out that enforcing the fine tuned boundary conditions generates large values for the derivatives of the n-point functions w.r.t. the renormalisation group scale.

We think our results are robust, in the sense that we expect them to hold beyond the mean field limit. This is due in particular to the fact that we find the trivial solution as expected. From the technical side the main point to be mentioned is that when going beyond the mean field limit, there appears a function  $\dot{C}^\alpha(p)$  in the FEs, see (12), which is replaced by 1 in the limit. Generally we have  $0 < \dot{C}^\alpha(p) = e^{-\alpha(p^2+m^2)} \leq 1$ , which means that taking into account this factor leads to a contraction of the respective term in FEs which should be a controllable modification. We also mention that the critical behaviour in statistical physics is exactly described by the mean field approximation in  $d > 4$  dimensions [5, 1], as was first pointed out by Ginzburg [6].

The conclusion of our findings is that irrelevant terms in the language of the renormalisation group, can nevertheless completely modify the behaviour of a theory, at least when we add an infinity of such terms,

The paper is organized as follows : In section 2 we introduce the flow equations. In section 3 we perform the mean field limit. Section 4 is at the heart of this paper. We study various types of smooth solutions of mean field FEs. In 4.1 we study solutions for which we impose certain smallness and smoothness conditions at the UV boundary. Then in 4.2 we study a class of strictly positive solutions of similar type. In 4.3 we also impose bounds on the initial conditions which are sufficient to make the starting regularised path integral well defined. In 4.4 we study the trivial solution. In section 5 we develop the corresponding formalism for one-particle irreducible functions and obtain results corresponding to section 4.1 for those.

## 2 The flow equations

We consider a self-interacting scalar field on four dimensional Euclidean space. We adopt the renormalisation group flow equation framework [29, 30, 26, 22]. In the following we will give a brief review of the general formalism and define the objects of interest for the purpose of this paper. See [20, 12, 9] for more comprehensive reviews of the flow equation approach within our context.

### 2.1 The flow equations for the effective action

We start formulating our theory with ultraviolet (UV) cutoff and infrared (IR) cutoff in the standard path integral formalism. This requires two main ingredients:

1. We define the regularised momentum space propagator as

$$C^{\alpha_0, \alpha}(p; m) = \frac{1}{p^2 + m^2} [\exp(-\alpha_0(p^2 + m^2)) - \exp(-\alpha(p^2 + m^2))] . \quad (1)$$

Upon removal of the cutoffs, i.e. in the limit  $\alpha_0 \rightarrow 0$  (UV),  $\alpha \rightarrow \infty$  (IR), we indeed recover the free propagator  $\frac{1}{p^2+m^2}$ . For the Fourier transform, we use the convention

$$f(x) = \int_p \hat{f}(p) e^{ipx} := \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} e^{ipx} \hat{f}(p) \quad (2)$$

so that in position space

$$C^{\alpha_0, \alpha}(x-y; m) = \int_p e^{ip(x-y)} C^{\alpha_0, \alpha}(p; m) \quad \text{using the shorthand} \quad \int_p := \int \frac{d^4 p}{(2\pi)^4}. \quad (3)$$

2. The interaction Lagrangian is supposed to be of the form <sup>2</sup>

$$L_0(\varphi) = \int d^4 x \sum_{n \in 2\mathbb{N}} c_{0,n}(\alpha_0) \varphi^n(x), \quad (4)$$

where the constants  $c_{0,n}$  should be such that

$$-\infty < K < L_0(\varphi) < \infty \quad \forall \varphi \in \text{supp } \mu^{\alpha, \alpha_0} \quad (5)$$

for some finite real constant  $K$ . The *basic field*  $\varphi$  is assumed to be in the support of the normalised Gaussian measure  $\mu^{\alpha, \alpha_0}$  with covariance (1) <sup>3</sup>. In order to obtain a well defined limit of the quantities of interest, the constants  $c_{0,n}$  generally need to be chosen as appropriate functions of the ultraviolet cutoff  $\alpha_0$ .

The correlation (= Schwinger =  $n$ -point) functions of  $n$  basic fields with cutoff are defined by the expectation values

$$\begin{aligned} \langle \varphi(x_1) \cdots \varphi(x_n) \rangle &\equiv \mathbb{E}_{\mu^{\alpha, \alpha_0}} \left[ \exp(-L_0) \varphi(x_1) \cdots \varphi(x_n) \right] / Z^{\alpha, \alpha_0} \\ &= \int d\mu^{\alpha, \alpha_0} \exp(-L_0) \varphi(x_1) \cdots \varphi(x_n) / Z^{\alpha, \alpha_0}. \end{aligned} \quad (6)$$

This expression is simply the standard Euclidean path-integral, but with the free part in the Lagrangian absorbed into  $d\mu^{\alpha, \alpha_0}$ . The normalisation factor  $Z^{\alpha, \alpha_0}$  is chosen so that  $\langle 1 \rangle = 1$ . For finite values of the cutoffs  $0 < \alpha_0 < \alpha < \infty$  and on imposing a finite (space) volume, the functional integral (6) exists in the nonperturbative sense. In the perturbative theory it has been shown that one can remove the cutoffs,  $\alpha_0 \rightarrow 0$  and  $\alpha \rightarrow \infty$ , for a suitable choice of the running couplings  $c_{0,n}(\alpha_0)$  at each given but fixed order in the number of loops. The correct behaviour of these couplings (in terms of bounds) is determined from the FEs which are a system of differential equations in the parameter  $\alpha$  for the Schwinger functions.

These differential equations are written most conveniently in terms of the hierarchy of ‘‘connected, amputated Schwinger functions’’ (CAS functions), whose generating functional is given by the following convolution <sup>4</sup> of the Gaussian measure with the exponentiated interaction,

$$-L^{\alpha_0, \alpha} := \log \left[ \mu^{\alpha_0, \alpha} \star \exp(-L_0) \right] - \log Z^{\alpha_0, \alpha}. \quad (7)$$

<sup>2</sup>Since we will pass to the mean field limit soon, we do not introduce a momentum dependent wave function renormalisation term  $b_0 (\partial \varphi(x))^2$  here.

<sup>3</sup> See the Appendix to Part I of [7] for mathematical details about Gaussian functional integrals.

<sup>4</sup>The convolution is defined in general by  $(\mu^{\alpha_0, \alpha} \star F)(\varphi) = \int d\mu^{\alpha_0, \alpha}(\varphi') F(\varphi + \varphi')$ .

The full Schwinger functions can be recovered from the CAS functions in the end. One can expand the functionals  $L^{\alpha_0, \alpha}$  as formal power series in terms of Feynman diagrams with  $\ell$  loops,  $n$  external legs and propagator  $C^{\alpha_0, \alpha}(p)$ . One can show that, indeed, only connected diagrams with an even number of external legs contribute, and that the (free) propagators on the external legs are removed. While we will not use diagrammatic decompositions in terms of Feynman diagrams, we start from analyzing the functional (7) in momentum space, expanded in moments, i.e. powers of  $\varphi$

$$L^{\alpha_0, \alpha}(\varphi) := \sum_{n \in 2\mathbb{N}} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} \bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n). \quad (8)$$

Here no statement is made about the convergence of this series. By performing the Fourier transformation in (4) we find the relation

$$(2\pi)^4 c_{0,n}(\alpha_0) \delta^4\left(\sum_{i=1}^n p_i\right) = \bar{\mathcal{L}}_n^{\alpha_0, \alpha_0}(p_1, \dots, p_n). \quad (9)$$

Translation invariance of the CAS functions in position space implies that  $\bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  is supported at  $p_1 + \dots + p_n = 0$  (momentum conservation), and thus only depends on  $n - 1$  independent four momenta. We write

$$\bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) = \delta^4\left(\sum_{i=1}^n p_i\right) \mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) \quad (10)$$

so that

$$c_{0,n}(\alpha_0) = (2\pi)^{-4} \mathcal{L}_n^{\alpha_0, \alpha_0}(p_1, \dots, p_n). \quad (11)$$

We use the convention that the variable  $p_n$  is determined in terms of the remaining  $n - 1$  four vectors by momentum conservation, i.e.  $p_n = -p_1 - \dots - p_{n-1}$ . One should keep in mind, however, that the functions  $\mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  are in fact fully symmetric under permutation of  $p_1, \dots, p_n$ .

To obtain the flow equations for the CAS functions, we take the  $\alpha$ -derivative of (7):

$$\partial_\alpha L^{\alpha_0, \alpha} = \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi}, \dot{C}^\alpha \star \frac{\delta}{\delta \varphi} \right\rangle L^{\alpha_0, \alpha} - \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi} L^{\alpha_0, \alpha}, \dot{C}^\alpha \star \frac{\delta}{\delta \varphi} L^{\alpha_0, \alpha} \right\rangle + \partial_\alpha \log Z^{\alpha_0, \alpha}. \quad (12)$$

Here we use the following notation: We write  $\dot{C}^\alpha$  for the derivative  $\partial_\alpha C^{\alpha_0, \alpha}$ , which, as we note, does not depend on  $\alpha_0$ . Further, by  $\langle \cdot, \cdot \rangle$  we denote the standard scalar product in  $L^2(\mathbb{R}^4, d^4x)$ , and  $\star$  stands for convolution in  $\mathbb{R}^4$ . As an example,

$$\left\langle \frac{\delta}{\delta \varphi}, \dot{C}^\alpha \star \frac{\delta}{\delta \varphi} \right\rangle = \int d^4x d^4y \dot{C}^\alpha(x-y; m) \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)} \quad (13)$$

is sometimes called the ‘‘functional Laplace operator’’. We can now write the flow equation (12) in an expanded version as

$$\begin{aligned} \partial_\alpha \mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) &= \binom{n+2}{2} \int_k \dot{C}^\alpha(k; m) \mathcal{L}_{n+2}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n) \\ &- \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 \mathbb{S} \left[ \mathcal{L}_{n_1, l_1}^{\alpha_0, \alpha}(p_1, \dots, p_{n_1-1}, q) \dot{C}^\alpha(q; m) \mathcal{L}_{n_2, l_2}^{\alpha_0, \alpha}(-q, p_{n_1}, \dots, p_n) \right], \end{aligned} \quad (14)$$

with  $q = p_{n_1} + \dots + p_n = -p_1 - \dots - p_{n_1-1}$ , and where  $\mathbb{S}$  is the symmetrisation operator acting on functions of the momenta  $(p_1, \dots, p_n)$  by taking the mean value over all permutations  $\pi$  of  $1, \dots, n$  satisfying  $\pi(1) < \pi(2) < \dots < \pi(n_1 - 1)$  and  $\pi(n_1) < \dots < \pi(n)$ . We also note that for the theory proposed through (4), only even moments (i.e. even in  $n, n_1, n_2$ ) will be nonvanishing due to the symmetry  $\varphi \rightarrow -\varphi$ . Furthermore note that  $\mathcal{L}_{2,0}^{\alpha_0, \alpha}$  vanishes identically since the free propagator is absorbed in the covariance, consistently with (14). The infinite system of equations (14) then constitutes an infinite dimensional nonlinear dynamical system.

The CAS functions are defined uniquely as a solution to these differential equations only after we impose suitable boundary conditions. Noting that  $L^{\alpha_0, \alpha_0} = L_0$ , these are fixed through the choice of the constants  $c_{0,n}$  in  $L_0$ , (4). The CAS functions are then obtained by integrating the flow equations subject to the boundary conditions. For an existence and uniqueness proof in the context of perturbation theory see e.g. [10, 12, 20]. For farther reaching results like e.g. large momentum bounds [14], bounds on large orders in perturbation theory [13], applications to finite temperature field theory [15], application to nonabelian gauge theories [4], or a proof of convergence of the operator product expansion [9], we refer to the respective references. The transition to Minkowski space is analysed in [16].

### 3 The mean field limit of the flow equations

The flow equations constitute an infinite dimensional nonlinear dynamical system. The system of functions  $\mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  is defined on configuration spaces whose dimension also goes to infinity for  $n \rightarrow \infty$ . Since this system is complicated we start analysing a simplified dynamical system, where the functions  $\mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  are replaced by constants  $A_n^{\alpha_0, \alpha}$ . This amounts to setting all external momenta equal to zero in (14) and to *suppose* that the functions  $\mathcal{L}_n^{\alpha_0, \alpha}(0, \dots, 0, k, -k)$  are  $k$ -independent. It thus corresponds to a mean field limit of the flow equations. There is hope that this simplification captures essential aspects of the behaviour of the full dynamical system which is in particular based on the fact that the simplification amounts to replacing the derived propagator in the second term on the r.h.s of (14) by 1. In fact we have

$$0 < \dot{C}^\alpha(q; m) = e^{-\alpha(q^2 + m^2)} \leq e^{-\alpha m^2} \leq 1. \quad (15)$$

So the full system is obtained from the simplified one by contracting the second term on the r.h.s. in a momentum dependent manner. Controlling this contraction seems not to be out of range though there are hard technical problems, in particular due to the fact that the  $n$ -point functions we want to construct have to respect Bose symmetry and euclidean invariance.

A second rather mild simplification will consist in choosing  $m = 0$  and in restricting in exchange our analysis to the interval  $\alpha \in [\alpha_0, 1]$  to avoid infrared problems. It should then be a straightforward extension of the present analysis to take the limit  $\alpha \rightarrow \infty$  while keeping  $m > 0$ . Our simplified dynamical system is thus obtained from (14) by setting all external momenta and  $m$  equal to zero and setting

$$A_n^{\alpha_0, \alpha} := \mathcal{L}_n^{\alpha_0, \alpha}(0, \dots, 0). \quad (16)$$

The system reads for  $n \in 2\mathbb{N}$

$$\partial_\alpha A_n^{\alpha_0, \alpha} = \binom{n+2}{2} c_\alpha A_{n+2}^{\alpha_0, \alpha} - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\alpha_0, \alpha} A_{n_2}^{\alpha_0, \alpha}, \quad (17)$$



where the sum, here and subsequently, is always over even values of  $n_1, n_2$  only. Furthermore

$$c_\alpha := \frac{c}{\alpha^2}, \quad \text{with } c := \frac{1}{16\pi^2} \quad (18)$$

so that the  $c_\alpha$  is the value at  $m = 0$  of

$$c_\alpha(m) := \int_k \dot{C}^\alpha(k; m) = \frac{1}{16\pi^2} \frac{1}{\alpha^2} e^{-\alpha m^2}. \quad (19)$$

As we said (17) is obtained from (14) by suppressing the momentum dependence of the functions  $\mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  so that the  $k$ -integral can be carried out explicitly.

It is useful to factor out the basic scaling behaviour w.r.t.  $\alpha$  and combinatoric factors on setting

$$A_n^{\alpha_0, \alpha} =: \alpha^{n/2-2} \frac{1}{n} a_n(\alpha), \quad (20)$$

where we suppressed the variable  $\alpha_0$ . In terms of the functions  $a_n(\alpha)$  our dynamical system can be rewritten as

$$a_{n+2}(\alpha) = \frac{1}{(n+1)c} \sum_{n_1+n_2=n+2} a_{n_1}(\alpha) a_{n_2}(\alpha) + \frac{n-4}{n(n+1)c} a_n(\alpha) + \frac{2}{n(n+1)c} \alpha \partial_\alpha a_n(\alpha). \quad (21)$$

This system permits to construct the functions  $a_n(\alpha)$  inductively in  $n$  if the function  $a_2(\alpha)$  is known. We make another change of variables in order to also factor out the  $1/c$  factors

$$a_n(\alpha) = c^{\frac{2-n}{2}} f_n(\alpha) \quad \text{with the definition } f_n(\mu) := \alpha^{2-\frac{n}{2}} c^{\frac{n-2}{2}} n A_n^{\alpha_0, \alpha}, \quad (22)$$

where we introduced the logarithmic variable  $\mu := \ln(\frac{\alpha}{\alpha_0})$ . The system (21) can be rewritten

$$f_{n+2} = \frac{1}{n+1} \sum_{n_1+n_2=n+2} f_{n_1} f_{n_2} + \frac{n-4}{n(n+1)} f_n + \frac{2}{n(n+1)} \partial_\mu f_n, \quad \mu \in [0, \ln \frac{1}{\alpha_0}]. \quad (23)$$

Making the functions  $f_2$  and  $f_4$  more explicit, we can also write

$$f_4 = \frac{1}{3} f_2 (f_2 - 1) + \frac{1}{3} \partial_\mu f_2, \quad (24)$$

$$f_{n+2} = \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} f_{n_1} f_{n_2} + \frac{1}{n+1} f_n [2f_2 + 1 - \frac{4}{n}] + \frac{2}{n(n+1)} \partial_\mu f_n, \quad n \geq 4. \quad (25)$$

Smooth solutions of the dynamical system (24), (25) are fixed if we fix the smooth function  $f_2(\mu)$ . In perturbative quantum field theory one primarily considers the flow of the four-point function which is represented by  $f_4(\mu)$ . From (24) we realise that we may first fix  $f_4(\mu)$  and then solve the differential equation (24) for  $f_2(\mu)$  to obtain a solution for  $f_2(\mu)$ .

At this stage we add a few general remarks in relation with the structure of the system (24), (25).

- The first remark concerns what one might call the *combinatorial instability* of the system. When trying to figure out an asymptotic behaviour of the  $f_n$  as functions of  $n$ , it turns out that, due to the prefactors the terms

$$f_{n+2}, \quad \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} f_{n_1} f_{n_2}$$

are dominant for realistic inductive hypotheses concerning the dependence on  $n$ , unless one would take into account cancellations of terms of opposite sign, which typically is beyond scope. As a consequence solving the system by iteration, starting from a first educated guess and integrating successively, typically does not define a convergent procedure.

- As a consequence of the previous statement we rather proceed in a different way: We start by fixing  $f_2(\mu)$  and construct the higher  $n$ -point functions from the two-point function. This will permit to find smooth solutions of the system (25), which are interesting also from the physical point of view. When asking the question whether this procedure is useful for the full system (14), the problem one is faced with is how to define a function  $\mathcal{L}_{n+2}^{\alpha_0, \alpha}(p_1, \dots, p_{n+2})$  out of the integral

$$\int_k \dot{C}^\alpha(k; m) \mathcal{L}_{n+2}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n),$$

once the functions  $\mathcal{L}_{n'}^{\alpha_0, \alpha}(p_1, \dots, p_{n'})$ ,  $n' \leq n$ , are known. This function has to be Bose symmetric, symmetric under the euclidean group, in particular translation invariant. It also should have good analyticity properties as required by a full-fledged quantum field theory which can be analytically continued to Minkowski space. A central and presumably hard challenge is to identify the conditions which determine these functions uniquely in agreement with the axioms of quantum field theory.

- We also mention in this context the so-called hierarchy problem of scalar field theory. It consists in the observation that in perturbative scalar field theory the two-point function diverges quadratically with the UV cutoff  $\Lambda_0 = \alpha_0^{-1/2}$ , as suggested by (22). In fact it is the only term diverging stronger than logarithmically in perturbation theory, even when inspecting the whole of the standard model of particle physics. It is then argued that this divergence leads to a fine-tuning problem when viewing  $\Lambda_0$  as a very high energy scale (“the Planck mass”) since fixing the mass of the Higgs particle associated to the scalar field at its much lower physical value requires fine-tuning of the corresponding counter term. Consequently this quadratic divergence is often cited as a motivation for supersymmetric (or other) extensions of the standard model where the perturbative divergences are only logarithmic. Once we look at the rescaled system (24), (25) - the same rescaling can be performed for the full system (14) - this quadratic divergence disappears. The rescaling is natural since it leads to a scale free system. So from this point of view the hierarchy problem appears to be a pseudo-problem of the perturbative treatment, whereas on the other hand supersymmetric cancellations appear to be due to a subtle fine-tuning procedure.

## 4 Solutions of the mean field equations

We will consider solutions of (24), (25) which are smooth functions of the renormalisation group scale  $\mu$  in the interval  $[0, \ln \frac{1}{\alpha_0}]$ . The existence of the ultraviolet limit means that the system of solutions has a finite limit for  $\alpha = 1$  (“when all degrees of freedom have been integrated out”) when the UV cutoff  $1/\alpha_0$  is sent to infinity. In other words claiming the existence of a mean field solution of the FEs in the UV limit is tantamount to prove that

$$\text{the limits } \lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) \text{ exist for all } n, \quad \text{where } \mu_{max} = \ln \frac{1}{\alpha_0}. \quad (26)$$

The solutions studied in 4.1 are the simplest to obtain. For these solutions we however do not control the signs of the  $n$ -point functions, not even at  $\mu = 0$ , i.e. for the bare action. We find a bare action which is nonpolynomial, and its moments are not necessarily positive. From the functional integral point of view the existence of the bare action for an arbitrary field configuration in the support of the Gaussian measure is therefore not assured. And for the (mean) field configurations for which the bare action exists, we do not know whether it is uniformly bounded from below. We will show that there exist globally bounded solutions which are monotonically increasing as functions of  $\mu$  and vanish at  $\mu = 0$  when taking the UV limit  $\alpha_0 \rightarrow 0$ . They are thus asymptotically free in the ultraviolet. The existence of such solutions is unexpected from the conventional wisdom point of view.

In section 4.2 we will then study solutions with strictly positive boundary conditions at  $\mu = 0$  for all  $n$ -point functions. So the bare action is nonpolynomial, and all of its moments have positive coefficients. The bare action is bounded from below (by 0) whenever it is well-defined. The solutions we obtain are again ultraviolet asymptotically free. Still the bare action (restricted to finite volume) is not well-defined for all admissible field configurations since it may diverge due to its nonpolynomial character.

We therefore study in section 4.3 solutions the boundary conditions of which, while being again nonpolynomial, can be resummed into bounded functions of the field variable and thus lead to well-defined bare actions in the (finite volume) path integral. These actions are also bounded from below so that the (regularised) path integral can be shown to exist. The solutions from 4.3 constitute subclasses of those considered in 4.1. We show in particular that there exist UV asymptotically free solutions with well-defined path integral. The proof requires much sharper restrictions on the couplings than those needed in 4.1.

Finally we study the boundary conditions of pure  $\phi^4$  theory in section 4.4. The solutions corresponding to these boundary conditions have alternating signs (at least for small  $\mu$ ) and large  $\mu$ -derivatives which is related to the aforementioned fine-tuning of the boundary conditions.

We shall find that with the exception of 4.3, the upper bounds on the coupling constants required in the existence proofs of the solutions are quite moderate when compared to constructive field theory upper bounds which typically are “astronomically small” (like exponentials of a very big negative number) due to the high complexity of the contributions from iterated cluster expansions. The upper bounds from 4.3 are astronomically small and not really made explicit. This is because the proof of Theorem 1 is delicate. So we did not try to optimise the bounds w.r.t. the size of the couplings, also for the sake of readability. But more reasonable upper bounds should be attainable with reasonable effort.

## 4.1 Bounded mean field solutions

The simplest solutions of (24), (25) are those for which

$$\partial_\mu f_2 \equiv 0. \quad (27)$$

It then follows directly from (24), (25) that

$$\partial_\mu f_n \equiv 0 \quad \forall n \quad (28)$$

so that we obtain the  $\mu$  independent system

$$f_4 = \frac{1}{3} f_2 (f_2 - 1), \quad f_{n+2} = \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} f_{n_1} f_{n_2} + \frac{1}{n+1} f_n \left[ 2f_2 + 1 - \frac{4}{n} \right], \quad n \geq 4. \quad (29)$$

The solutions of (29) are fully determined on imposing the value of  $f_2$ .

The solutions of (29) are **scale invariant**, they do not show any corrections to the canonical scaling factored out in (20). It is not possible to maintain this condition beyond the mean field limit since  $\mu$ -independence cannot be preserved once we introduce the  $\alpha$ - (and thus  $\mu$ -) dependent propagator  $\hat{C}^\alpha$ . So the solutions of (29) are of limited interest. Controlling them is essentially trivial. We consider different cases as regards the value of  $f_2$ .

a)  $0 < |f_2| \leq \varepsilon \ll 1$

In this regime we find that  $f_4 = \mathcal{O}(\varepsilon)$ , with sign opposite to that of  $f_2$ ,  $f_6 = \mathcal{O}(\varepsilon^2)$  with negative sign, and  $f_n > 0$ ,  $f_n = \mathcal{O}(\varepsilon^2)$  for  $n \geq 8$ . So we have an action bounded from below. This regime is not perturbative, in the sense that  $f_n$  is not of increasing order in  $\varepsilon$  for increasing  $n$ . The  $|f_n|$  for  $n \geq 6$  are bounded by  $\varepsilon^2$  times a numerical coefficient becoming small ( $\ll 1$ ) rapidly for increasing  $n$ .

b)  $0 < f_2 < 1$

In this case  $f_4 < 0$ . Generally the  $f_n$  may have either sign, depending on  $n$ .

c)  $f_2 = 1$

We find  $f_n = 0 \quad \forall n \geq 4$ , i.e. a ‘‘free theory’’.

d)  $f_2 > 1$

By induction on  $n$  one finds that the coefficients  $f_n$  of the system (29) satisfy

i)  $f_n > 0$ , so the action is bounded from below.

ii)  $f_n$  are strictly increasing when viewed as functions of  $f_2$  and geometrically bounded by a constant to the power  $n$ . In particular for  $f_2 = 1 + \varepsilon$  with  $0 < \varepsilon \ll 1$  we find  $f_4 = \frac{1}{3}(1 + \varepsilon)\varepsilon$ ,

$f_6 = \frac{1}{15}(1 + \varepsilon)^2\varepsilon$ ,  $f_n = \mathcal{O}(\varepsilon) \quad \forall n \geq 4$ . The  $|f_n|$  for  $n \geq 6$  are bounded by  $\varepsilon^2$  times a numerical coefficient becoming small ( $\ll 1$ ) rapidly for increasing  $n$ .

e)  $f_2 < 0$

In this case we do not control the signs of the  $f_n$ . The  $|f_n|$  may become large in modulus for large  $f_2$ .

We now study more general solutions for which all  $|f_n|$  are bounded by 1. We consider a smooth two-point function satisfying for  $0 < \delta < 1$

$$-K_1 \delta \leq f_2(0) \leq -\delta < 0, \quad |\partial_\mu^l f_2(\mu)| \leq \frac{K_1^l \delta^{l+1}}{(l+1)^2} l! \quad \forall \mu \in [0, \ln \frac{1}{\alpha_0}] \quad \text{and} \quad \forall l \geq 0, \quad (30)$$

where  $K_1 > 1$  is a positive constant. We restrict ourselves for simplicity and definiteness to the interval  $0 < \delta < 1$ , but larger values could be analysed similarly. We note that the sign of

$f_2(0)$  in (30) is in agreement with the sign of the mass counter term in perturbative  $\phi^4$ -theory at lowest order.

**Proposition 1:** For suitable  $K \geq \sup(K_1, 4)$  and  $f_2(\mu)$  satisfying (30), the functions  $f_n(\mu)$  solving (25) are smooth and satisfy for  $\mu \in [0, \ln \frac{1}{\alpha_0}]$ ,  $n \in 2\mathbb{N} + 2$ ,  $l \geq 0$

$$|\partial_\mu^l f_n(\mu)| \leq \frac{K^{n+l-2} \delta^{l+1}}{(l+1)^2} \frac{(n+l-2)!}{(n-2)!}. \quad (31)$$

*Proof:* The proof is by induction in  $n+l \geq 2$ .

The bounds hold for the two-point function by assumption (30). Verification of the bounds on  $\partial_\mu^l f_4(\mu)$  using (24) is straightforward and simpler than the general case  $n \geq 4$ . So we leave this case to the reader. For  $n \geq 4$  we insert the induction hypothesis on the r.h.s. of (25), derived  $l$  times w.r.t.  $\mu$ . This gives the bound

$$\begin{aligned} & \frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2, n_i \geq 4 \\ l_1+l_2=l}} \binom{l}{l_1} \frac{K^{n+l-2}}{(l_1+1)^2 (l_2+1)^2} \frac{(n_1+l_1-2)!}{(n_1-2)!} \frac{(n_2+l_2-2)!}{(n_2-2)!} \\ & + \frac{\delta^{l+1}}{n+1} \sum_{l_1+l_2=l} \binom{l}{l_1} \frac{K^{n+l_1-2}}{(l_1+1)^2} \frac{(n+l_1-2)!}{(n-2)!} \left[ \delta \frac{K^{2+l_2-2}}{(l_2+1)^2} 2(2+l_2-2)! + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \\ & + \frac{2\delta^{l+2}}{n(n+1)} \frac{K^{n+l+1-2}}{(l+2)^2} \frac{(n+l+1-2)!}{(n-2)!}. \end{aligned} \quad (32)$$

Using the standard bound (all entries are supposed to be nonnegative integers)

$$\binom{l}{l_1} \binom{n-2}{n_1-2} \leq \binom{n-2+l}{n_1-2+l_1} \quad (33)$$

we obtain the following estimate for (32)

$$\begin{aligned} & \frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2, n_i \geq 4 \\ l_1+l_2=l}} \frac{K^{n+l-2}}{(l_1+1)^2 (l_2+1)^2} \frac{(n+l-2)!}{(n-2)!} \\ & + \frac{\delta^{l+1}}{n+1} \sum_{l_1+l_2=l} \frac{K^{n+l-2}}{(l_1+1)^2} \frac{(n+l-2)!}{(n-2)!} \left[ \frac{2\delta}{(l_2+1)^2} + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \\ & + \frac{2\delta^{l+2}}{n(n+1)} \frac{K^{n+l+1-2}}{(l+2)^2} \frac{(n+l+1-2)!}{(n-2)!}. \end{aligned} \quad (34)$$

Choosing  $K$  sufficiently large such that for  $n \geq 6$

$$\frac{\delta}{n+1} \sum_{\substack{n_1+n_2=n+2, n_i \geq 4 \\ l_1+l_2=l}} \frac{1}{(l_1+1)^2 (l_2+1)^2} \leq \frac{1}{3} \frac{K^2}{(l+1)^2} \frac{(n+l)(n+l-1)}{n(n-1)}$$

and such that

$$\frac{1}{n+1} \sum_{l_1+l_2=l} \frac{1}{(l_1+1)^2 (l_2+1)^2} \left[ 2\delta + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \leq \frac{K^2}{2(l+1)^2} \frac{(n+l)(n+l-1)}{n(n-1)}$$

and such that

$$\frac{2\delta}{n(n+1)} \frac{1}{(l+2)^2} \leq \frac{1}{6} K \frac{1}{(l+1)^2} \frac{n+l}{n(n-1)}$$

we find that (32) is bounded by

$$\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{6}\right) \frac{\delta^{l+1} K^{n+l}}{(l+1)^2} \frac{(n+l)!}{n!} .$$

One can straightforwardly convince oneself  $K = 4$  is admissible for  $\delta = 1$ ,  $K_1 \leq 4$  and that smaller values of  $K$  are allowed if  $\delta < 1$ ,  $K_1 < 4$ .  $\square$

Going back to the dynamical system (17) we obtain from the set of smooth functions  $f_n(\mu)$  the system of smooth functions  $A_n^{\alpha_0, \alpha}$ . If the functions  $f_n(\mu)$  satisfy the bounds from Proposition 1, then the  $A_n^{\alpha_0, \alpha}$  satisfy the bounds

$$|A_n^{\alpha_0, \alpha}| \leq \delta \left(\frac{\alpha K^2}{c}\right)^{\frac{n-2}{2}} \frac{1}{\alpha n} \quad \text{for } 0 < \alpha_0 \leq \alpha \leq 1 . \quad (35)$$

This bound is uniform in  $\alpha_0$ .

In order to show that there is a subclass of solutions among those from Proposition 1 which describe a mean field theory in the sense that the solutions  $f_n(\mu_{max})$  have a well-defined limit for  $\alpha_0 \rightarrow 0$ ,  $\mu_{max} \rightarrow \infty$ , we choose

$$f_2(\mu) = -\delta(\mu), \quad \delta(\mu_{max}) = \delta, \quad \partial_\mu \delta(\mu) = \beta \delta^2(\mu), \quad 0 < \delta, \beta < \frac{1}{2}, \quad \mu \in [0, \ln \frac{1}{\alpha_0}] . \quad (36)$$

The well-known solution is

$$\delta(\mu) = \frac{\delta}{1 + (\mu_{max} - \mu)\beta \delta} . \quad (37)$$

Evidently (36) verifies the assumptions of Proposition 1. We have in particular

$$\lim_{\mu_{max} \rightarrow \infty} f_2(\mu_{max}) = -\delta, \quad \lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_2(\mu_{max}) = -\beta^l l! \delta^{l+1} . \quad (38)$$

By straightforward induction in  $n + l$ , proceeding as in the proof Proposition 1 we then find that the limits  $\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max})$ ,  $n \geq 4$ , also exist and obey the bounds of Proposition 1. We collect our findings in

**Proposition 2:** *Among the solutions from Proposition 1 there are nontrivial asymptotically free solutions, for which hold*

$$f_2(\mu_{max}) < 0, \quad f_4(\mu_{max}) > 0, \quad (39)$$

$$\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) \text{ exists } \quad \forall n \in 2\mathbb{N}, \quad (40)$$

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_n(0) = 0 \quad \forall n \in 2\mathbb{N}, \quad l \in \mathbb{N}_0 . \quad (41)$$

*Proof.* The second inequality in (39) is true if  $\frac{1}{3} \delta(\mu_{max})(\delta(\mu_{max}) + 1) - \partial_\mu \delta(\mu_{max}) > 0$ , which is the case for  $\delta, \beta$  bounded as in (36). The last statement (41) again follows by induction proceeding as in the proof of Proposition 1.  $\square$

We also note that solutions of the type (36), but with  $\beta < 0$  negative, will lead to trivial theories, namely we find that

$$\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) = 0 \quad (42)$$

if  $\delta(0)$  is fixed to be positive and not too large. In fact one obtains in this case

$$\delta(\mu) = \frac{\delta(0)}{1 - \mu \beta \delta(0)} \quad (43)$$

which vanishes for  $\mu = \mu_{max} \rightarrow \infty$  and then implies the vanishing of all  $f_n$  in this limit, which can be proven again inductively as Proposition 1. We do not work out this point further here. We will come back to the triviality question in 4.4.

## 4.2 Mean field solutions with positive bare values

The bare actions constructed from the solutions  $f_n(0)$  in 4.1 are generally not bounded from below. In this subsection we look at solutions for which all  $f_n$  are positive and monotonic

$$f_n(\mu) \geq 0, \quad \partial_\mu f_n(\mu) \geq 0. \quad (44)$$

These properties assure positivity of bare action whenever it is well-defined.

We obtain smooth solutions of (24), (25) satisfying the conditions (44) on considering functions  $f_2(\mu)$  such that

$$f_2(\mu) = 1 + \delta(\mu), \quad 0 < \delta(\mu) < 1, \quad \partial_\mu \delta = \beta \delta^2(\mu), \quad 0 < \beta < 1, \quad \mu \in [0, \ln \frac{1}{\alpha_0}]. \quad (45)$$

The main difference between (45) and (36) is that  $f_2$  in (45) is not of order  $\delta$ . So the solutions studied are nonperturbative from the beginning.

Rewriting (24) as

$$3 f_4(\mu) = (1 + \delta(\mu)) \delta(\mu) + \partial_\mu \delta(\mu), \quad (46)$$

we see that the relations (45), (46) imply

$$\partial_\mu^l f_4(\mu) \geq 0 \quad \forall l. \quad (47)$$

**Proposition 3:** For suitable  $K > 1$  and  $f_2(\mu) = 1 + \delta(\mu)$  smooth, satisfying (45), the functions  $f_n(\mu)$  are smooth, and satisfy for  $\mu \in [0, \ln \frac{1}{\alpha_0}]$ ,  $n \in 2\mathbb{N} + 2$ ,  $l \geq 0$

$$0 < \partial_\mu^l f_n(\mu) \leq \delta(\mu) \frac{K^{n+l-2}}{(l+1)^2} \frac{(n+l-2)!}{(n-2)!}. \quad (48)$$

*Proof:*

The proof is by induction, in  $n+l \geq 2$ . Positivity follows immediately by inspecting the r.h.s of (25). The bound does not contain higher powers of  $\delta(\mu)$  as in Proposition 1 since  $f_2$  is

no more of order  $\delta$ . Otherwise the proof follows strictly that of Proposition 1. So we do not rewrite it. We find again that  $K = 4$  is an admissible value.  $\square$

From (17) and Proposition 3 we find bounds for the  $A_n^{\alpha_0, \alpha}$

$$0 < A_n^{\alpha_0, \alpha} < (\delta_{n,2} + \delta(\mu)) \left( \frac{\alpha K^2}{c} \right)^{\frac{n-2}{2}} \frac{1}{\alpha n} \quad \text{for } 0 < \alpha_0 \leq \alpha \leq 1 \quad (49)$$

which are uniform in the UV cutoff  $\alpha_0$ . The initial data  $f_n(0) \geq 0$  assure the positivity of all moments of the bare action, which obey the bounds (49).

The solutions studied in Proposition 3 are again asymptotically free. When choosing  $\delta = \delta(\mu_{max}) > 0$  fixed, we have statements analogous to (37, 38, 39, 40, 41):

$$\delta(\mu) = \frac{\delta}{1 + (\mu_{max} - \mu)\beta\delta} . \quad (50)$$

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_2(\mu_{max}) = \delta_{l,0} + \beta^l l! \delta^{l+1}, \quad \lim_{\mu_{max} \rightarrow \infty} f_4(\mu_{max}) = \frac{1}{3}(1 + \delta)\delta + \beta\delta^2, \quad (51)$$

$$\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) \quad \text{exists and is positive} \quad \forall n \in 2\mathbb{N}, \quad (52)$$

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_2(0) = \delta_{l,0}, \quad \lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_n(0) = 0 \quad \forall n \in 2\mathbb{N} + 2, l \in \mathbb{N}_0. \quad (53)$$

The asymptotically free solutions we have considered seem to be quite special. Still, as regards the UV limit, the basic possibilities are nontrivial asymptotically free or safe (i.e. scale independent) solutions, or trivial solutions which are free at  $\mu = \mu_{max}$ . An interesting task left for the future is to analyse the different classes of solutions more systematically.

In perturbative quantum field theory one generally analyses in a first place the scaling behaviour of the four-point function, and not that of the two-point function. When imposing  $f_4(\mu)$  and then solving the differential equation for the two-point function (46), which is of Riccati type, we find as a particular solution

$$f_2(\mu) = -3f_4(\mu). \quad (54)$$

This implies that we can find all solutions of the Riccati equation. For example, the one satisfying  $f_2(0) = 0$  is given by

$$f_2(\mu) = \frac{3f_4(0) e^{\int_0^\mu (6f_4(\mu') + 1) d\mu'}}{1 + 3f_4(0) \int_0^\mu d\mu' e^{\int_0^{\mu'} (6f_4(\mu'') + 1) d\mu''}} - 3f_4(\mu). \quad (55)$$

It satisfies

$$f_2(\mu) \geq 0, \quad \partial_\mu f_2(\mu) \geq 0, \quad f_2(0) = 0.$$

Higher order derivatives of this solution are however not positive for all values of  $\mu$ . It is therefore not clear whether the corresponding solutions of (25) satisfy Proposition 3 for  $l = 0$ . Since the solution (55) has a vanishing mass counter term, it may well be that the ultraviolet limit for this solution does not exist.

An interesting project would be to find out whether one can construct along these lines asymptotically free solutions, in particular for the four-point function, satisfying Proposition 3 which are such that the bare action is well defined and bounded from below.



### 4.3 Solutions of bounded action

The bare actions of the solutions constructed so far are nonpolynomial and generally not well-defined on the whole of the support of the Gaussian measure. The solutions we will construct in this section satisfy sufficiently strong bounds in order to assure well-defined bare actions bounded from below. Since the estimates become more delicate the upper bounds on the coupling constants required are much more restrictive. The solutions from this section will in fact be subclasses of those from section 4.1.

The bare functional (4) has the form  $L_0(\varphi) = \sum_{n \in 2\mathbb{N}} \int d^4x c_{0,n} \varphi^n(x)$ . The constants  $c_{0,n}$  are related to the  $A_n(\alpha_0)$  via

$$c_{0,n} = \frac{1}{(2\pi)^4} A_n(\alpha_0)$$

as can be seen from (9), (10), (16). For the  $A_n(\alpha_0)$  we have deduced the bounds (35) resp. (49). The functional  $L_0(\varphi)$  is well-defined for all  $\varphi$  in

$$\mathcal{D}(\alpha_0) = \left\{ \varphi \mid \varphi \in \bigcap_{n \in 2\mathbb{N}} L^n(\mathbb{R}^4, d^4x), L_0(\varphi) < \infty \right\}.$$

If the bounds (35) resp. (49) hold, the set  $\mathcal{D}(\alpha_0)$  contains

$$\mathcal{M}_\varepsilon(\alpha_0) := \left\{ \varphi \mid \varphi \in \bigcap_{n \in 2\mathbb{N}} L^n(\mathbb{R}^4, d^4x), \limsup \|\varphi\|_n < \left(\frac{c}{\alpha_0 K^2}\right)^{1/2} - \varepsilon \right\} \subset \mathcal{D}(\alpha_0)$$

for arbitrarily small positive  $\varepsilon$ . The sets  $\mathcal{M}_\varepsilon(\alpha_0)$  do not exhaust the support of the measure  $\mu(C^{\alpha_0, \alpha})$  for finite  $\alpha_0$ . One might then be tempted to introduce one more regularisation by setting

$$\begin{aligned} V(\varphi) &\equiv e^{-L_0(\varphi)}, & \text{if } \varphi \in \text{supp } \mu(C^{\alpha_0, \alpha}) \cap \mathcal{D}(\alpha_0), \\ V(\varphi) &\equiv 0, & \text{if } \varphi \in \text{supp } \mu(C^{\alpha_0, \alpha}) - \mathcal{D}(\alpha_0). \end{aligned}$$

But  $V(\varphi)$  is not differentiable w.r.t.  $\varphi$ , and it is thus no more possible to derive the FEs from the path integral by partial integration. In fact boundary terms appear where the potential  $V(\varphi)$  is cut off.

To impose boundedness from the beginning we instead study bare actions, still supposed to be local, which are of the form

$$\boxed{L_0(\varphi) = \sum_{n \in 2\mathbb{N}} \int d^4x \tilde{\mathcal{L}}_n \sin(\alpha_0^{n/2} \varphi^n) \alpha_0^{-n/2}}. \quad (56)$$

Since we are interested in the bare action we only consider the FEs for the functions  $\tilde{\mathcal{L}}_n$  and their  $\alpha$ -derivatives evaluated at  $\alpha = \alpha_0$ . When going to zero external momentum they take the

form

$$\begin{aligned}
& \sum_{\substack{v \geq 0 \\ (1+2v)n'=n}} \alpha_0^{n'v} \frac{(-1)^v}{(2v+1)!} \partial_\alpha \tilde{\mathcal{L}}_{n'} + \sum_{\substack{v \geq 1 \\ (1+2v)n'=n}} \alpha_0^{vn'-1} v n' \frac{(-1)^v}{(2v+1)!} \tilde{\mathcal{L}}_{n'} \\
&= \frac{c}{2} (n+2)(n+1) \alpha_0^{-2} \tilde{\mathcal{L}}_{n+2} + \frac{c}{2} \alpha_0^{-2} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n+2}} \alpha_0^{vn'} n' (n'-1) \frac{(-1)^v}{(2v)!} \tilde{\mathcal{L}}_{n'} \\
&+ \frac{c}{2} \alpha_0^{-2} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n+2}} \alpha_0^{vn'} n'^2 \frac{(-1)^v}{(2v-1)!} \tilde{\mathcal{L}}_{n'} \\
&- \frac{1}{2} \sum_{\substack{n_1+n_2=n+2, v_1, v_2 \geq 0 \\ (1+2v_1)n'=n_1 \\ (1+2v_2)n''=n_2}} \alpha_0^{n'v_1+n''v_2} n_1 n_2 \frac{(-1)^{v_1+v_2}}{(2v_1)!(2v_2)!} \tilde{\mathcal{L}}_{n'} \tilde{\mathcal{L}}_{n''} .
\end{aligned} \tag{57}$$

The additional terms appearing as compared to (14) and (17), carry  $v \geq 1$  in the various sums. They stem from higher order terms on expanding the sine function in (56). So (57) follows directly from (14), (17), when expressing the  $\mathcal{L}_n$  in terms of the  $\tilde{\mathcal{L}}_{n'}$ . We note that

$$\tilde{\mathcal{L}}_2 = \mathcal{L}_2^{\alpha_0, \alpha_0} . \tag{58}$$

As before we introduce dimensionless functions  $a(n)$  via the definition

$$\tilde{\mathcal{L}}_n =: \frac{1}{n} \alpha_0^{n/2-2} a(n) . \tag{59}$$

This gives the following FEs for the  $a(n)$ , evaluated at  $\alpha = \alpha_0$  or equivalently at  $\mu = 0$

$$\begin{aligned}
a(n+2) &= \frac{1}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n+2}} (n'-1) \frac{(-1)^{v-1}}{(2v)!} a(n') + \frac{1}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n+2}} n' \frac{(-1)^{v-1}}{(2v-1)!} a(n') \\
&+ \frac{1}{(n+1)c} \sum_{\substack{n_1+n_2=n+2, v_1, v_2 \geq 0 \\ (1+2v_1)n'=n_1 \\ (1+2v_2)n''=n_2}} (-1)^{v_1+v_2} \frac{(1+2v_1)(1+2v_2)}{(2v_1)!(2v_2)!} a(n') a(n'') \\
&+ \frac{2}{(n+1)c} \frac{n-4}{2n} a(n) + \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n}} \frac{n'-4}{2n'} \frac{(-1)^v}{(2v+1)!} a(n') \\
&+ \frac{2}{(n+1)c} \frac{1}{n} \partial_\mu a(n) + \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n}} \frac{1}{n'} \frac{(-1)^v}{(2v+1)!} \partial_\mu a(n') \\
&+ \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n'=n}} (-1)^v \frac{v}{(2v+1)!} a(n') .
\end{aligned} \tag{60}$$

For  $n = 2$  we obtain simply

$$a(4) = \frac{1}{3c} [a(2)(a(2) - 1) + \partial_\mu a(2)] \tag{61}$$

in agreement with (21) for  $n = 2$ .

We write for shortness

$$a(n, l) := \partial_\mu^l a(n). \quad (62)$$

Our bound will be expressed in terms of the decomposition of  $n$  in prime numbers. We write this decomposition for general  $n$  in hopefully obvious notation as

$$n = 2^{p_2(n)} \cdot 3^{p_3(n)} \cdot 5^{p_5(n)} \cdot 7^{p_7(n)} \cdot 11^{p_{11}(n)} \dots \quad (63)$$

We also define

$$\begin{aligned} B(n, 0) &:= B(n) := [2^{p_2(n)/4} \cdot 3^{p_3(n)/4} \cdot 5^{p_5(n)/2} \cdot 7^{9p_7(n)/8} \cdot 11^{9p_{11}(n)/8} \dots]^{-1}, \\ B(n, l) &:= B(n) \frac{(n+l)!}{n!}, \quad B_\varepsilon(n, l) := B(n, l) \varepsilon^{l+1}. \end{aligned} \quad (64)$$

**Lemma 1:** For fixed  $\varepsilon > 0$  sufficiently small, there exists  $\varepsilon'$  satisfying  $0 < \varepsilon' \leq \varepsilon$  such that if

$$|a(2, l)| \leq B_{\varepsilon'}(2, l) \quad \forall l \geq 0, \quad (65)$$

then

$$|a(n, l)| \leq B_\varepsilon(n, l) \quad \forall l \geq 0, \quad n \geq 2. \quad (66)$$

*Proof.* We proceed by induction in  $N = n + l \geq 2$ , going up in  $l$  for fixed  $N$ . We are not ambitious on the size of  $\varepsilon, \varepsilon'$ . On inspecting (61), (60) it is obvious that the bound (66) holds for all  $(n, l)$  with  $n + l \leq N_0$ , for  $N_0$  fixed and  $\varepsilon'$  chosen sufficiently small depending on  $N_0$  and  $\varepsilon$ . We will not derive an explicit upper bound on  $\varepsilon'(\varepsilon, N_0) > 0$ , satisfying ourselves with the existence statement, but comment on the size of  $N_0$  in the proof. To improve the upper bounds on  $\varepsilon, \varepsilon'$  one has to consider small values of  $N$  explicitly, and to bound separately particular cases, where  $B(n+2)$  is much bigger than  $B(n)$ . This is the case for example for  $n+2 = 2^k$  with  $k$  large, where  $B(n+2) = (n+2)^{-1/4}$  whereas  $B(n)$  may be smaller than  $1/n$ .

We consider (60) and bound inductively the l.h.s. of this equation in terms of the r.h.s. We will treat  $\mu$ -derivatives of (60) afterwards. We bound successively the terms on the r.h.s. of (60).

- 1st term

$$\begin{aligned} & \left| \frac{1}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n+2}} (n'-1) \frac{(-1)^{v-1}}{(2v)!} a(n') \right| \leq \frac{n+2}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n+2}} \frac{1}{1+2v} \frac{|a(n')|}{(2v)!} \\ & \leq \frac{n+2}{n+1} \left( \frac{3^{1/4}}{3!} + \frac{5^{1/2}}{5!} + \frac{7^{9/8}}{7!} + \frac{9^{1/4}}{9!} + \dots \right) B_\varepsilon(n+2) \\ & \leq \frac{n+2}{n+1} (0.22 + 0.02 + 0.002 + \dots) B_\varepsilon(n+2) \leq 0.25 \frac{n+2}{n+1} B_\varepsilon(n+2). \end{aligned} \quad (67)$$

In the second line we use the identity

$$B_\varepsilon(n') = (1+2v)^{p_v} B_\varepsilon(n+2) \quad \text{for } n+2 = (1+2v)n'.$$

Here the exponent  $p_v$  is to be read from the definition (64). It varies between  $1/4$  and  $9/8$  depending on the value of  $1+2v$ .

- 2nd term

$$\begin{aligned}
& \left| \frac{1}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n+2}} n' \frac{(-1)^{v-1}}{(2v-1)!} a(n') \right| \\
& \leq \frac{n+2}{n+1} \left( 3^{-3/4} + \frac{5^{-1/2}}{3!} + \frac{7^{1/8}}{5!} + \frac{9^{-3/4}}{7!} + \dots \right) B_\varepsilon(n+2) \\
& \leq (0.44 + 0.075 + 0.011 + \dots) \frac{n+2}{n+1} B_\varepsilon(n+2) \leq 0.53 \frac{n+2}{n+1} B_\varepsilon(n+2).
\end{aligned} \tag{68}$$

- 3rd term

$$\begin{aligned}
& \left| \frac{1}{(n+1)c} \sum_{\substack{n_1+n_2=n+2 \\ v_1, v_2 \geq 0 \\ (1+2v_1)n' = n_1 \\ (1+2v_2)n'' = n_2}} (-1)^{v_1+v_2} \frac{(1+2v_1)(1+2v_2)}{(2v_1)!(2v_2)!} a(n') a(n'') \right| \\
& \leq \frac{\varepsilon^2}{(n+1)c} \sum_{v_1, v_2 \geq 0} \frac{(1+2v_1)(1+2v_2)}{(2v_1)!(2v_2)!} 2 \sum_{n_1 \leq n_2} \sum_{n' \leq \frac{n+2}{2(1+v_1)}} B(n') \left[ \frac{1}{2} \frac{n+2}{1+2v_2} \right]^{-1/4}.
\end{aligned} \tag{69}$$

We have

$$\sum_{n' \leq \frac{n+2}{2(1+v_1)}} B(n') \leq \sum_{\mu, \nu, \rho \geq 0} 2^{-\frac{\mu}{4}} 3^{-\frac{\nu}{4}} 5^{-\frac{\rho}{2}} \sum_{n' \geq 7} \left( \frac{1}{n'} \right)^{\frac{9}{8}} \leq \underbrace{\frac{8}{(1-2^{-1/4})(1-3^{-1/4})(1-5^{-1/2})}}_{=: K} \tag{70}$$

so that if we choose  $\varepsilon$  sufficiently small to assure

$$\frac{1}{(n+1)c} 2^{5/4} K \sum_{v_1, v_2} \frac{(1+2v_1)(1+2v_2)^{5/4}}{(2v_1)!(2v_2)!} \varepsilon \leq \frac{1}{30} \frac{1}{(n+2)^{7/8}}, \tag{71}$$

(69) is bounded by

$$\frac{1}{30} \frac{1}{(n+2)^{9/8}} \varepsilon \leq \frac{1}{30} B_\varepsilon(n+2). \tag{72}$$

One may note that the sum appearing in (71) is bounded by 10.

- 4th term

$$\begin{aligned}
& \left| \frac{2}{(n+1)c} \frac{n-4}{2n} a(n) + \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n}} \frac{n'-4}{2n'} \frac{(-1)^v}{(2v+1)!} a(n') \right| \\
& \leq \frac{\varepsilon}{(n+2)c} n^{-1/4} \underbrace{\left( 1 + \frac{3^{1/4}}{3!} + \frac{5^{1/2}}{5!} + \frac{7^{9/8}}{7!} + \dots \right)}_{\leq 5/4} \leq \frac{1}{30} B_\varepsilon(n+2)
\end{aligned} \tag{73}$$

for  $n$  sufficiently large such that

$$\left( \frac{n+2}{n} \right)^{1/8} n^{-1/8} \frac{25}{2c} \leq 1. \tag{74}$$

- 5th term

$$\begin{aligned}
\frac{2}{(n+1)c} \left| \sum_{\substack{v \geq 0 \\ (1+2v)n' = n}} \frac{1}{n'} \frac{(-1)^v}{(2v+1)!} a(n', 1) \right| &\leq \frac{2}{(n+1)c} \left( \sum_{\substack{v \geq 0 \\ (1+2v)n' = n}} \frac{n'+1}{n'} \frac{(2v+1)^{1/4}}{(2v+1)!} \right) \frac{\varepsilon^2}{n^{1/4}} \\
&\leq \frac{8}{3c} \frac{(n+2)^{9/8}}{(n+1)n^{1/4}} \frac{\varepsilon^2}{(n+2)^{9/8}} \leq \frac{1}{30} \frac{\varepsilon}{(n+2)^{9/8}} \leq \frac{1}{30} B_\varepsilon(n+2),
\end{aligned} \tag{75}$$

using that the sum is bounded by  $4/3$  and that for  $\varepsilon$  sufficiently small  $\frac{8}{3c} \frac{(n+2)^{9/8}}{(n+1)n^{1/4}} \varepsilon \leq \frac{1}{30}$ .

- 6th term

$$\begin{aligned}
\frac{2}{(n+1)c} \left| \sum_{\substack{v \geq 1 \\ (1+2v)n' = n}} (-1)^v \frac{v}{(2v+1)!} a(n') \right| &\leq \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n}} \frac{v}{(2v+1)!} B_\varepsilon(n') \\
&\leq \frac{2}{(n+1)c} \underbrace{\left( \frac{3^{1/4}}{3!} + \frac{2 \cdot 5^{1/2}}{5!} + \frac{3 \cdot 7^{9/8}}{7!} + \dots \right)}_{\leq 3/10} B_\varepsilon(n) \\
&\leq \frac{3}{5c} \frac{(n+2)^{9/8}}{(n+1)n^{1/4}} B_\varepsilon(n+2) \leq \frac{1}{15} B_\varepsilon(n+2) \quad \text{for} \quad \frac{9}{c} \frac{(n+2)^{9/8}}{(n+1)n^{1/4}} \leq 1.
\end{aligned} \tag{76}$$

We then collect

$$0.78 \frac{n+2}{n+1} + \frac{1}{10} + \frac{3}{30} < 1.$$

We thus have inductively proven the assertion

$$a(n+2) \leq B_\varepsilon(n+2). \tag{77}$$

The most stringent lower bound on  $n$  (which equals  $N_0$  for  $l = 0$ ) comes from (74). This and the other lower bounds on  $n$  and upper bounds on  $\varepsilon$  can be relaxed on distinguishing a number of different cases which we will not do here.

It is straightforward to verify the assertion for the  $a(4, l)$  by bounding inductively the  $\mu$ -

derivatives of (61). When taking  $\mu$ -derivatives of (60) we get

$$\begin{aligned}
a(n+2, l) &= \frac{1}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n+2}} (n'-1) \frac{(-1)^{v-1}}{(2v)!} a(n', l) + \frac{1}{n+1} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n+2}} \frac{(n'-1)^2}{n'} \frac{(-1)^{v-1}}{(2v-1)!} a(n', l) \\
&+ \frac{1}{(n+1)c} \sum_{l'+l''=l} \binom{l}{l'} \sum_{\substack{n_1+n_2=n+2 \\ v_1, v_2 \geq 0 \\ (1+2v_1)n' = n_1 \\ (1+2v_2)n'' = n_2}} (-1)^{v_1+v_2} \frac{(1+2v_1)(1+2v_2)}{(2v_1)!(2v_2)!} a(n', l') a(n'', l'') \\
&+ \frac{2}{(n+1)c} \frac{n-4}{2n} a(n, l) + \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n}} \frac{n'-4}{2n'} \frac{(-1)^v}{(2v+1)!} a(n', l) \\
&+ \frac{2}{(n+1)c} \frac{1}{n} a(n, l+1) + \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n}} \frac{1}{n'} \frac{(-1)^v}{(2v+1)!} a(n', l+1) \\
&+ \frac{2}{(n+1)c} \sum_{\substack{v \geq 1 \\ (1+2v)n' = n}} (-1)^v \frac{v}{(2v+1)!} a(n', l).
\end{aligned} \tag{78}$$

Going from  $l$  to  $l+1$ , the inductive bound for the l.h.s., i.e. for  $a(n+2, l)$ , is multiplied by a factor of  $n+2+l+1$ . The respective bounds on the linear terms on the r.h.s. take factors of

1)  $n'+l+1$ , 2)  $n'+l+1$ , 4)  $n+l+1$ , 5)  $n'+l+1$ , 6)  $n+l+2$ , 7)  $n'+l+2$ , 8)  $n'+l+1$ .

All these factors are strictly smaller than the one for the l.h.s. so that the inductive verification of the bound for  $l > 0$  follows directly from its verification for  $l = 0$ . For the quadratic term (the third term) we use the bound (33) which gives

$$\binom{l}{l'} \frac{(n'+l')!}{n'!} \frac{(n''+l'')!}{n''!} \leq l! \binom{n+2+l}{n+2}. \tag{79}$$

The factors  $\frac{(n'+l')!}{n'!}$ ,  $\frac{(n''+l'')!}{n''!}$  stem from the inductive bounds on the  $a(n', l')$ ,  $a(n'', l'')$ . The expression on the r.h.s. corresponds to the factorials appearing in the definition of  $B(n+2, l)$ . The sum over  $l', l''$  then gives a factor of  $l+1$  which is again smaller than  $n+2+l+1$ . The remaining part of the bound is established as for the third item (69).  $\square$

We note that (for  $\varepsilon'$  sufficiently small) the assumptions (65) imply the assumptions (30)<sup>5</sup> of Proposition 1. By choosing the signs of  $a(2, l)$  appropriately we can also verify the assumptions of Proposition 2. As a consequence of Propositions 1 and 2 and of Lemma 1, therefore we have proven

**Theorem 1:** *For  $\varepsilon' > 0$  sufficiently small, the solutions of bounded action (56) obeying (65) verify the conditions of Proposition 1. For appropriate choices of the  $a(2, l)$  they also verify Proposition 2. This implies the existence of asymptotically free scalar fields of bounded action in the mean field limit.*

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<sup>5</sup>Remember that  $a(2, l) = \partial_\mu^l f_2(\mu)|_{\mu=0}$ .

## 4.4 The trivial solution

It has been proven by Fröhlich [5] and Aizenman [1] under mild assumptions that the pure  $\varphi_4^4$ -theory is trivial, i.e. interaction free. To be precise the result applies to the continuum limit of lattice regularised (even) pure  $\varphi_4^4$ -theory under the assumption that the theory has infinite wave function renormalisation. There are no restrictions on the size of the  $\varphi_4^4$  coupling. The result is also valid in more than four dimensions without any restriction on the wave function renormalisation. We note that beyond four dimensions it is also known that the critical behaviour of the theory is exactly described by the mean field approximation [6], [5], [1]. The fact that the continuum limit is interaction free is proven by showing that the truncated (i.e. connected) four-point function vanishes in this limit. By inequalities due to Glimm, Jaffe [8] and to Newman [21], the vanishing of the truncated four-point function implies the vanishing of the truncated higher  $n$ -point functions as well. The triviality result seems quite robust and has also been confirmed by a comprehensive analysis including numerical work [17]. The hypothesis on the wave function renormalisation has never been verified mathematically. Nor can we shed light on this hypothesis in our context since we work from the beginning in the mean field approximation.

The boundary conditions of pure  $\varphi_4^4$ -theory appear to be particularly simple from the point of view of Ising type lattice models. On the other hand they appear to be very special from the point of view of the renormalisation group, where one analyses the infinite dynamical system of flow equations for the  $n$ -point functions. In fact they correspond to a fine-tuning problem since it seems unnatural that the infinite number of trajectories  $f_n(\mu)$ ,  $n > 4$ , all pass through 0 at the same value of  $\mu$ , namely at  $\mu = 0$ . This is even more true in the full theory where the boundary conditions require that all these trajectories pass through 0 at  $\mu = 0$  for *all* values of the momentum or position arguments. It will turn out that enforcing these conditions tends to make grow higher derivatives of the  $f_n(\mu)$  more rapidly with increasing  $n$  than in case of the solutions we have considered so far. In any case, since we want to argue that our considerations grasp important aspects of scalar field theory, it is important to look at the pure  $\varphi_4^4$ -theory in our mean field context. It will turn out that we can construct the trivial solution quite explicitly for all values of the renormalisation group parameter and sufficiently small bare coupling, thus basically confirming the above cited results [5], [1].

We start with a bare action

$$L_0(\varphi) = \sum_{n=2,4} \int d^4x c_{0,n} \varphi^n(x) \quad (80)$$

From (80) we obtain using (11), (16) and (22)

$$f_2(0) = \alpha_0 2(2\pi)^4 c_{0,2}, \quad f_4(0) = 4c(2\pi)^4 c_{0,4} = 4\pi^2 c_{0,4}, \quad f_n(0) = 0, \quad n > 4. \quad (81)$$

As a consequence of the pure  $\varphi_4^4$  boundary conditions we have

**Lemma 2:** *For smooth solutions  $f_n(\mu)$  of (24), (25) respecting the boundary conditions (81)*

$$\partial_\mu^l f_n(0) = 0 \quad \text{for } n \geq 6 \text{ and } 0 \leq l \leq \frac{n}{2} - 3. \quad (82)$$

*Proof.* We proceed as usual by induction, in  $N = n + l \in \mathbb{N}$ , going up in  $l$  for fixed  $N$  and starting at  $N = 6$ . For  $N = 6$  the assertion just corresponds to the boundary condition

$$f_6(0) = 0 .$$

For  $N > 6$  (25), derived  $l$  times at  $\mu = 0$ , together with the induction hypothesis implies for  $l < \frac{n}{2} - 3$

$$\partial_\mu^{l+1} f_n(0) = 0$$

since all other terms appearing in (23) derived  $l$  times w.r.t.  $\mu$  vanish by induction. We note in particular that for the products

$$\partial_\mu^{l_1} f_{n_1}(0) \partial_\mu^{l_2} f_{n_2}(0)$$

with  $l_1 + l_2 = l$  and  $n_1 + n_2 = n + 2$ , the condition  $l < \frac{n}{2} - 3$  implies that either  $l_1 \leq \frac{n_1}{2} - 3$  or  $l_2 \leq \frac{n_2}{2} - 3$ .  $\square$

As a consequence of Lemma 2 we can write smooth solutions verifying (81) as

$$f_n(\mu) = \mu^{\frac{n}{2}-2} g_n(\mu) , \quad n \geq 4 , \quad (83)$$

where the  $g_n(\mu)$  are smooth. The system (25) can then be rewritten as

$$\begin{aligned} \mu^2 g_{n+2} &= \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1} g_{n_2} \\ &+ \mu \frac{1}{n+1} g_n (2f_2 + 1 - \frac{4}{n}) + \frac{n-4}{n(n+1)} g_n + \frac{2}{n(n+1)} \mu \partial_\mu g_n , \quad n \geq 4 . \end{aligned} \quad (84)$$

Expanding the  $g_n$  and  $f_2$  in a (for the moment formal) Taylor series around  $\mu = 0$

$$g_n(\mu) = \sum_{k \geq 0} g_{n,k} \mu^k , \quad f_2(\mu) = \sum_{k \geq 0} f_{2,k} \mu^k \quad (85)$$

we find for the coefficients from (24) and (25)

$$g_{4,k} = \frac{1}{3} \left( (k+1) f_{2,k+1} - f_{2,k} + \sum_{0 \leq v \leq k} f_{2,v} f_{2,k-v} \right) . \quad (86)$$

$$\begin{aligned} g_{n+2,k} &= \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{0 \leq v \leq k+2} g_{n_1,v} g_{n_2,k+2-v} + \frac{2}{n+1} \sum_{0 \leq v \leq k+1} g_{n,v} f_{2,k+1-v} \\ &+ \frac{1}{n+1} g_{n,k+1} \left( 1 - \frac{4}{n} \right) + \frac{n+2k}{n(n+1)} g_{n,k+2} , \quad n \geq 4 \end{aligned} \quad (87)$$

which can be rewritten

$$f_{2,k+1} = \frac{1}{k+1} \left[ 3g_{4,k} + f_{2,k} - \sum_{0 \leq v \leq k} f_{2,v} f_{2,k-v} \right] , \quad (88)$$



$$\begin{aligned}
g_{n,k+2} = & -\frac{n-4}{n+2k} g_{n,k+1} - \frac{2n}{n+2k} \sum_{0 \leq v \leq k+1} g_{n,v} f_{2,k+1-v} \\
& - \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{0 \leq v \leq k+2} g_{n_1,v} g_{n_2,k+2-v} + \frac{n(n+1)}{n+2k} g_{n+2,k}.
\end{aligned} \tag{89}$$

Regularity of the system (84) at  $\mu = 0$  also implies for  $n \geq 4$

$$\begin{aligned}
\frac{n-4}{n} g_{n,0} + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,0} &= 0, \\
\frac{2}{n} g_{n,1} + \frac{n-4}{n} g_{n,1} + 2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,1} + g_{n,0} (2f_{2,0} + 1 - \frac{4}{n}) &= 0.
\end{aligned} \tag{90}$$

If we choose freely  $f_{2,0}, g_{4,0}$ , the last two equations (90) fix all other  $g_{n,0}, g_{n,1}$ . All terms with  $f_{2,k}$  with  $k \geq 1$  and  $g_{n,k}$  with  $k \geq 2$  are then determined through (88), (89).

**Lemma 3:** *We consider smooth solutions  $f_n(\mu)$  of (24), (25) respecting the boundary conditions (81) and assume that*

$$|f_{2,0}| \leq \frac{\varepsilon}{4}, \quad 0 \leq f_{4,0} = g_{4,0} \leq \frac{\varepsilon}{32} \tag{91}$$

for  $0 < \varepsilon \leq 10^{-2}$ . Then

$$|f_{2,1}| \leq \frac{\varepsilon}{2}, \quad |g_{4,1}| \leq \frac{\varepsilon^2}{32} \tag{92}$$

and for  $n \geq 6$

$$|g_{n,0}| \leq \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2}, \quad |g_{n,1}| \leq \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}. \tag{93}$$

The constants  $g_{n,0}$  are alternating in sign:

$$g_{n,0} = (-1)^{n/2} |g_{n,0}|. \tag{94}$$

*Proof.* For  $f_{2,1}$  we find explicitly from (88)

$$f_{2,1} = 3f_{4,0} - f_{2,0}(f_{2,0} - 1) \leq \frac{\varepsilon}{2}.$$

Similarly from (90)

$$g_{4,1} = 4g_{4,0}f_{2,0} \quad \text{so that} \quad |g_{4,1}| \leq \frac{\varepsilon^2}{32}.$$

We then proceed as usual by induction  $n$ , treating first  $g_{n,0}$ . For  $n \geq 6$  we find from (90)

$$|g_{n,0}| \leq \frac{n}{n-4} \frac{1}{4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{\varepsilon^{\frac{n}{2}-1}}{n_1^2 (n+2-n_1)^2} \leq \frac{1}{2} \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}.$$

For  $n \geq 6$  also

$$|g_{n,1}| \leq \frac{2n}{n-2} \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{\varepsilon^{\frac{n}{2}-1}}{n_1^2 (n+2-n_1)^2} + \frac{n}{n-2} \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2} \left( \frac{\varepsilon^2}{2} + 1 - \frac{4}{n} \right) \leq \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}.$$

The two previous bounds can be verified explicitly for  $n \leq 10$ . For  $n \geq 12$  we use

$$\begin{aligned} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4, n_i \in 2\mathbb{N}}} \frac{1}{n_1^2 (n+2-n_1)^2} &\leq \frac{1}{16} \sum_{\substack{n_1+n_2=\frac{n}{2}+1 \\ n_i \geq 2, n_i \in \mathbb{N}}} \frac{1}{n_1^2 (\frac{n}{2}+1-n_1)^2} \leq \frac{1}{8} \sum_{\substack{n_1 \leq \frac{n}{2}+1 \\ n_1 \geq 3}} \frac{4}{n_1^2 (n+2)^2} + \frac{1}{8} \frac{4}{4(n-2)^2} \\ &\leq \frac{1}{2(n+2)^2} \underbrace{\left( \zeta(2) - \frac{5}{4} \right)}_{\leq 1/4} + \frac{1}{8(n-2)^2} \leq \frac{1}{8} \left( \frac{1}{(n+2)^2} + \frac{1}{(n-2)^2} \right) \end{aligned}$$

and

$$\frac{n}{n-2} \frac{1}{8} \left( \frac{1}{(n+2)^2} + \frac{1}{(n-2)^2} \right) \leq \frac{1}{2n^2} (1 - \varepsilon^2).$$

The statement on the signs follows from (90) by induction in  $n$ , using that  $g_{4,0} > 0$ .  $\square$

**Lemma 4:** *Under the same assumptions as in Lemma 3 we have the bounds*

$$|g_{n,k}| \leq 2^{k-2} \varepsilon^{n/2-1} \left( k + \frac{n-4}{2} \right)!, \quad |f_{2,k}| \leq 2^k \varepsilon |k-1|!. \quad (95)$$

*Proof.* We proceed by induction going up in  $N = n + k$  using (89). For  $g_{n,1}$ ,  $g_{n,0}$  and  $f_{2,1}$ ,  $f_{n,0}$  we use the bounds from Lemma 3. We obtain from (89), (93) and (4)

$$\begin{aligned} |g_{n,k+2}| &\leq 2^k \varepsilon^{\frac{n}{2}-1} \left[ \frac{n-4}{2(n+2k)} (k+1 + \frac{n-4}{2})! + \frac{\varepsilon n}{n+2k} \sum_{0 \leq v \leq k+1} (v + \frac{n-4}{2})! |k-v|! \right. \\ &\quad \left. + \frac{n}{4(n+2k)} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{0 \leq v \leq k+2} (v + \frac{n_1-4}{2})! (k+2-v + \frac{n_2-4}{2})! \right. \\ &\quad \left. + \frac{n(n+1)\varepsilon}{4(n+2k)} (k + \frac{n-2}{2})! \right] \quad (96) \end{aligned}$$

$$\begin{aligned} &\leq 2^k \varepsilon^{\frac{n}{2}-1} (k+2 + \frac{n-4}{2})! \left[ \frac{n-4}{2n} \frac{2}{n} + \frac{2\varepsilon n}{n} \frac{2}{n} + \frac{n}{4n} \frac{n}{2} \frac{2}{n} + \frac{n(n+1)\varepsilon}{4n} \frac{2}{n} \right] \\ &\leq 2^k \varepsilon^{\frac{n}{2}-1} (k+2 + \frac{n-4}{2})!. \quad (97) \end{aligned}$$

We used

$$\sum_{0 \leq v \leq n-a} (n-v)! v! \leq 2n! \quad \text{for } a \in \mathbb{N}, \quad a \leq n$$

and

$$\sum_{0 \leq v \leq k} (a+v)!(b+k-v)! \leq \sum_{0 \leq v \leq k} (A+v)!(A+k-v)! \quad \text{with } A = \sup(a,b), \quad a, b \in \mathbb{N}.$$

For  $n = 2$  the bound follows from (88) and Lemma 3. □

We note that the bounds we derived are not sufficient to prove convergence of the Taylor expansion around  $\mu = 0$ , in contrast to the bounds (31). So (85) still stand as formal power series. We think the factorial behaviour of the bounds is not far from optimal and trace the large size of the derivatives back to the particular boundary conditions. We now want to show that there exist smooth solutions corresponding to the formal power series (85).

**Proposition 4:** *There exist smooth solutions  $f_n(\mu)$  of (24), (25) respecting the boundary conditions (81). They vanish in the limit  $\mu_{max} = \ln \frac{1}{\alpha_0} \rightarrow \infty$ .*

*Proof.* We study two-point functions  $f_2(\mu)$  of the form

$$f_2(\mu) = \sum_{n \geq 1} a_n \frac{x_n^{n-1}}{1 + x_n^n}, \quad \text{where } x_n = n\mu \quad \text{and} \quad |a_n| < 1. \quad (98)$$

This ansatz is the most important ingredient in our construction of the trivial solution. If it is well-defined, then all the  $f_n(\mu)$  and thus all the  $g_n(\mu)$  are determined as functions of  $f_2(\mu)$  as follows from (24), (25). Expanding as in (85)

$$f_2(\mu) = \sum_{k \geq 0} f_{2,k} \mu^k,$$

we find for the Taylor coefficients

$$f_{2,k} = (k+1)^k \sum_{\rho=1}^{k+1} a_{\{\frac{k+1}{\rho}\}} (-1)^{\rho-1} \left(\frac{1}{\rho}\right)^k. \quad (99)$$

Here we set  $a_0 := 0$ , and for integers  $n, m$

$$\left\{\frac{n}{m}\right\} := \begin{cases} \frac{n}{m} & , \quad \text{if } \frac{n}{m} \in \mathbb{N} \\ 0 & , \quad \text{otherwise} \end{cases}. \quad (100)$$

We have in particular

$$f_{2,0} = a_1, \quad f_{2,1} = 2a_2 - a_1. \quad (101)$$

Choosing  $f_{2,0}$  and  $f_{4,0}$  such that the assumptions of Lemma 3 are fulfilled, Lemma 4 implies for smooth solutions of (24), (25) respecting the boundary conditions (81)

$$|f_{2,k}| \leq 2^k \varepsilon |k-1|!.$$

We then claim that the coefficients  $a_n$  in (98) obey the bounds

$$|a_n| \leq 4 \left(\frac{3}{4}\right)^n \varepsilon. \quad (102)$$

The claim is easily verified for  $a_1$  to  $a_3$  using Lemmata 3 and 4. For  $n \geq 3$  we obtain inductively from (99):

$$\begin{aligned} |a_{n+1}| &\leq 2^n \frac{(n-1)!}{(n+1)^n} \varepsilon + \sum_{\rho=2}^{n+1} |a_{\{\frac{n+1}{\rho}\}}| \left(\frac{1}{\rho}\right)^n \\ &\leq \left( \left(\frac{2}{e}\right)^n \frac{1}{n^2} + 4 \sum_{\rho=2}^{n+1} \left(\frac{3}{4}\right)^{\frac{n+1}{\rho}} \left(\frac{1}{\rho}\right)^n \right) \varepsilon \leq 4 \left(\frac{3}{4}\right)^{n+1} \varepsilon \end{aligned} \quad (103)$$

using that

$$\sum_{\rho=2}^{n+1} \left(\frac{3}{4}\right)^{\frac{n+1}{\rho}} \left(\frac{1}{\rho}\right)^n \leq \sum_{\rho=2}^{n+1} \left(\frac{1}{\rho}\right)^n \leq \zeta(n) - 1 \leq 2^{1-n}.$$

The bound (102) implies absolute convergence of the series in (98), uniformly in  $\mu$  so that  $f_2(\mu)$  is smooth and well-defined for  $0 \leq \mu \leq \ln \frac{1}{\alpha_0}$ . The free choice of  $f_{2,0}$  and  $f_{4,0}$  fixes  $a_1$  and  $a_2$ . All  $a_n$ ,  $n \geq 3$  are uniquely determined by (88), (89) and (90) as a consequence of the boundary conditions (81) and the smoothness condition.

Uniform absolute convergence in  $[0, \infty)$  of the series (98) and its derivatives

$$\sum_{n \geq 1} a_n \partial_\mu^l \frac{x_n^{n-1}}{1+x_n^n}$$

and the fact that

$$\lim_{\mu \rightarrow \infty} \partial_\mu^l \frac{x_n^{n-1}}{1+x_n^n} = 0$$

imply

$$\lim_{\mu \rightarrow \infty} \partial_\mu^l f_2(\mu) = 0 \quad \forall l \geq 0. \quad (104)$$

The functions  $\partial_\mu^l f_n(\mu)$  for  $n \geq 4$  are then determined from  $\partial_\mu^l f_2(\mu)$  through (24), (25). Proceeding by induction in  $n \geq 4$  one finds straightforwardly

- The solutions  $f_n(\mu)$  are smooth bounded functions of  $\mu$ .
- Together with all derivatives they have vanishing limits for  $\mu_{max} \rightarrow \infty$ , i.e.  $\alpha_0 \rightarrow 0$ :

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_n(\mu_{max}) = 0. \quad (105)$$

□

We collect the previous findings in the following

**Theorem 2:** *Triviality of weakly coupled mean field pure  $\phi_4^4$ -theory:*

*For the boundary conditions (81), setting*

$$0 \leq c_{0,4} \leq \frac{\varepsilon}{2^7 \pi^2}, \quad |c_{0,2}| \leq \Lambda_0^2 \frac{\varepsilon}{2^7 \pi^4}, \quad 0 < \varepsilon \leq 10^{-2},$$

*the solutions of the mean field flow equations vanish in the UV limit  $\mu_{max} \rightarrow \infty$ , i.e. on removing the UV cutoff  $\Lambda_0^{-2} = \alpha_0 \rightarrow 0$ .*

We note that the upper limit on  $\varepsilon$  is certainly not optimal and could be easily improved. A more ambitious project, which however does not seem to be without reach either, is to include also large values of  $\varepsilon$  exceeding 1.

We close this section with two **general remarks**:

- *The Landau pole*

In perturbative field theory the triviality of pure  $\phi_4^4$ -theory reflects itself in the so-called Landau pole of the energy dependent coupling when going to high energies. This means that when we fix the physical coupling at low energies - at  $\mu = \mu_{max}$  in our setting - via

$$g(0) := f_2(\mu_{max}), \quad (106)$$

then

$$g(\lambda) := f_2(\mu_{max} - \lambda)$$

diverges at a finite value of  $\lambda$  unless we let  $g(0) \rightarrow 0$  which implies triviality. This is indeed the case for our solution. If we truncate for simplicity the expression (98) at lowest order setting

$$f_2(\mu) = a_1 \frac{1}{1 + \mu} \quad (107)$$

we get

$$g(\lambda) = \frac{g(0)}{1 - \beta g(0) \lambda} \quad \text{with} \quad \beta = \frac{1}{a_1}.$$

The Landau pole is situated at  $\lambda_L = \frac{1}{\beta g(0)}$ . In physical perturbation theory one normally chooses  $f_4(\mu_{max})$  to define  $g(0)$ , but this does not change the reasoning since  $f_2$  and  $f_4$  can be expressed in terms of each other and behave in a similar way for large  $\mu$ . Nor do the conclusions change when taking the full expression (98) instead of (107) since all entries in the absolutely convergent series in (98) behave similarly for  $\mu \rightarrow \infty$ . Since the perturbative truncations get out of control way before the Landau pole divergence occurs, perturbation theory does not allow to make hard statements about triviality.

- *Perturbation theory*

The solutions we considered in the previous sections are not perturbative, which is reflected in the fact that the bounds from Propositions 1, 3 and from section 4.3 do not involve a power proportional to  $n$  of the small parameters  $\delta$  or  $\varepsilon$ . For the trivial solution the perturbative behaviour w.r.t. the bare coupling is revealed by the factor of  $\varepsilon^{n/2-1}$  appearing in the bounds of Lemma 4. It should also be possible and would be interesting to reexpress the formal power series in  $\varepsilon$  as formal power series in the renormalized coupling  $g(0)$  (106) and to show that the coefficients of these series are termwise finite for  $\mu_{max} \rightarrow \infty$ . This would correspond to the perturbative renormalizability proof for  $\phi_4^4$ -theory. Our nonperturbative proof implies finiteness and even triviality for  $\mu_{max} \rightarrow \infty$ , but we did not analyse the expansion in powers of  $g(0)$ .

## 5 The flow equations for the effective potential

The Wilson flow equations for the effective action  $L^{\alpha_0, \alpha}(\varphi)$  can be transformed into flow equations for the effective potential, the generating functional of the one-particle irreducible (1PI) functions, on performing a Legendre transformation [27], [19], [16]. We denote the regularised effective potential as  $\Gamma^{\alpha_0, \alpha}(\Phi)$ . One expects that general results for a given field theory which

can be deduced from the effective action, can also be derived from the effective potential. Specific properties of the connected and of the 1PI functions may of course be different. In the mean field approximation the two schemes are no more strictly equivalent. Nevertheless, since we presume that our results on  $\varphi^4$ -theory are generic, we would like to confirm that the reasoning from section 4 can also be applied to the 1PI formalism. The analysis becomes more complicated so that we will restrict ourselves in this paper to a result analogous to Propositions 1 and 2 in section 4.1, namely we will show that there exist bounded nontrivial solutions to the flow equations which may be asymptotically free. A farther reaching analysis of the 1PI functions is left for the future.

The moments of the effective potential  $\Gamma^{\alpha_0, \alpha}(\Phi)$  or 1PI  $n$ -point functions are denoted as  $\Gamma_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$ . They obey the FEs [16]

$$\partial_\alpha \Gamma_n^{\alpha_0, \alpha}(p_1, \dots, p_n) = \quad (108)$$

$$\frac{1}{2} \int_p \frac{1}{1 + [\Gamma_2^{\alpha_0, \alpha} C^{\alpha_0, \alpha}](p)} \dot{C}^\alpha(k) \hat{\Gamma}_{n+2}^{\alpha_0, \alpha}(p, -p, p_1, \dots, p_n) \frac{1}{1 + [\Gamma_2^{\alpha_0, \alpha} C^{\alpha_0, \alpha}](-p)},$$

where the functions  $\hat{\Gamma}_{n+2}^{\alpha_0, \alpha}$  are given by

$$\hat{\Gamma}_{n+2}^{\alpha_0, \alpha}(p, -p, p_1, \dots, p_n) = \Gamma_{n+2}^{\alpha_0, \alpha}(p, -p, p_1, \dots, p_n) + \sum_{v=2}^{n/2} \sum_{\{b_j\}}^{(v)} (-1)^{v-1} \times$$

$$\mathbb{S} \left[ \prod_{k=1}^{v-1} \left( \Gamma_{b_k+2}^{\alpha_0, \alpha}(q'_{k-1}, p_{i_k+1}, \dots, p_{i_k+b_k}) C_{\Gamma_2}^{\alpha_0, \alpha}(q'_k) \right) \Gamma_{b_v+2}^{\alpha_0, \alpha}(q'_{v-1}, -p, p_{i_v+1}, \dots, p_{n-1}) \right] \quad (109)$$

with

$$q'_0 = p, \quad q'_k = p + \sum_{j=1}^{b_1+\dots+b_k} p_j, \quad k \geq 1, \quad b_j \in 2\mathbb{N}, \quad \sum_{j=1}^v b_j = n, \quad i_k = \sum_{j=1}^{k-1} b_j.$$

Here  $\sum_{\{b_j\}}^{(v)}$  indicates the sum over all partitions of  $n$  into  $v$  packets, the cardinality of the packets being an even integer. The symbol  $\mathbb{S}$  has the same meaning as in (14). Note that in the symmetric  $\varphi^4$ -theory all odd 1PI functions vanish. The function  $C_{\Gamma_2}^{\alpha_0, \alpha}(p)$  denotes the regularised complete two-point function

$$C_{\Gamma_2}^{\alpha_0, \alpha}(p) = \frac{C^{\alpha_0, \alpha}(p)}{1 + \Gamma_2^{\alpha_0, \alpha}(p, -p) C^{\alpha_0, \alpha}(p)}. \quad (110)$$

As compared to [16] we thus have resummed the two-point function insertions into (110).

## 5.1 The mean field limit of the 1PI functions

We want to analyse the mean field limit of (109). The procedure is analogous to that of section 3. The mean field dynamical system is obtained from (108) and (109) by replacing the

$\Gamma_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  by their zero momentum values. We call the corresponding mean field functions  $G_n^{\alpha_0, \alpha}$ . The flow equations for these functions write for  $n \in 2\mathbb{N}$

$$\partial_\alpha G_n^{\alpha_0, \alpha} = \frac{1}{2} \sum_{v=1}^{n/2} (-1)^{v-1} I_{v-1}^{\alpha_0, \alpha} \prod_{k=1}^v \sum_{\{b_k\}} \binom{n}{b_1 \dots b_v} G_{b_k+2}^{\alpha_0, \alpha}. \quad (111)$$

Here the integrals  $I_n^{\alpha_0, \alpha}$  are defined as

$$I_n^{\alpha_0, \alpha} = \int_p \dot{C}^\alpha(p) \left[ \frac{1}{1 + G_2^{\alpha_0, \alpha} C^{\alpha_0, \alpha}(p)} \right]^2 [C_{G_2}^{\alpha_0, \alpha}(p)]^n \quad (112)$$

with the definition

$$C_{G_2}^{\alpha_0, \alpha}(p) = \frac{C^{\alpha_0, \alpha}(p)}{1 + G_2^{\alpha_0, \alpha} C^{\alpha_0, \alpha}(p)}. \quad (113)$$

As in (109) we sum over all  $b_j$  such that

$$b_j \in 2\mathbb{N}, \quad \sum_{j=1}^v b_j = n.$$

As in section 4 we have replaced the mass parameter by 0, and will in turn restrict the values of  $\alpha$  to the interval  $[\alpha_0, 1]$ . To factor out the basic scaling behaviour w.r.t.  $\alpha$  we write

$$G_n^{\alpha_0, \alpha} =: \alpha^{n/2-2} g_n^{\alpha_0, \alpha}, \quad (114)$$

which gives the dynamical system

$$2\alpha \partial_\alpha g_n(\alpha) + (n-4)g_n(\alpha) = \sum_{v=1}^{n/2} (-1)^{v-1} \alpha^{3-v} I_{v-1}^{\alpha_0, \alpha} \prod_{k=1}^v \sum_{\{b_k\}} \binom{n}{b_1 \dots b_v} g_{b_k+2}(\alpha). \quad (115)$$

We set for  $v \geq 1$

$$\begin{aligned} J_{v-1}(\mu) &= \alpha^{3-v} I_{v-1}^{\alpha_0, \alpha} = \int_p \alpha^{3-v} \left[ \frac{1}{1 + G_2^{\alpha_0, \alpha} C^{\alpha_0, \alpha}(p)} \right]^2 \dot{C}^\alpha(p) [C_{G_2}^{\alpha_0, \alpha}(p)]^{v-1} \\ &= \int_q \dot{C}^1(q) \left[ \frac{1}{1 + g_2^{\alpha_0, \alpha} C^{\gamma, 1}(q)} \right]^2 [C_{g_2^{\gamma, 1}}^{\gamma, 1}(q)]^{v-1}, \end{aligned} \quad (116)$$

where  $0 \leq \gamma = \frac{\alpha_0}{\alpha} \leq 1$  and  $q^2 = \alpha p^2$ . Passing to the variable  $\mu = \ln \frac{\alpha}{\alpha_0} = \ln \frac{1}{\gamma} \geq 0$  and writing in shorthand  $g_n(\mu) := g_n^{\alpha_0, \alpha}$  we then get

$$2\partial_\mu g_n(\mu) + (n-4)g_n(\mu) = \sum_{v=1}^{n/2} (-1)^{v-1} J_{v-1}(\mu) \prod_{k=1}^v \sum_{\{b_k\}} \binom{n}{b_1 \dots b_v} g_{b_k+2}(\mu). \quad (117)$$

Setting finally

$$g_n(\mu) =: (n-2)! h_n(\mu), \quad \text{which implies } h_n(\mu) =: \alpha^{2-\frac{n}{2}} \frac{1}{(n-2)!} G_n^{\alpha_0, \alpha}, \quad (118)$$

we obtain the mean field 1PI flow equations in the form suited for our subsequent analysis

$$J_0(\mu) h_{n+2}(\mu) = \sum_{v=2}^{n/2} (-1)^v J_{v-1}(\mu) \sum_{\{b_k\}} \prod_{k=1}^v h_{b_k+2}(\mu) + \frac{2}{n(n-1)} \partial_\mu h_n(\mu) + \frac{n-4}{n(n-1)} h_n(\mu). \quad (119)$$

This system of equations is obviously more complicated than the system (24), (25) due to the appearance of the functions  $J_v$ . These functions are well-behaved and straightforward to control, as will be seen in the next section. We shall be able to make analogous statements to Propositions 1 and 2 for the 1PI functions. For farther reaching results corresponding to Proposition 3 and section 4.3, we would need even sharper bounds on arbitrary derivatives of these functions.

## 5.2 Solutions for the 1PI case

We establish bounds on smooth solutions of the dynamical system (119). The bounds are expressed in terms of positive constants  $K$ ,  $\delta$ ,  $\beta$ , and a smooth positive function  $\delta(\mu)$  satisfying  $\delta(\mu) \leq \delta$ . Our assumptions are similar as in (30), (36). We assume  $K$  to be sufficiently large and  $\delta$ ,  $\beta$  to be sufficiently small. The bounds will turn out to hold for

$$K \geq \frac{4}{c}, \quad \beta \leq 2, \quad \delta \leq \frac{c}{4}, \quad \text{with (as before) } c = \frac{1}{16\pi^2}. \quad (120)$$

We assume the following properties of the two-point function  $h_2(\mu)$ :

$$h_2(\mu) = -\delta(\mu), \quad \partial_\mu \delta(\mu) = \beta \delta^2(\mu), \quad 0 < \delta(\mu) \leq \delta \quad \forall \mu \in [0, \ln \frac{1}{\alpha_0}]. \quad (121)$$

From the assumptions (121) it follows that

$$\partial_\mu^l \delta(\mu) = \beta^l \delta^{l+1}(\mu) l! , \quad \partial_\mu^l \delta^N(\mu) = \beta^l \delta^{l+N}(\mu) \frac{(N+l)!}{N!} . \quad (122)$$

It is possible to make assumptions more general than (120), (121), for example  $\partial_\mu \delta(\mu) = \beta(\delta(\mu))$ , with  $\beta(x) = O(x^2)$ , and  $\beta$  analytic in a disc of sufficiently large radius, typically larger than  $1/\delta$ .

**Lemma 5:** *Under the assumptions (120), (121)*

$$\frac{c}{(1 + \delta(\mu))^2} \leq J_0(\mu) \leq \frac{c}{(1 - \delta(\mu))^2}, \quad (123)$$

and for  $v \in \mathbb{N}$

$$0 < J_v(\mu) \leq \frac{c(1-\gamma)^v}{(1-\delta(\mu))^{2+v}}. \quad (124)$$

*Proof.* The first bound is immediate from the definition (116) and since

$$C^{\gamma,1} = \int_\gamma^1 d\zeta e^{-\zeta q^2} \leq 1. \quad (125)$$



Regarding the second bound we write

$$\begin{aligned}
J_\nu(\mu) &= \int_q \dot{C}^1(q) \left[ \frac{1}{1 + h_2(\mu) C^{\gamma,1}(q)} \right]^2 [C_{h_2}^{\gamma,1}(q)]^\nu \\
&= \int_\gamma^1 d\gamma_1 \dots d\gamma_\nu \int_q e^{-(1+\gamma_1+\dots+\gamma_\nu)q^2} \left[ \frac{1}{1 + h_2(\mu) C^{\gamma,1}(q)} \right]^{2+\nu} \\
&\leq \frac{c}{(1 - \delta(\mu))^{2+\nu}} \int_\gamma^1 d\gamma_1 \dots d\gamma_\nu \frac{1}{(1 + \gamma_1 + \dots + \gamma_\nu)^2} \leq \frac{c (1 - \gamma)^\nu}{(1 - \delta(\mu))^{2+\nu}}.
\end{aligned} \tag{126}$$

□

**Lemma 6:** For  $n \in \mathbb{N}$

$$\left( x \frac{d}{dx} \right)^n = \sum_{\nu=1}^n a(n, \nu) x^\nu \frac{d^\nu}{dx^\nu}. \tag{127}$$

The positive integers  $a(n, \nu)$  satisfy

$$a(1, 1) = 1, \quad a(n, n) = a(n, 1) = 1. \tag{128}$$

$$a(n+1, \nu) = a(n, \nu) \nu + a(n, \nu-1), \quad 2 \leq \nu \leq n, \tag{129}$$

$$a(n, \nu) \leq 2^n \frac{n!}{\nu!}. \tag{130}$$

*Proof.* It is obvious that the expansion (127) holds with nonnegative integer coefficients. The relations (128) are also immediate. Furthermore

$$x \frac{d}{dx} \sum_{\nu=1}^n a(n, \nu) x^\nu \frac{d^\nu}{dx^\nu} = \sum_{\nu=1}^n \left[ a(n, \nu) \nu x^\nu \frac{d^\nu}{dx^\nu} + a(n, \nu) x^{\nu+1} \frac{d^{\nu+1}}{dx^{\nu+1}} \right],$$

which gives (129); (130) (which is not optimal) follows directly by induction on  $n + \nu \geq 2$ .

□

Lemma 6 says that for  $l \geq 1$

$$\partial_\mu^l = (-\gamma \partial_\gamma)^l = (-1)^l \sum_{\lambda=1}^l a(l, \lambda) \gamma^\lambda \frac{\partial^\lambda}{\partial \gamma^\lambda}. \tag{131}$$

We set for  $0 \leq \gamma \leq 1$

$$I_N(\gamma) := \int_\gamma^1 d\gamma_1 \dots d\gamma_N \frac{1}{(1 + \gamma_1 + \dots + \gamma_N)^2}. \tag{132}$$

**Lemma 7:**

$$\text{For } l \geq 1 \quad |\partial_\mu^l I_N(\gamma)| \leq \gamma 2^{N+2l+2} (l+1)! . \quad (133)$$

*Proof.* We have

$$\partial_\gamma^l I_N(\gamma) = \sum_{\lambda=1}^l \binom{l}{\lambda} (-1)^l \frac{N!}{(N-\lambda)!} \int_\gamma^1 d\gamma_{\lambda+1} \dots d\gamma_N \frac{(l-\lambda+1)!}{(1+\lambda\gamma+\gamma_{\lambda+1}+\dots+\gamma_N)^{2+l-\lambda}}$$

and

$$\begin{aligned} \int_\gamma^1 \frac{d\gamma_{1+\lambda} \dots d\gamma_N}{(1+\lambda\gamma+\gamma_{\lambda+1}+\dots+\gamma_N)^{2+l-\lambda}} &\leq \sum_{j=0}^{N-\lambda} \binom{N-\lambda}{j} 2^{-(N-\lambda)} \frac{1}{(1+\frac{j}{2})^{2+l-\lambda}} \\ &\leq 2^{2+l-N} \sum_{j=0}^{N-\lambda} \binom{N-\lambda}{j} \frac{1}{(2+j)^{2+l-\lambda}} . \end{aligned}$$

To obtain the last bound we split each integration interval  $[\gamma, 1]$  into its lower and upper half segment and bound the integrand for each choice of segments by its sup. Then

$$\begin{aligned} |\partial_\gamma^l I_N(\gamma)| &\leq 2^{2+l-N} \sum_{\lambda=1}^l \binom{l}{\lambda} \frac{N!}{(N-\lambda)!} (l-\lambda+1)! \sum_{j=0}^{N-\lambda} \binom{N-\lambda}{j} \frac{1}{(2+j)^{2+l-\lambda}} \\ &\leq \sum_{\lambda=1}^l \sum_{j=0}^N 2^{2+l-N+l+2N-\lambda} (l-\lambda+1)! \lambda! \frac{1}{(2+j)^{2+l-\lambda}} \\ &\leq 2^{2+2l+N} l! \sum_{\lambda=1}^l 2^{-\lambda} (\zeta(2+l-\lambda) - 1) \\ &\leq 2^{N+l+1} l! l , \end{aligned} \quad (134)$$

where we used the well-known bounds for the  $\zeta$ -function

$$\zeta(n) - 1 \leq 2^{1-n} \text{ for } n \geq 2 .$$

From (131), Lemma 6 and (134) we then get

$$\begin{aligned} |\partial_\mu^l I_N(\gamma)| &\leq 2^{N+1} \sum_{\lambda=1}^l 2^l \frac{l!}{\lambda!} 2^\lambda \lambda! \lambda \gamma^\lambda \\ &\leq 2^{N+l+1} l! \sum_{\lambda=1}^l \lambda (2\gamma)^\lambda \leq 2^{N+2l+2} \gamma (l+1)! . \end{aligned} \quad (135)$$

□

The bound of Lemma 7 can be improved if  $\gamma$  is close to 1.

**Lemma 8:** Under the assumptions (120), (121) and for  $l \geq 1$

$$\left| \partial_\mu^l J_\nu(\mu) \right| \leq \frac{\gamma}{\pi^2} 2^{2\nu+3l} (l+1)! , \quad \left| \partial_\mu^l J_0(\mu) \right| \leq \delta(\mu) \frac{\gamma}{2\pi^2} 2^{3l} (l+1)! . \quad (136)$$

*Proof.* Setting  $f(\mu) = \delta(\mu)C^{\gamma,1}(q)$  and expanding, we obtain for  $v \geq 1$

$$\frac{1}{[1 - f(\mu)]^{2+v}} = \sum_{N=0}^{\infty} \sum_{n_1+\dots+n_{2+v}=N} f(\mu)^N = \sum_{N=0}^{\infty} \binom{N+v+1}{N} \delta^N(\mu) [C^{\gamma,1}(q)]^N.$$

Using this expression in (116)

$$\begin{aligned} J_v(\mu) &= \int_q \dot{C}^1(q) \left[ \frac{1}{1 - \delta(\mu)C^{\gamma,1}(q)} \right]^2 [C_{h_2}^{\gamma,1}(q)]^v \\ &= \int_{\gamma}^1 d\gamma_1 \dots d\gamma_v \int_q e^{-(1+\gamma_1+\dots+\gamma_v)q^2} \left[ \frac{1}{1 - \delta(\mu)C^{\gamma,1}(q)} \right]^{2+v} \\ &= c \sum_{N=0}^{\infty} \binom{N+v+1}{N} \delta^N(\mu) \int_{\gamma}^1 d\gamma_1 \dots d\gamma_{v+N} \frac{1}{(1 + \gamma_1 + \dots + \gamma_{v+N})^2}, \end{aligned} \quad (137)$$

and deriving  $l$  times w.r.t.  $\mu$ , using (122) and Lemma 7, we then get the bounds

$$\begin{aligned} |\partial_{\mu}^l J_v(\mu)| &\leq c \sum_{N=0}^{\infty} \binom{N+v+1}{N} \sum_{\lambda=0}^l \binom{l}{\lambda} \beta^{\lambda} \delta^{N+\lambda}(\mu) \frac{(N+\lambda)!}{N!} \gamma 2^{N+v+2(l-\lambda)+2} (l-\lambda+1)! \\ &\leq c \gamma \sum_{N=0}^{\infty} 2^{2N+2v+3l+3} \delta^N(\mu) \sum_{\lambda=0}^l (\beta \delta(\mu))^{\lambda} 2^{-2\lambda} (l-\lambda+1)! \\ &\leq c \gamma 2^{2v+3l+3} \frac{1}{1-4\delta(\mu)} \frac{1}{1-\frac{1}{4}\beta\delta(\mu)} (l+1)! \leq \frac{\gamma}{\pi^2} 2^{2v+3l} (l+1)! . \end{aligned} \quad (138)$$

In the case  $v = 0$ ,  $l \geq 1$  we get

$$\begin{aligned} |\partial_{\mu}^l J_0(\mu)| &\leq c \sum_{N=0}^{\infty} \binom{N+1}{N} \sum_{\lambda=1}^l \binom{l}{\lambda} \beta^{\lambda} \delta^{N+\lambda}(\mu) \frac{(N+\lambda)!}{N!} \gamma 2^{N+2(l-\lambda)+2} (l-\lambda+1)! \\ &\leq c \gamma \sum_{N=0}^{\infty} 2^{2N+3l+3} \delta^N(\mu) \sum_{\lambda=1}^l (\beta \delta(\mu))^{\lambda} 2^{-2\lambda} (l-\lambda+1)! \\ &\leq c \gamma 2^{3l+3} \frac{1}{1-4\delta(\mu)} \frac{\frac{1}{4}\beta\delta(\mu)}{1-\frac{1}{4}\beta\delta(\mu)} (l+1)! \leq \delta(\mu) \frac{\gamma}{2\pi^2} 2^{3l} (l+1)! . \end{aligned} \quad (139)$$

We can make explicit a factor of  $\frac{1}{4}\beta\delta(\mu)$  in the last line, since there is no contribution with  $\lambda = 0$  in the second sum in (139) as compared to (138).

□

**Lemma 9:**

$$\sum_{\{l_j\}} \prod_{j=1}^v \frac{1}{(l_j+2)(l_j+1)} \leq \frac{(3/2)^v}{(l+2)(l+1)}, \quad \text{where } v \geq 1, \quad l, l_j \in \mathbb{N}, \quad \sum_{j=1}^v l_j = l. \quad (140)$$

*Proof.* The statement is evident for  $v = 1$ . For  $v = 2$  we have

$$\begin{aligned}
& \sum_{l_1=0}^l \frac{1}{(l_1+2)(l_1+1)} \frac{1}{(l-l_1+2)(l-l_1+1)} = \sum_{l_1=0}^l \left( \frac{1}{l_1+1} - \frac{1}{l_1+2} \right) \left( \frac{1}{l-l_1+1} - \frac{1}{l-l_1+2} \right) \\
&= \sum_{l_1=0}^l \left[ \frac{1}{l+2} \left( \frac{1}{l_1+1} + \frac{1}{l-l_1+1} \right) - \frac{1}{l+3} \left( \frac{1}{l_1+2} + \frac{1}{l-l_1+1} \right) \right. \\
&\quad \left. - \frac{1}{l+3} \left( \frac{1}{l_1+1} + \frac{1}{l-l_1+2} \right) + \frac{1}{l+4} \left( \frac{1}{l_1+2} + \frac{1}{l-l_1+2} \right) \right] \\
&= 2 \left( \frac{1}{l+2} - \frac{1}{l+3} \right) \sum_{l_1=0}^l \frac{1}{l_1+1} - 2 \left( \frac{1}{l+3} - \frac{1}{l+4} \right) \sum_{l_1=0}^l \frac{1}{l_1+2} \\
&= \frac{1}{(l+2)(l+3)} \left( 1 - \frac{2}{l+2} \right) + \left[ \frac{1}{(l+2)(l+3)} - \frac{1}{(l+3)(l+4)} \right] \sum_{l_1=0}^l \frac{2}{l_1+2} \\
&\leq \frac{1}{(l+2)(l+1)} \left[ \frac{l}{l+3} + \frac{4(l+1)}{(l+3)(l+4)} \ln(l+1) \right] \leq \frac{3}{2} \frac{1}{(l+1)(l+2)}.
\end{aligned}$$

The 3<sup>rd</sup> and 4<sup>th</sup> lines are obtained by expanding the products and using  $\frac{1}{ab} = \left(\frac{1}{a} + \frac{1}{b}\right) \frac{1}{a+b}$ . The expression in square brackets in the last line reaches its maximal value 1.446 for  $l = 10$ .

We then get by induction  $v-1 \rightarrow v \geq 3$

$$\begin{aligned}
& \sum_{\{l_j\}} \prod_{j=1}^v \frac{1}{(l_j+2)(l_j+1)} \leq \sum_{1 \leq l_v \leq l-v+1} \left(\frac{3}{2}\right)^{v-1} \frac{1}{(l-l_v+2)(l-l_v+1)} \frac{1}{(l_v+2)(l_v+1)} \\
&\leq \sum_{1 \leq l_v \leq l-1} \left(\frac{3}{2}\right)^{v-1} \frac{1}{(l-l_v+2)(l-l_v+1)} \frac{1}{(l_v+2)(l_v+1)} \leq \left(\frac{3}{2}\right)^v \frac{1}{(l+2)(l+1)},
\end{aligned} \tag{141}$$

where we applied the previous bound (141) again. □

As a consequence of of Lemma 9 we obtain immediately

**Lemma 10:**

$$\sum_{\{b_k\}} \prod_{k=1}^v \frac{1}{(b_k+2)(b_k+1)} \leq \frac{(3/2)^v}{(n+2)(n+1)}, \text{ where } v \geq 2, b_k \in 2\mathbb{N}, \sum_{k=1}^v b_k = n. \tag{142}$$

**Proposition 5:** Under the assumptions (120), (121)

$$h_4(\mu) = J_0^{-1} (\delta(\mu) - \beta \delta^2(\mu)) > \frac{(1 - \delta(\mu))^2}{c} (\delta(\mu) - \beta \delta^2(\mu)) > 0. \tag{143}$$

We set

$$\mathcal{B}(n, l; \mu) := \frac{\delta^2(\mu) K^{n+l-2}}{(n+2)(n+1)(l+2)(l+1)} (n+l-2)!. \tag{144}$$

For  $n = 4$ ,  $l \geq 1$  and for  $n \geq 6$  we have the bounds

$$\left| \partial_\mu^l h_n(\mu) \right| \leq \mathcal{B}(n, l; \mu), \quad |h_4(\mu)| \leq \mathcal{B}(4, 0) \frac{4c}{\delta(\mu)}. \quad (145)$$

*Proof.* For the four-point function the dynamical system gives

$$\partial_\mu^l [J_0(\mu) h_4(\mu)] = \partial_\mu^{l+1} h_2(\mu) - \partial_\mu^l h_2(\mu) = \beta^l \delta^{l+1}(\mu) l! - \beta^{l+1} \delta^{l+2}(\mu) (l+1)!.$$

Then (143) directly follows from (120) and Lemma 5. For  $n = 4$ ,  $l \geq 1$  we then get by induction in  $l$

$$\begin{aligned} \left| J_0 \partial_\mu^l h_4(\mu) \right| &\leq \beta^l \delta^{l+1}(\mu) l! + \beta^{l+1} \delta^{l+2}(\mu) (l+1)! \\ &+ \delta(\mu) \frac{1}{2\pi^2} \sum_{\lambda=1}^l \binom{l}{\lambda} 2^{3\lambda} (\lambda+1)! K^{l-\lambda+2} \sup(4c, \delta(\mu)) \delta(\mu) \frac{(l-\lambda)!}{60} \\ &\leq \beta^l \delta^{l+1}(\mu) l! + \beta^{l+1} \delta^{l+2}(\mu) (l+1)! + 4c \delta^2(\mu) \frac{K^{2+l} l!}{120 \pi^2} \underbrace{\sum_{\lambda=1}^l \left(\frac{8}{K}\right)^\lambda (\lambda+1)}_{\leq 24/K}. \end{aligned} \quad (146)$$

The factor  $[\sup\{4c, \delta(\mu)\}]^v = (4c)^v$  is due to the fact that underived four-point functions allow for an additional factor of  $4c$  in the bound, whereas derived functions allow for an additional factor of  $\delta$ , as follows from (144) and (145). Then the assertion is true if

$$\frac{(1 + \delta^2(\mu))}{c} \left( \beta^l \delta^{l-1}(\mu) + \beta^{l+1} \delta^l(\mu) (l+1) + \frac{4cK}{5\pi^2} K^l \right) \leq \frac{K^{2+l}}{30}$$

which holds due to (120).

We then proceed by induction in  $n+l$  for  $n \geq 4$ . We apply the induction hypothesis to (119), derived  $l$  times w.r.t.  $\mu$  which can be written as

$$\begin{aligned} J_0(\mu) \partial_\mu^l h_{n+2}(\mu) &= - \sum_{\lambda=0}^{l-1} \binom{l}{\lambda} (\partial_\mu^{l-\lambda} J_0(\mu)) \partial_\mu^\lambda h_{n+2}(\mu) \\ &+ \sum_{v=2}^{n/2} (-1)^v \sum_{\lambda=0}^l \binom{l}{\lambda} \partial_\mu^{l-\lambda} J_{v-1}(\mu) \sum_{\{b_k\}} (\partial_\mu^\lambda \prod_{k=1}^v h_{b_k+2}(\mu)) \\ &+ \frac{2}{n(n-1)} \partial_\mu^{l+1} h_n(\mu) + \frac{n-4}{n(n-1)} \partial_\mu^l h_n(\mu). \end{aligned} \quad (147)$$

We now bound successively the contributions from the four terms on the r.h.s. of this equation.

- The 1st term is bounded by

$$\begin{aligned}
& \left| \sum_{\lambda=0}^{l-1} \binom{l}{\lambda} (\partial_{\mu}^{l-\lambda} J_0(\mu)) \partial_{\mu}^{\lambda} h_{n+2}(\mu) \right| \\
& \leq \sum_{\lambda=0}^{l-1} \binom{l}{\lambda} \frac{\delta(\mu)}{2\pi^2} 2^{3(l-\lambda)} (l-\lambda+1)! \frac{\delta^2(\mu) K^{n+\lambda} (n+\lambda)!}{(n+4)(n+3)(l+2)(l+1)} \\
& \leq \frac{\delta(\mu)}{2\pi^2} \frac{\delta^2(\mu) K^n 8^l}{(n+4)(n+3)(l+2)(l+1)} \underbrace{\sum_{\lambda=0}^{l-1} \left(\frac{K}{8}\right)^{\lambda} (n+\lambda)!}_{\leq (K/8)^l (n+l)!} \leq \frac{\delta(\mu)}{2\pi^2} \mathcal{B}(n+2, l; \mu).
\end{aligned} \tag{148}$$

- The 2nd term is bounded by

$$\begin{aligned}
& \left| \sum_{v=2}^{n/2} (-1)^v \sum_{\lambda=0}^l \binom{l}{\lambda} \partial_{\mu}^{l-\lambda} J_{v-1}(\mu) \sum_{\{b_k\}} (\partial_{\mu}^{\lambda} \prod_{k=1}^v h_{b_k+2}(\mu)) \right| \\
& \leq \sum_{v=2}^{n/2} \sum_{\lambda=0}^l \binom{l}{\lambda} \frac{1}{\pi^2} 2^{2(v-1)+3(l-\lambda)} (l-\lambda+1)! K^{n+\lambda} \\
& \quad \times \sum_{\{b_k\}, \{\lambda_k\}} \binom{\lambda}{\lambda_1 \dots \lambda_v} \prod_{k=1}^v \frac{K^{b_k+\lambda_k} \delta^v(\mu) [\sup\{4c, \delta(\mu)\}]^v}{(b_k+4)(b_k+3)(\lambda_k+2)(\lambda_k+1)} (b_k+\lambda_k)! \\
& \leq \sum_{v=2}^{n/2} \sum_{\lambda=0}^l \frac{1}{\pi^2} 2^{2(v-1)+3(l-\lambda)} l! (l-\lambda+1) K^{n+\lambda} \frac{(n+\lambda)!}{\lambda!} (4c\delta(\mu))^v \\
& \quad \times \frac{(3/2)^v}{(\lambda+2)(\lambda+1)} \frac{(3/2)^v}{(n+4)(n+3)} \\
& \leq \frac{1}{4\pi^2} \frac{K^n 8^l l!}{(n+4)(n+3)} \underbrace{\sum_{v=2}^{n/2} [36c\delta(\mu)]^v}_{\leq 2(36c\delta(\mu))^2} \underbrace{\sum_{\lambda=0}^l \frac{(n+\lambda)! (l-\lambda+1) (K/8)^{\lambda}}{(\lambda+2)!}}_{\leq 2(K/8)^l (n+l)!/l!} \\
& \leq \frac{1}{\pi^2} \frac{(36c\delta(\mu))^2 K^{n+l}}{(n+4)(n+3)(l+2)(l+1)} (n+l)! \leq \frac{(36c)^2}{\pi^2} \mathcal{B}(n+2, l; \mu).
\end{aligned} \tag{149}$$

We have used Lemmata 8, 9 and 10 to sum over the  $b_k$  and over the  $\lambda_k$ ,  $\sum \lambda_k = \lambda$ . Furthermore we used

$$\prod_{k=1}^v \frac{(b_k+\lambda_k)!}{\lambda_k!} \leq \frac{(n+\lambda)!}{\lambda!}.$$

- The 3rd term is bounded by

$$\left| \frac{2}{n(n-1)} \partial_{\mu}^{l+1} h_n(\mu) \right| \leq \frac{2}{n(n-1)} \mathcal{B}(n, l+1) \leq \frac{1}{20K} \mathcal{B}(n+2, l). \tag{150}$$

- The 4th term is bounded by

$$\left| \frac{n-4}{n(n-1)} \partial_{\mu}^l h_n(\mu) \right| \leq \frac{n-4}{n(n-1)} \mathcal{B}(n, l) \leq \frac{1}{K^2} \mathcal{B}(n+2, l). \tag{151}$$

The claim then follows since

$$\frac{(1 + \delta^2(\mu))}{c} \left( \frac{\delta(\mu)}{2\pi^2} + \frac{(36c)^2}{\pi^2} + \frac{1}{20K} + \frac{1}{K^2} \right) \leq 1.$$

□

Proceeding in the same way as we did in proving Proposition 2 as a consequence of Proposition 1, we may deduce from Proposition 5 that *the smooth solutions we have constructed are nontrivial and asymptotically free for  $\beta > 0$* . As stated before we have no result for the 1PI functions so far, assuring the existence of solutions with bounded action in the sense of 4.3. In a first moment it seems that the boundary conditions for the 1PI functions  $\Gamma_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$  are easy to analyse because for  $\alpha = \alpha_0$  the  $C_{\Gamma_2}^{\alpha_0, \alpha_0}(p)$  vanish so that we are only left with the contribution  $\nu = 1$  in (109). But the construction of smooth solutions requires control of all derivatives of the  $h_n(\mu)$ . To make further reaching statements this requires more stringent bounds on all derivatives of the  $J_\nu(\mu)$  and of  $J_0^{-1}(\mu)$ .

In conclusion we hope that progress will be made in the future on the issues raised by this paper. Technical improvements should allow to control larger values of the couplings, to prove sharper bounds and to take the limit  $\alpha \rightarrow \infty$  while introducing a finite mass  $m$ . They may also permit to extend all the results for the moments of the effective action to those of the effective potential. Better control might also help to establish a kind of phase diagramme which characterises the different types of solutions in their dependence on respective classes of boundary conditions. It seems natural to us to focus on smooth solutions of the FEs if the regulators are chosen to be smooth with respect to the flow parameter. But this restriction might also deserve further attention. The most interesting and most challenging problem is certainly to extend our reasoning beyond the mean field limit.

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