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HAUSDORFF DIMENSION OF LIMIT SETS FOR PROJECTIVE ANOSOV REPRESENTATIONS

OLIVIER GLORIEUX, DANIEL MONCLAIR, NICOLAS THOLOZAN

Abstract. We study the relation between critical exponents and Hausdorff dimensions of limit sets for projective Anosov representations. We prove that the Hausdorff dimension of the symmetric limit set in $\mathbf{P}(\mathbb{R}^n) \times \mathbf{P}(\mathbb{R}^{n*})$ is bounded between two critical exponents associated respectively to a highest weight and a simple root.

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1. Introduction

1.1. **Critical exponents and Hausdorff dimension.** Let Γ be a discrete group of isometries of a metric space (X, d). A well-known metric invariant of Γ is its *critical exponent*, which measures the exponential growth rate of its orbits. It can be defined by

$$\delta_\Gamma = \limsup_{R \to +\infty} \frac{1}{R} \log \left(\mathrm{Card}\{g \in \Gamma \mid d(x, g \cdot x) \leq R\} \right) \ ,$$

where x is any base point in X.

When (X, d) is the hyperbolic space $\mathbb{H}^n$ and Γ is *convex-cocompact* (i.e. acts cocompactly on a non-empty convex subset of $\mathbb{H}^n$), Sullivan [Sul79] proved that the critical exponent of Γ equals the Hausdorff dimension of the

limit set of Γ inside $\partial_\infty \mathbb{H}^n$. The proof relies on the Ahlfors regularity of the *Patterson–Sullivan measure* on the limit set.

This famous theorem has known a number of generalizations. See for instance [DOP00, Rob03, Coo93] for generalizations to other discrete groups acting on hyperbolic spaces. This paper is mainly interested in extensions to other non-positively curved geometries. A fairly general version of Sullivan's theorem was given by Coornaert for a discrete group Γ acting convex cocompactly on a Gromov hyperbolic space X (see [Coo93, Corollaire 7.6]). In this setting, the critical exponent equals the Hausdorff dimension of the limit set of Γ in $\partial_\infty X$ measured with respect to the Gromov metric on the boundary (see Section 3.1). When X is a Riemannian manifold with variable negative curvature, this metric may differ from the visual metric on the boundary. For instance, the Gromov metric on the boundary of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ coincides with the Carnot–Caratheodory metric of the unit sphere in $\mathbb{C}^n$.

There have also been several important works generalizing Patterson–Sullivan theory to discrete subgroups of a semi-simple Lie group G of higher rank acting on its symmetric space X [Lin04, Qui02b]. A new feature of the higher rank is the existence of several critical exponents corresponding to several G-invariant "metrics" on X. Quint studied in [Qui02b] the dependence of those critical exponents on such a choice and constructed analogs of Patterson–Sullivan measures on the space G/P_{min}, where P_{min} is a minimal parabolic subgroup.

The recently developed theory of Anosov subgroups of higher rank Lie groups motivates a further investigation of these generalizations. Anosov subgroups are in many aspects the "right" generalization of convex cocompact groups in rank 1. In particular, they are Gromov hyperbolic, and their Gromov boundary is realized geometrically as a limit set in some flag variety G/P. It is natural to ask how the Hausdorff dimension of this limit set relates to the different critical exponents of the group.

1.2. **Statement of the results.** The present work focuses on *projective Anosov subgroups* of $\mathrm{SL}(n,\mathbb{R})$. We explain in paragraph 2.3 that general Anosov subgroups of a semi-simple Lie group G can be seen as projective Anosov groups via a suitably chosen linear representation.

Let Γ be a projective Anosov subgroup of $\mathrm{SL}(n,\mathbb{R})$. Then Γ is Gromov hyperbolic and comes with two injective equivariant maps $\xi : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^n)$ and $\xi^* : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^{n*})$. We denote by ξ^{sym} the map $(\xi,\xi^*) : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^n)\times\mathbf{P}(\mathbb{R}^{n*})$. If moreover Γ preserves a proper convex subset of $\mathbf{P}(\mathbb{R}^n)$, then Γ is *strongly projectively convex-cocompact* in the sense of [DGK17].

Given $g \in \mathrm{SL}(n,\mathbb{R})$, define $\mu_i(g)$ as the logarithm of the i-th eigenvalue of $\sqrt{gg^T}$ (in decreasing order). We define the *simple root critical exponent* of Γ by

$$\delta_{1,2}(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R}\log\left(\mathrm{Card}\{\gamma\in\Gamma \mid \mu_1(\gamma)-\mu_2(\gamma)\le R\}\right)$$

and the *Hilbert critical exponent* of Γ by

$$\delta_{1,n}(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R}\log\left(\mathrm{Card}\{\gamma\in\Gamma \mid \mu_1(\gamma)-\mu_n(\gamma)\le R\}\right) \ .$$

These critical exponents are relevant for different reasons: the projective Anosov property means that $\mu_1(\gamma) - \mu_2(\gamma)$ grows linearly with the word length of γ, so $\delta_{1,2}(\Gamma)$ can be seen as a "measure" of the Anosov property. The critical exponent $\delta_{1,n}(\Gamma)$ is the critical exponent associated to the Hilbert metric on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ seen as the projectivization of the cone of positive definite quadratic forms on $\mathbb{R}^n$. Our main result compares these two critical exponents with the Hausdorff dimension of $\xi^{sym}(\partial_\infty\Gamma)$.

Our first comparison result between Hausdorff dimensions concerns strongly projectively convex cocompact subgroups of $\mathrm{SL}(n,\mathbb{R})$, introduced by Crampon and Marquis [CM14]. It is shown in [DGK17] that these groups are projective Anosov.

Theorem 1.1. *Let Γ be a strongly projectively convex cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$. Then*

$$2\delta_{1,n}(\Gamma) \leq \mathrm{DimH}(\xi^{sym}(\partial_\infty\Gamma)) \leq \delta_{1,2}(\Gamma) \ .$$

For projective Anosov subgroups that are not convex cocompact, composing with the representation of $\mathrm{SL}(n,\mathbb{R})$ into $\mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^n))$ gives the following weaker result:

Corollary 1.2. *Let Γ be a projective Anosov subgroup of $\mathrm{SL}(n,\mathbb{R})$. Then*

$$\delta_{1,n}(\Gamma) \leq \mathrm{DimH}(\xi^{sym}(\partial_\infty\Gamma)) \leq \delta_{1,2}(\Gamma) \ .$$

Note that Theorem 1.1 is "sharp" in the sense that if Γ is a convex cocompact subgroup in $\mathrm{SO}(n-1,1) \subset \mathrm{SL}(n,\mathbb{R})$, then

$$2\delta_{1,n}(\Gamma) = \mathrm{DimH}(\xi^{sym}(\partial_\infty\Gamma)) = \delta_{1,2}(\Gamma) \ .$$

Corollary 1.2 is weaker since $\delta_{1,n}(\Gamma)$ is always less or equal to $\frac{1}{2}\delta_{1,2}(\Gamma)$. However, it cannot be sharpened in full generality. For instance, let Γ be a cocompact lattice in $\mathrm{SL}(2,\mathbb{R})$ and let ρ_{irr} and ρ_{red} denote respectively the irreducible and reducible representations of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(3,\mathbb{R})$. Then $\rho_{irr}(\Gamma)$ and $\rho_{red}(\Gamma)$ are projective Anosov with limit set a smooth curve (of Hausdorff dimension 1). However, their critical exponents differ:

- $\rho_{irr}(\Gamma)$ is convex cocompact and
$$2\delta_{1,3}(\rho_{irr}(\Gamma)) = \delta_{1,2}(\rho_{irr}(\Gamma)) = 1 \ .$$
- $\rho_{red}(\Gamma)$ is not convex cocompact and
$$\delta_{1,3}(\rho_{red}(\Gamma)) = \frac{1}{2}\delta_{1,2}(\rho_{red}(\Gamma)) = 1 \ .$$

Let us discuss further these results.

Lower inequality. The main motivation for the lower inequality in Theorem 1.1 was to generalize the following theorem of Crampon:

Theorem 1.3 ([Cra11])**.** *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a Gromov hyperbolic group acting properly discontinuously and cocompactly on a strictly convex open domain Ω in $\mathbf{P}(\mathbb{R}^n)$. Then*

$$2\delta_{1,n} \leq n-2 \ ,$$

with equality if and only if Γ is conjugate to a subgroup of $\mathrm{SO}(n-1,1)$ (in which case Ω is projectively equivalent to the hyperbolic space $\mathbb{H}^{n-1}$).

In that case, Γ is projective Anosov, $\xi(\partial_\infty\Gamma)$ is the boundary of Ω and $\xi^*(\partial_\infty\Gamma)$ the boundary of the dual convex set. One can show that $\xi^{sym}(\partial_\infty\Gamma)$ is a Lipschitz manifold of dimension $n-2$, hence $\mathrm{DimH}(\partial_\infty\Gamma) = n-2$. Theorem 1.1 thus recovers Crampon's inequality as a particular case.

Initially, we hoped to get a lower bound on Hausdorff dimension of $\xi(\partial_\infty\Gamma)$. But several attempts with slightly different methods always led to a "symmetric" version of the limit set. This raised the following question:

Question 1.4. *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a projective Anosov subgroup. Do we have*

$$\mathrm{DimH}(\xi(\partial_\infty\Gamma)) = \mathrm{DimH}(\xi^*(\partial_\infty\Gamma)) = \mathrm{DimH}(\xi^{sym}(\partial_\infty\Gamma)) \ ?$$

While our naïve intuition leaned towards a positive answer, the following case might actually provide a counter-example:

Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$, $u : \Gamma \to \mathbb{R}^2$ a function satisfying the cocycle relation

$$u(\gamma\eta) = u(\gamma) + \gamma \cdot u(\eta) \ ,$$

and let ρ_u be the representation of Γ into $\mathrm{SL}(3,\mathbb{R})$ given by

$$\rho_u(\gamma) = \begin{pmatrix} \gamma & u(\gamma) \\ 0 & 1 \end{pmatrix} \ .$$

Then $\rho_u(\Gamma)$ is projective Anosov. Let $\xi_u : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^3)$ and $\xi_u^* : \partial_\infty\Gamma \to \mathbf{P}(\mathbb{R}^{3*})$ denote the boundary maps associated to $\rho_u(\Gamma)$. Then $\xi_u(\partial_\infty\Gamma) = \xi_0(\partial_\infty\Gamma)$ is a projective line. On the other side, the dual limit set $\xi_u^*(\partial_\infty\Gamma)$ is not a projective line as soon as u is not a coboundary[1], and some numerical simulations seem to show that $\xi_u^*(\partial_\infty\Gamma)$ has typically Hausdorff dimension> 1.

There are situations where the equality in Question 1.4 is known to be true: If Γ preserves a non-degenerate quadratic form $\mathbf{q}$ on $\mathbb{R}^n$, then ξ^* is the image of ξ by the isomorphism $\mathbb{R}^n \simeq \mathbb{R}^{n*}$ defined by $\mathbf{q}$, and therefore

$$\mathrm{DimH}(\xi(\partial_\infty\Gamma)) = \mathrm{DimH}(\xi^*(\partial_\infty\Gamma)) = \mathrm{DimH}(\xi^{sym}(\partial_\infty\Gamma)) \ .$$

In that case we also have that $\mu_n(\gamma) = -\mu_1(\gamma)$ for all $\gamma \in \Gamma$, so that

$$2\delta_{1,n}(\Gamma) = \delta_1(\Gamma) \stackrel{\mathrm{def}}{=} \limsup_{R\to+\infty} \frac{1}{R} \log\left(\mathrm{Card}\{\gamma \in \Gamma \mid \mu_1(\gamma) \leq R\}\right) \ .$$

Those projectively convex cocompact groups preserving a non degenerate quadratic form are precisely the $\mathbb{H}^{p,q}$-convex cocompact groups introduced in [DGK18], whose critical exponent was studied by the first two authors in [GM18]. In this setting, Theorem 1.1 gives an alternative proof of the inequality

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) \leq \mathrm{DimH}(\Lambda(\Gamma))$$

in [GM18, Theorem 1.2].

A rigidity statement in that context was obtained by Collier–Tholozan–Toulisse in [CTT17] for $\mathbb{H}^{2,q}$-convex cocompact surface groups, which are the images of fundamental groups of closed surfaces by maximal representations

[1]Recall that a cocycle u is a coboundary if there exists $v \in \mathbb{R}^2$ such that $u(\gamma) = \gamma \cdot v - v$ for all $\gamma \in \Gamma$.

into $\mathrm{SO}(2, q+1)$. Their limit set is a Lipschitz curve (of Hausdorff dimension 1), and they prove that the critical exponent δ_1 is ≤ 1, with equality if and only the group is contained in $\mathrm{SO}(2,1) \times \mathrm{SO}(q)$ (up to conjugation and finite index). Together with Crampon's theorem, this leads us to formulate the following conjecture:

Conjecture 1.5. *Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a projectively convex cocompact subgroup. If $2\delta_{1,n} = \mathrm{DimH}(\xi^{sym}(\partial_\infty \Gamma))$, then Γ is conjugate to a subgroup of $\mathrm{SO}(n-1, 1)$.*

Note finally that Potrie–Sambarino proved in [PS17] a similar but stronger inequality for Hitchin representations of surface groups. If Γ is the fundamental group of a closed surface and $\rho : \Gamma \to \mathrm{SL}(n, \mathbb{R})$ is a Hitchin representation, then $\rho(\Gamma)$ is projective Anosov and $\xi_\rho(\partial_\infty \Gamma)$ is a $\mathcal{C}^1$ curve, of Hausdorff dimension 1. However, they prove that

$$2\delta_{1,n}(\rho(\Gamma)) \leq \frac{2}{n-1} \ .$$

with equality if and only if $\rho = m_{irr} \circ j$ where $j : \Gamma \to \mathrm{SL}(2, \mathbb{R})$ is Fuchsian and $m_{irr} : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ is irreducible.

Upper inequality. The upper inequality $\mathrm{DimH}(\xi^{sym}(\partial_\infty \Gamma)) \leq \delta_{1,2}(\Gamma)$ is proven independently by Pozzetti–Sambarino–Wienhard in [PSW19]. There, they also find a sufficient criterion for this inequality to be an equality. This criterion is satisfied by many families of Anosov groups, showing in particular that the equality can be stable under small deformations of Γ. Their work generalizes a result of Potrie–Sambarino for surface groups embedded in $\mathrm{SL}(n, \mathbb{R})$ via a Hitchin representation. They are in stark contrast with the rigidity phenomena for the $\delta_{1,n}$ discussed above.

Here we merely give an example where equality holds:

Theorem 1.6. *Let Γ be the fundamental group of a closed surface of genus greater than 1 and let j_1 and j_2 be two Fuchsian representations of Γ into $\mathrm{SL}(2, \mathbb{R})$. Then $j_1 \otimes j_2(\Gamma) \subset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \subset \mathrm{SL}(4, \mathbb{R})$ is projective Anosov, $\xi^{sym}_{j_1 \otimes j_2}(\partial_\infty \Gamma)$ is a Lipschitz curve and*

$$\delta_{1,2}(j_1 \otimes j_2(\Gamma)) = 1 \ .$$

The groups to which this theorem applies are the fundamental groups of globally hyperbolic Cauchy compact anti-de Sitter spacetimes studied by Mess [Mes07]. They form a connected component in the space of surface groups embedded in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \simeq \mathrm{SO}(2, 2)$. To our knowledge, this class of example is not covered by the main result of Pozzetti–Sambarino–Wienhard.

On the other hand, a Fuchsian group of $\mathrm{SL}(2, \mathbb{R})$ embedded reducibly in $\mathrm{SL}(3, \mathbb{R})$ gives an example where $\mathrm{DimH}(\xi^{sym}(\Gamma)) < \delta_{1,2}$. Determining a necessary and sufficient criterion for the equality to hold seems difficult.

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2. Background

2.1. **Hausdorff dimension.** Let (X, d) be a metric space. For $s > 0$, the s-dimensional Hausdorff measure of X is defined by

$$H^s(X,d) = \lim_{\varepsilon \to 0} \inf\{\sum_i r_i^s \mid X \subset \bigcup_{i \in I} B(x_i, r_i) \mid r_i \leq \varepsilon\}$$

where the infimum is taken over all countable covers of X by balls of radius less than ε. One can show that there exists a critical parameter $s_0 > 0$ such that $H^s(X,d) = +\infty$ for all $s < s_0$ and $H^s(X,d) = 0$ for all $s > s_0$. This number s_0 is called the Hausdorff dimension of (X,d) and is denoted by $\mathrm{DimH}(X,d)$.

In this paper, we will compare the Hausdorff dimension of different metrics on a compact set. The following proposition summarizes the comparison properties that we will need. It follows easily from the definition.

Proposition 2.1. *Let d and d' be two distances on a space X. If there exists C and $\alpha > 0$ such that*

$$d' \leq C d^\alpha \ ,$$

then

$$\mathrm{DimH}(X,d') \leq \frac{1}{\alpha} \mathrm{DimH}(X,d) \ .$$

Proof. Denote respectively by $B_d(x,r)$ and $B_{d'}(x,r)$ the balls of center x and radius r for d and d'. Take $s > \mathrm{DimH}(X,d)$. Since $H^s(X,d) = 0$, for every $\varepsilon > 0$ we can find a covering

$$X = \bigcup_{i \in I} B_d(x_i, r_i)$$

with $r_i < \left(\frac{\varepsilon}{C}\right)^{1/\alpha}$ such that

$$\sum_{i \in I} r_i^s < \varepsilon \ .$$

Since $d' \leq C d^\alpha$, we have

$$B_d(x_i, r_i) \subset B_{d'}(x_i, C r_i^\alpha) \ .$$

Writing $r_i' = C r_i^\alpha$, we get that $\bigcup_{i \in I} B_{d'}(x_i, r_i')$ covers X and $r_i' < \varepsilon$. Finally, we have

$$\sum_{i \in I} {r_i'}^{\frac{s}{\alpha}} = C^{\frac{s}{\alpha}} \sum_{i \in I} r_i^s < C^{\frac{s}{\alpha}} \varepsilon \ .$$

Hence $\frac{s}{\alpha} \geq \mathrm{DimH}(X,d')$. Since this is valid for any $s > \mathrm{DimH}(X,d)$, we obtain that $\mathrm{DimH}(X,d) \geq \alpha \, \mathrm{DimH}(X,d')$. □

In particular, if d and d' are bi-Lipschitz, then

$$\mathrm{DimH}(X,d) = \mathrm{DimH}(X,d') \ .$$

Assume now that X is a compact subset of a smooth manifold M. Any two Riemannian metrics on M are bi-Lipschitz equivalent in a neighbourhood of

X. Hence the Hausdorff dimension of X with the induced distance is independent of the choice of such a metric. We denote this Hausdorff dimension by $\mathrm{DimH}(X)$ and we have:

$$\mathrm{DimH}(X) = \mathrm{DimH}(X, d)$$

where d is the distance induced by any Riemannian metric on M.

2.2. **Cartan and Jordan projections.**

2.2.1. *Cartan subspaces and restricted roots.* We present in this subsection the basic structure theory of semi-simple real Lie groups. A detailed exposition of this theory can be found in [Ebe96].

Let G be a real semisimple Lie group with finite center, K a maximal compact subgroup of G and $X = G/K$ the symmetric space of G. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{k} \subset \mathfrak{g}$ the Lie algebra of K. Let $\mathfrak{p}$ denote the orthogonal of $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. A *Cartan subspace* $\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{p}$.

A *restricted root* is a non-zero linear form α on $\mathfrak{a}$ for which there exists $u \in \mathfrak{g}$, $u \neq 0$ such that

$$\mathrm{ad}_a(u) = \alpha(a)u$$

for all $a \in \mathfrak{a}$. We will denote by Δ the set of restricted roots.

The *Weyl group* $W(\mathfrak{a})$ is the finite group $N(\mathfrak{a})/Z(\mathfrak{a})$, where $N(\mathfrak{a})$ and $Z(\mathfrak{a})$ denote respectively the normalizer and the centralizer of $\mathfrak{a}$ in K. The kernels of the restricted roots cut $\mathfrak{a}$ into fundamental domains for the action of $W(\mathfrak{a})$. Choosing a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Delta} \ker \alpha$, we define the set of *positive roots* Δ^+ as the roots that are positive on this connected component, and the *Weyl chamber* as the closure of this connected component, i.e.

$$\mathfrak{a}_+ = \{b \in \mathfrak{a} \mid \alpha(b) \geq 0 \text{ for all } \alpha \in \Delta^+\} .$$

With those choices, the *simple roots* are the positive roots that are not a positive linear combination of other positive roots. They form a basis of $\mathfrak{a}^*$. We denote by Δ_s the set of simple roots.

Finally there is a unique element $w \in W(\mathfrak{a})$ such that $-w$ preserves $\mathfrak{a}^+$. The transformation $-w$ is an involution called the *opposition involution* and denoted by i. The opposition involution preserves Δ_s.

Main example. The main example we will be interested in here is when G is the group $\mathrm{SL}(n, \mathbb{R})$. A canonical choice for a maximal compact subgroup K is the subgroup $\mathrm{SO}(n, \mathbb{R})$ of orthogonal matrices. The symmetric space $X_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ can be identified with the space of scalar products on $\mathbb{R}^n$ up to scaling, with the standard scalar product as base point o.

The Lie algebra $\mathfrak{k}$ is the space of anti-symmetric matrices and its orthogonal $\mathfrak{p} \subset \mathfrak{sl}(n, \mathbb{R})$ is the space of symmetric matrices of trace 0. A canonical choice of Cartan subspace is $\mathfrak{a} = \{\text{Diagonal matrices of trace } 0\} = \{\mathrm{diag}(\lambda_1, ..., \lambda_n), \sum_i \lambda_i = 0\}$. The Weyl group is the symmetric group $\mathfrak{S}_n$ acting by permuting the eigenvalues. Denote by $\varepsilon_i \in \mathfrak{a}^*$ the linear form on $\mathfrak{a}$ corresponding to the i-th eigenvalue. The restricted roots are the $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$, for $1 \leq i, j \leq n$. A canonical choice of Weyl chamber is $\mathfrak{a}^+ = \{\text{Diagonal matrices with ordered eigenvalues}\} = \{\mathrm{diag}(\lambda_1, ..., \lambda_n),\ \lambda_1 \geq \lambda_2 \geq$

$\ldots \geq \lambda_n$, $\sum_i \lambda_i = 0\}$, with associated set of positive restricted roots $\{\alpha_{i,j}, 1 \leq i < j \leq n\}$. The simple roots are the roots $\alpha_{i,i+1}$, $1 \leq i \leq n-1$. Finally, the opposition involution i maps $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ to $\mathrm{diag}(-\lambda_n, \ldots, -\lambda_1)$.

2.2.2. *Cartan projections.* From now on, we always assume a fixed choice of

- a maximal compact subgroup K,
- a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{p}$,
- a Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$, with associated positive roots Δ^+ and simple roots Δ_s.

Note that the choice of a maximal compact subgroup K corresponds to the choice of a base point $o = \mathrm{Fix}(K)$ in the symmetric space X.

Theorem 2.2 (Cartan decomposition)**.** *For every $g \in G$, there is a unique vector $\mu(g) \in \mathfrak{a}_+$ such that*

$$g = k \exp(\mu(g)) k'$$

for some $k, k' \in K$. The map $\mu : G \to \mathfrak{a}_+$ is called the Cartan projection.

Remark 2.3. *The Cartan projections of g and its inverse are related by the following formula:*

$$\mu(g^{-1}) = i(\mu(g)) \ .$$

This relation characterizes the opposition involution.

The Cartan projection allows to define a "vector valued distance" on the symmetric space X. If x and y are two points in X, we define

$$\mu(x, y) = \mu(g^{-1} h) \ ,$$

where g and h are elements of G such that $g \cdot o = x$ and $h \cdot o = y$. This is a vector valued distance in the following sense: if $\| \cdot \|$ is a $W(\mathfrak{a})$-invariant norm on $\mathfrak{a}$, then

$$(x, y) \mapsto \|\mu(x, y)\|$$

is a G-invariant Finsler distance on X. In particular, if $\| \cdot \|_{eucl}$ is the Euclidean norm on $\mathfrak{a}$ given by the Killing form, then

$$\|\mu(x, y)\|_{eucl} = d_R(x, y) \ ,$$

where d_R is the symmetric Riemannian distance of X.

Benoist showed that the Cartan projection satisfies a generalized triangle inequality:

Proposition 2.4. [Ben97] *For every compact subset L of G, there is a constant $C > 0$ such that*

$$\|\mu(lgl') - \mu(g)\|_{eucl} \leq C$$

for all $g \in G$ and all $l, l' \in L$.

In particular, given two points x and $y \in X$, there is a constant C such that

$$\|\mu(x, z) - \mu(y, z)\|_{eucl} \leq C$$

for all $z \in X$.

2.2.3. *Jordan projections.* For a restricted root $\alpha \in \Delta$ we denote by $\mathfrak{g}_\alpha := \{u \in \mathfrak{g}, \mathrm{ad}_a(u) = \alpha(a)u, \forall a \in \mathfrak{a}\}$ the corresponding eigenspace. We denote by $A^+ := \exp(\mathfrak{a}^+)$ and $N = \exp(\oplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha)$.

An element of G is called elliptic (resp. hyperbolic, unipotent) if it is conjugated to an element of K (resp. A^+, N).

Theorem 2.5 (Jordan decomposition)**.** [Hel78, Theorem 2.19.24] *There is a unique triple (g_e, g_h, g_p) of commuting elements, such that g_e is elliptic, g_h hyperbolic and g_p unipotent, that satisfies: $g = g_e g_h g_p$.*

Definition 2.6. *The Jordan projection of g is the element $\lambda(g) \in \mathfrak{a}^+$ such that g_h is conjugated to* $\exp(\lambda(g))$.

While the Cartan projection depends on the choice of a base point in X, the Jordan projection is a conjugacy invariant. One has the following alternative definition of $\lambda(g)$:

Proposition 2.7. *For every $g \in G$,*

$$\lambda(g) = \lim_{n\to+\infty} \frac{1}{n}\mu(g^n)$$

Remark 2.8. *Similarly to the Cartan projection, we have the following relation:*

$$\lambda(g^{-1}) = i(\lambda(g)) \ .$$

Main example. The Cartan decomposition for $\mathrm{SL}(n,\mathbb{R})$ is usually called the polar decomposition, and the Cartan projection associates to a matrix $g \in \mathrm{SL}(n,\mathbb{R})$ the logarithm of the eigenvalues of $\sqrt{gg^t}$ in decreasing order. We will denote by $\mu_i(g) = \varepsilon_i(\mu(g))$ the i-eigenvalue of the Cartan projection of g.

The decomposition $g = g_e g_h g_p$ in that case is sometimes called the Dunford decomposition. The Jordan projection associates to g the logarithms of the moduli of the complex eigenvalues of g, in decreasing order. We will denote similarly by $\lambda_i(g) = \varepsilon_i(\lambda(g))$ the i-th eigenvalue of the Jordan projection of g.

2.3. **Anosov groups.** Anosov subgroups of higher rank Lie groups have been introduced by Labourie [Lab06] as a reasonable generalization of convex-cocompact subgroups in rank 1. The original definition for deformations of uniform lattices in rank 1 was extended by Guichard and Wienhard to Gromov hyperbolic groups. More recently, Gueritaud–Guichard–Kassel–Wienhard [GGKW17] and Kapovich–Leeb–Porti [KLP17] independently gave a characterization of Anosov subgroups in terms of their Cartan projections. While the first team assumes *a priori* that the group is hyperbolic, the second team shows moreover that their condition implies Gromov hyperbolicity. Here, we use their characterization as a definition, that can be found under this formulation in [Gui17]

Let G be a semisimple Lie group. Fix a choice of K, $\mathfrak{a}$ and $\mathfrak{a}_+$ as before. Let Θ be a non-empty subset of the set of simple roots Δ_s.

Definition 2.9. *A finitely generated group $\Gamma \subset G$ is called* Θ-Anosov *if there exist constants $C, C' > 0$ such that*

$$\theta(\mu(\gamma)) \geq C|\gamma| - C'$$

for all $\gamma \in \Gamma$ and all $\theta \in \Theta$. (Here, $|g|$ denotes the word length of g with respect to a finite generating set.)

The definition implies in particular that Γ is discrete and quasi-isometrically embedded in G. One of the nice features of this definition is that it forces Γ to have some "negatively curved behaviour":

Theorem 2.10. [KLP18, Theorem 6.15] *Let $\Gamma \subset G$ be a Θ-Anosov subgroup, for some non-empty subset Θ of Δ_s. Then Γ is Gromov hyperbolic.*

Remark 2.11. *Since Γ is invariant by $g \mapsto g^{-1}$ this definition readily implies that a Θ-Anosov subgroup is also $i(\Theta)$-Anosov, and thus Θ^{sym}-Anosov, where $\Theta^{sym} = \Theta \cup i(\Theta)$. There is thus no loss of generality in assuming that Θ is invariant by the opposition involution.*

Main example. Let us describe more properties of Anosov subgroups in a specific case. In the next section, we will explain how to reduce the general case to this specific case.

Definition 2.12. *A finitely generated group $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ is called* projective Anosov *if there exist constants $C, C' > 0$ such that*

$$\mu_1(\gamma) - \mu_2(\gamma) \geq C|\gamma| - C',$$

for all $\gamma \in \Gamma$.

Remark 2.13. *By definition, a projective Anosov subgroup is Θ-Anosov for $\Theta = \{\alpha_{1,2}\}$. Since the opposition involution sends $\alpha_{1,2}$ to $\alpha_{n-1,n}$, projective Anosov subgroups are actually Θ^{sym}-Anosov for $\Theta^{sym} = \{\alpha_{12}, \alpha_{n-1,n}\}$*

The group $\mathrm{SL}(n, \mathbb{R})$ acts on the projective space $\mathbf{P}(\mathbb{R}^n) = \{\text{lines in } \mathbb{R}^n\}$ and on the "dual" projective space $\mathbf{P}(\mathbb{R}^{n*}) = \{\text{hyperplanes in } \mathbb{R}^n\}$. Recall that the *Gromov boundary* $\partial_\infty \Gamma$ of a Gromov hyperbolic group Γ is a compact metrizable space on which Γ acts by homeomorphisms. The following theorem says that the Gromov boundary of a projective Anosov subgroup is "realized" in the projective space:

Theorem 2.14 ([Lab06],[KLP17])**.** *Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a projective Anosov subgroup. Then there exist Γ-equivariant maps $\xi : \partial_\infty \Gamma \to \mathbf{P}(\mathbb{R}^n)$ and $\xi^* : \partial_\infty \Gamma \to \mathbf{P}(\mathbb{R}^{n*})$ such that*

$$\xi(\eta) \subset \xi^*(\eta') \Longleftrightarrow \eta = \eta' \ .$$

Moreover, every element $\gamma \in \Gamma$ of infinite order has a unique eigenvalue of highest module, with corresponding eigenspace $\xi(\gamma_+)$ (where γ_+ denotes the attracting fixed point of γ in $\partial_\infty \Gamma$).

2.3.1. *Fundamental weights and fundamental representations.* Here, we explain how to interpret the Θ-Anosov property as several projective Anosov properties, via linear representations of the Lie group G. The content of this section is already described in [GGKW17, Section 3].

Let $\langle \cdot \,|\, \cdot \rangle$ denote a scalar product on $\mathfrak{a}^*$ invariant under the Weyl group action.

Definition 2.15. *The* fundamental weight w_θ *associated to a simple root* θ *is the unique element of* $\mathfrak{a}^*$ *such that*

$$2\frac{\langle w_\theta \,|\, \alpha\rangle}{\langle \alpha \,|\, \alpha\rangle} = \delta_{\alpha,\theta} \ ,$$

where $\delta_{\alpha,\theta}$ *is the Kronecker symbol.*

The classical representation theory of semi-simple Lie algebras gives the following:

Lemma 2.16. *For every* $\theta \in \Delta_s$ *there is an integer* $n_\theta \geq 2$*, an integer* k_θ *and an irreducible representation* $\rho_\theta : G \to \mathrm{SL}(n_\theta, \mathbb{R})$ *mapping* K *into* $\mathrm{SO}(n_\theta)$ *and such that*

- $\theta(\mu(g)) = \mu_1(\rho_\theta(g)) - \mu_2(\rho_\theta(g))$,
- $k_\theta w_\theta(\mu(g)) = \mu_1(\rho_\theta(g))$,
- $\theta \circ i(\mu(g)) = \mu_{n_\theta - 1}(\rho_\theta(g)) - \mu_{n_\theta}(\rho_\theta(g))$,
- $w_{\theta\circ i}(\mu(g)) = -\mu_{n_\theta}(\rho_\theta(g))$

for all $g \in G$*. We call this* ρ_θ *the* fundamental representation[2].

Remark 2.17. *The equalities above hold when replacing Cartan projections with Jordan projections.*

Remark 2.18. *The fundamental representation* $\rho_{\theta\circ i}$ *is dual to the the representation* ρ_θ*.*

The following proposition easily follows from the definitions of Anosov representation:

Proposition 2.19. *Let* Γ *be a finitely generated subgroup of* G *and* Θ *be a non-empty subset of* Δ_s*. Then* Γ *is* Θ*-Anosov if and only if* $\rho_\theta(\Gamma)$ *is projective Anosov for all* $\theta \in \Theta$*.*

Example 2.20. *For* $G = \mathrm{SL}(n, \mathbb{R})$*, let* θ_i *denote the simple root* $\alpha_{i,i+1}$ *then the fundamental weight* $w_i = w_{\theta_i}$ *associated to* θ_i *is the linear form* $\varepsilon_1 + \ldots + \varepsilon_i$*, and the fundamental representation* ρ_{θ_i} *is the representation of dimension* C_n^i *given by the action of* $\mathrm{SL}(n, \mathbb{R})$ *on* $\Lambda^i(\mathbb{R}^n)$*.*

Taking tensor products of fundamental representations, one obtains representations for which $\mu_1 - \mu_2$ captures the behaviour of several simple roots at once. Given Θ a non-empty subset of simple roots, denote by $\rho_\theta : G \to \mathrm{SL}(V_\theta)$ the fundamental representations associated to each $\theta \in \Theta$ and by

$$\rho_\Theta = \bigotimes_{\theta\in\Theta} \rho_\theta : G \to \mathrm{SL}\left(\bigotimes_{\theta\in\Theta} V_\theta\right)$$

the tensor product representation.

Proposition 2.21. *For all* $g \in G$*, we have*

- $\mu_1(\rho_\Theta(g)) = \sum_{\theta\in\Theta} k_\theta w_\theta(\mu(g))$,
- $\mu_1(\rho_\Theta(g)) - \mu_2(\rho_\Theta(g)) = \inf_{\theta\in\Theta} \theta(\mu(g))$.

[2]These properties actually do not characterize a representation, but they do if we assume moreover that k_θ is minimal. When G is the split real form of a complex Lie group (such as $\mathrm{SL}(n, \mathbb{R})$), we can have $k_\theta = 1$.

As a corollary we obtain the following:

Corollary 2.22. *A subgroup $\Gamma \subset G$ is Θ-Anosov if and only if $\rho_\Theta(\Gamma)$ is projective Anosov.*

The behaviour of the limit maps under these tensor products is given by the following proposition. Let $\Gamma \subset G$ be a Θ-Anosov subgroup and let $\xi_\theta : \partial_\infty\Gamma \to \mathbf{P}(V_\theta)$ be the boundary map associated to $\rho_\theta(\Gamma)$, seen as a projective Anosov subgroup of of $\mathrm{SL}(V_\theta)$.

Proposition 2.23. *The boundary map ξ_Θ associated to $\rho_\Theta(\Gamma)$ sends a point $x \in \partial_\infty\Gamma$ to*

$$\xi_\Theta(x) = \bigotimes_{\theta\in\Theta} \xi_\theta(x) \in \mathbf{P}\left(\bigotimes_{\theta\in\Theta} V_\theta\right) .$$

Remark 2.24. *The boundary map ξ_Θ takes values in the algebraic set of pure tensors in $\mathbf{P}\left(\bigotimes_{\theta\in\Theta} V_\theta\right)$ which is canonically isomorphic to $\prod_{\theta\in\Theta} \mathbf{P}(V_\theta)$.*

Example 2.25. *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a projective Anosov subgroup. Then Γ is Θ-Anosov with $\Theta = \{\alpha_{1,2}, \alpha_{n-1,n}\}$. Let V denote the space $\mathbb{R}^n$ seen as the standard representation of $\mathrm{SL}(n,\mathbb{R})$. Then $\rho_\Theta : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(V \otimes V^*)$ is the tensor product of the standard representation and its dual.*

If $\xi : \partial_\infty\Gamma \to \mathbf{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbf{P}(V^*)$ denote the boundary maps from Theorem 2.14, then the boundary map associated to ρ_Θ is the map

$$\xi^{sym} = (\xi, \xi^*) : \partial_\infty\Gamma \to \mathbf{P}(V) \times \mathbf{P}(V^*) \subset \mathbf{P}(V \otimes V^*) .$$

We will see in Section 2.4 that projective Anosov representations are deeply connected to the notion of convex-cocompact subgroups of $\mathrm{PGL}(n,\mathbb{R})$ as defined by Danciger-Guéritaud-Kassel [DGK17].

For future use, we introduce the following notations. Given a subset Θ of Δ_s, we define

$$C(\Theta) = \bigcup_{\theta\in\Theta} \{v \in \mathfrak{a}_+ \mid \theta(v) = 0\}$$

and

$$C^*(\Theta) = \mathrm{Span}_{\mathbb{R}_+}(\Theta) .$$

Define also

$$\mathfrak{a}_+(\Theta) = \mathfrak{a}_+ \setminus C(\Theta)$$

and

$$\mathfrak{a}_+^*(\Theta) = \mathfrak{a}_+^* \setminus C^*(\Delta_s - \Theta) = \{\varphi \in \mathfrak{a}_+^* \mid \varphi_{|\mathfrak{a}_+(\Theta)} > 0\}.$$

Remark 2.26. *The motivation to consider such a subset of linear forms comes from the counting of elements of the group, as we will see in the next section. For a Θ-Anosov subgroup Γ, we know that the Cartan projections of elements of Γ lie in a closed cone contained in $\mathfrak{a}_+(\Theta)$. In particular, $\varphi(\mu(g))$ grows linearly with $|g|$ for all $\varphi \in \mathfrak{a}_+^*(\Theta)$.*

Main example. Let G be $\mathrm{SL}(n,\mathbb{R})$ and $\Theta = \{\alpha_{1,2}, \alpha_{n-1,n}\}$. We then have

$$C(\Theta) := \{v \in \mathfrak{a}_+ \,|\, \alpha_{1,2}(v) = 0\} \cup \{v \in \mathfrak{a}_+ \,|\, \alpha_{n-1,n}(v) = 0\}.$$

Thus the set $\mathfrak{a}_+(\Theta)$ consists of diagonal matrices for which there is a spectral gap between the two highest, and between the two lowest eigenvalues. Finally, $\mathfrak{a}_+^*(\Theta)$ is the set of linear forms on $\mathfrak{a}$ which are strictly positive on $\mathfrak{a}^+$ except maybe on the walls of the Weyl chambers defined by the equality of the two highest (resp. smallest) eigenvalues. In coordinates this means that any linear form $\varphi \in \mathfrak{a}_+^*(\Theta)$ can be written as $\varphi = \sum_{i=1}^{n-1} x_i \alpha_{i,i+1}$, for $x_i \in \mathbb{R}^{n-1}$ where $x_i > 0$ for all $i \in \{2, ..., n-2\}$ and $x_i \geq 0$ for $i \in \{1, n-1\}$.

2.3.2. *Critical exponents and entropies.* The critical exponent for a discrete group of isometries of a metric space is the exponential growth rate of the orbit of a basepoint. In the case of a discrete subgroup Γ of a higher rank semisimple Lie group G acting on its symmetric space X, one can define a several critical exponents for each G-invariant distance on X, and more generally for every choice of a way of measuring the "size" of Cartan projections. Following Quint [Qui02a], we focus here on non-negative linear forms on the Weyl chamber.

Definition 2.27. *Let Γ be a discrete subgroup of G and φ a linear form on $\mathfrak{a}$ which is non-negative on the Weyl chamber. We define the* φ-critical exponent *of Γ as*

$$\begin{aligned} \delta_\varphi(\Gamma) &= \limsup_{R\to\infty} \frac{1}{R} \log\left(\mathrm{Card}\{\gamma \in \Gamma \,|\, \varphi(\mu(\gamma)) \leq R\}\right) \\ &= \inf\{s > 0 \,|\, \sum_{\gamma\in\Gamma} e^{-s\varphi(\mu(\gamma))} < +\infty\} \ . \end{aligned}$$

In full generality, $\delta_\varphi(\Gamma)$ has no reason to be finite. However, for finitely generated groups, Quint showed in [Qui02a] that $\delta_\varphi(\Gamma)$ is finite as soon as φ is positive on the *limit cone* of Γ, defined as

$$\mathrm{Cone}(\Gamma) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\gamma\in\Gamma, \|\gamma\|\geq n} \mu(\gamma)} \ .$$

Applying his results to the case of Anosov representations gives the following

Proposition 2.28 (Quint, [Qui02a])**.** *Let Θ be a non-empty subset of Δ_s. Then*

$$\delta_\varphi(\Gamma) < +\infty$$

for every linear form φ in $\mathfrak{a}_+^*(\Theta)$ and every Θ-Anosov subgroup Γ. Moreover, the map

$$\varphi \mapsto \delta_\varphi$$

is convex and homogeneous of degree -1 on $\mathfrak{a}_+^*(\Theta)$.

In a similar way, one can consider the exponential growth rate of the Jordan projections.

Definition 2.29. *Let Γ be a discrete subgroup of G and φ a linear form on $\mathfrak{a}$ which is non-negative on the Weyl chamber. We define the φ-*entropy *of Γ as*

$$\begin{aligned} h_\varphi(\Gamma) &= \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{Card}\{[\gamma] \in [\Gamma] \mid \varphi(\lambda(\gamma)) \leq R\} \\ &= \inf\{s > 0 \mid \sum_{[\gamma]\in[\Gamma]} e^{-s\varphi(\lambda(\gamma))} < +\infty\} \ , \end{aligned}$$

where $[\Gamma]$ denotes the set of conjugacy classes in G.

The term "entropy" comes from the analogy with the geodesic flow of a closed negatively curved manifold, whose closed orbits are in bijection with conjugacy classes in the fundamental group, and whose topological entropy equals the exponential growth rate of lengths of closed orbits. In the case where φ is a linear combination of the fundamental weights $w_\theta, \theta \in \Theta$, this is more than an analogy. From the work of A. Sambarino [Sam14], one see that it is possible to associate to a Θ-Anosov subgroup Γ of G a flow on a compact metric space, whose orbits are in bijection with conjugacy classes in Γ, and such that the length of the orbit associated to g is given by $\varphi(\lambda(g))$. This flow has a hyperbolicity property, and its topological entropy is h_φ see [BCLS15].

For sufficiently nice discrete groups of isometries of a negatively curved manifold, the critical exponent equals the entropy. For a Zarisky dense Θ-Anosov group, Sambarino obtained in [Sam14] precise counting estimates for

$$\mathrm{Card}\{\gamma \in \Gamma \mid w_\theta(\mu(\gamma)) \leq R\} \ ,$$

implying in particular that $h_\varphi(\Gamma) = \delta_\varphi(\Gamma)$ when φ is a linear combination of the fundamental weights $\{w_\theta, \theta \in \Theta\}$. The tools he uses, however, do not seem to apply to simple root critical exponents in general. Here we prove the equality $\delta_\varphi = h_\varphi$ whenever we manage to generalize the classical arguments that work in negative curvature.

For the sake of clarity, let us first state our result in the main case of interest for us.

Theorem 2.30. *Let Γ be a projective Anosov subgroup of $\mathrm{SL}(n,\mathbb{R})$. Then*

- $h_{1,2}(\Gamma) \leq \delta_{1,2}(\Gamma)$,
- $h_{1,n}(\Gamma) = \delta_{1,n}(\Gamma)$.

If Γ is moreover Zariski dense in $\mathrm{SL}(n,\mathbb{R})$, then

- $h_{1,2}(\Gamma) = \delta_{1,2}(\Gamma)$.

This theorem will be a particular case of a more general result for Θ-Anosov subgroups of a semi-simple Lie group G. Let $\overline{C}^*(\Theta)$ be the set of non-negative linear combinations of the simple roots $\theta \in \Theta$, $W(\Theta)$ the span of $\{w_\theta, \theta \in \Theta\}$, and define

$$D^*(\Theta) = \{\varphi = \alpha + \beta, \alpha \in \overline{C}^*(\Theta), \beta \in W(\Theta)\} \subset \mathfrak{a}^* \ .$$

Theorem 2.31. *Let Γ be a Θ-Anosov subgroup of G. Then*

- $h_\varphi(\Gamma) \leq \delta_\varphi(\Gamma)$ *for all* $\varphi \in D^*(\Theta) \cap \mathfrak{a}^*_+(\Theta)$,

- $h_\varphi(\Gamma) = \delta_\varphi(\Gamma)$ *for all* $\varphi \in W(\Theta)$.

If Γ *is moreover Zariski dense in* G*, then*

- $h_\varphi(\Gamma) = \delta_\varphi(\Gamma)$ *for all* $\varphi \in D^*(\Theta) \cap \mathfrak{a}^*_+(\Theta)$.

The conditions on φ might look exotic, but they will appear naturally in view of Lemma 2.35.

Main example. Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a projective Anosov subgroup, which is thus Θ-Anosov for $\Theta = \{\alpha_{1,2}, \alpha_{n-1,n}\}$. The fundamental weights associated to $\alpha_{1,2}$ and $\alpha_{1,n}$ are respectively ε_1 and ε_n. Therefore, $\alpha_{1,2}$ belongs to $D^*(\Theta) \cap \mathfrak{a}^*_+(\Theta)$ and $\alpha_{1,n}$ belongs to $W(\Theta) \cap \mathfrak{a}^*_+(\Theta)$. Thus Theorem 2.31 implies Theorem 2.30.

Remark 2.32. *Note the second part of Theorem 2.31 actually holds as soon as the Zariski closure of* Γ *is semi-simple (by simply restricting to the Zariski closure). A typical example where we don't know whether the equality* $\delta_{1,2} = h_{1,2}$ *holds is a deformation of a projective Anosov subgroup of* $\mathrm{SL}(n, \mathbb{R})$ *inside* $\mathrm{Aff}(\mathbb{R}^n) \subset \mathrm{SL}(n+1, \mathbb{R})$.

Let us introduce the notion of *proximality.*

Definition 2.33. *We say that a matrix* $g \in \mathrm{SL}(n, \mathbb{R})$ *is* proximal *if* g *preserves a splitting of* $\mathbb{R}^n$ *as* $\mathbb{R}u \oplus H$*, where* u *is an eigenvector for the eigenvalue* $\lambda_1(g)$ *and if the spectral radius* $\lambda_2(g)$ *of* $g_{|H}$ *is strictly less than* $\lambda_1(g)$*. We says that* g *is* ε-proximal *if, moreover, the line* $\mathbb{R}u$ *and the hyperplane* H *form an angle greater than* ε*.*

If θ *is a simple root of* G*, we say that* $g \in G$ *is* (θ, ε)-proximal *if* $\rho_\theta(g)$ *is* ε*-proximal. If* Θ *is a subset of* Δ_s*, we say that* $g \in G$ *is* (Θ, ε)-proximal *if* g *is* (θ, ε)*-proximal for all* $\theta \in \Theta$*. Finally, we says that* g *is* ε-loxodromic *if* g *is* (Δ_s, ε)*-proximal.*

We need to compare the Cartan and Jordan projections of proximal elements. This is the purpose of Lemma 2.35. We will use the following topological result:

Lemma 2.34. *Let* G *be a locally compact group acting transitively on a Hausdorff space* X *and let* x *be a point in* X*. Then for all compact subset* K *of* X *there exists a compact* K' *of* G *such that* $K \subset K' \cdot x$*.*

Proof. Let C be a compact neighborhood of the identity in G. Since G acts transitively on X, $\bigcup_{g \in G} gC \cdot x$ covers K. By compactness of K we can extract a finite cover, $K \subset \bigcup_{i \in \{1,m\}} g_i C \cdot x$, for some $g_i \in G$.

Then $K' = \bigcup_{i \in \{1,m\}} g_i C$ fulfills the conclusion of the Lemma. □

Lemma 2.35. *Let* θ *be a simple root of* G *and fix* $\varepsilon > 0$*. Then there exists a constant* C *(depending on* ε*) such that for every* (θ, ε)*-proximal element* g*, we have*

$$|w_\theta(\mu(g)) - w_\theta(\lambda(g))| < C$$

and

$$\theta(\mu(g)) < \theta(\lambda(g)) + C \ .$$

Proof. Taking the fundamental linear representation ρ_θ, it is sufficient to prove there exists $C > 0$ such that for any $g \in \mathrm{SL}(n, \mathbb{R})$ which is $(\alpha_{1,2}, \varepsilon)$-proximal,

$$|\mu_1(g) - \lambda_1(g))| < C,$$

and

$$(\mu_1 - \mu_2)(g) \leq (\lambda_1 - \lambda_2)(g) + C.$$

The subset of $\mathbf{P}(\mathbb{R}^n) \times \mathbf{P}(\mathbb{R}^{n*})$ consisting of pairs of a line ℓ and a hyperplane H such that $d(\ell, H) \geq \varepsilon$ is a compact subset of the set of pairs (ℓ, H) which are in general position. Since $\mathrm{SL}(n, \mathbb{R})$ is locally compact and acts transitively on this latter set, by Lemma 2.34, there exists a compact set $K' \subset \mathrm{SL}(n, \mathbb{R})$ such that for all $(\alpha_{12}, \varepsilon)$-proximal $g \in \mathrm{SL}(n, \mathbb{R})$, there exists $P \in K'$ such that

$$\begin{cases} P \cdot g^+ = (PgP^{-1})^+ = [e_1] \\ P \cdot H^-(g) = H^-(PgP^{-1}) = \mathrm{span}(e_2, ..., e_n) \end{cases}$$

We then have $\lambda_1(g) = \lambda_1(PgP^{-1}) = \mu_1(PgP^{-1})$ and $\lambda_2(g) = \lambda_2(PgP^{-1}) = \lambda_1(\widehat{PgP^{-1}})$ where $\widehat{PgP^{-1}}$ is the restriction of PgP^{-1} to $\mathrm{span}(e_2, ..., e_n)$. Note also that $\mu_1(\widehat{PgP^{-1}}) = \mu_2(PgP^{-1})$. Since any matrix h satisfies $\lambda_1(h) \leq \mu_1(h)$, we find that $\lambda_2(PgP^{-1}) \leq \mu_2(PgP^{-1})$. Hence

$$(\mu_1 - \mu_2)(PgP^{-1}) \leq (\lambda_1 - \lambda_2)(g) \ .$$

By Proposition 2.4, there is a constant C' (depending only on K') such that

$$|\mu_1(PgP^{-1}) - \mu_1(g)| < C'$$

and

$$|\mu_2(PgP^{-1}) - \mu_2(g)| < C' \ .$$

It follows that $(\mu_1 - \mu_2)(g) \leq (\lambda_1 - \lambda_2)(g) + 2C'$.

Finally, we also have

$$\lambda_1(g) \leq \mu_1(g) \leq \mu_1(PgP^{-1}) + C' = \lambda_1(PgP^{-1}) + C' = \lambda_1(g) + C'.$$

□

A result of Abels-Margulis-Soifer states that in a Zariski dense subgroup, it is possible to make all elements ε-loxodromic up to left multiplication by a finite set.

Lemma 2.36 ([AMS95, Theorem 6.8])**.** *Let Γ be a Zariski dense subgroup of G. Then there exists $\varepsilon > 0$ and a finite subset F of Γ such that, for every $\gamma \in \Gamma$, there exists $f \in F$ such that $f\gamma$ is ε-loxodromic.*

If Γ is not assumed to be Zariski dense, then it may not contain loxodromic elements. However, if Γ is Θ-Anosov, it certainly contains Θ-proximal elements, and we have the analoguous statement:

Lemma 2.37 ([GW12, Theorem 5.9])**.** *Let Γ be (not necessarily Zariski dense) Θ-Anosov subgroup of G. Then there exists $\varepsilon > 0$ and a finite subset F of Γ such that, for every $\gamma \in \Gamma$, there exists $f \in F$ such that $f\gamma$ is (Θ, ε)-proximal.*

We will also need to control the number of Θ-proximal elements in a conjugacy class. This will reduce to intrinsic geometric properties of hyperbolic groups.

Let d be the distance on Γ associated to a finite set of generators and let d_∞ be a Gromov distance on $\partial_\infty \Gamma$. For every $[\gamma] \in [\Gamma]$, define

$$l([\gamma]) = \inf_{x \in \Gamma} d(x, \gamma x) \ .$$

We call an element $\gamma \in \Gamma$ *η-hyperbolic* if γ has an attracting fixed point γ_+ and a repelling fixed point γ_- in $\partial_\infty\Gamma$ such that $d_\infty(\gamma_-, \gamma_+) > \eta$.

Lemma 2.38 (See [CK02, Paragraph 7])**.** *There exists $\eta_0 > 0$ such that, for $\eta < \eta_0$, there is a constant C such that every $[\gamma] \in [\Gamma]$ contains at least one and at most $Cl(\gamma)$ elements that are η-hyperbolic.*

Corollary 2.39. *Let Γ be a Θ-Anosov subgroup of G. Then there exists $\varepsilon > 0$ and $C > 0$ such that every conjugacy class of an infinite order element $[\gamma] \in [\Gamma]$ contains at least one and at most $D\|\lambda(\gamma)\|_{eucl} + D$ elements that are (Θ, ε)-proximal.*

Proof. Let d be the distance on Γ associated to a finite set of generators, and let d_∞ be a Gromov distance on $\partial_\infty\Gamma$. Let η_0 be such that Lemma 2.38 applies. Since Γ is Θ-Anosov, there exists $\varepsilon > 0$ such that $\gamma \in \Gamma$ is (Θ, ε)-proximal whenever $d_\infty(\gamma_-, \gamma_+) > \eta_0$. Conversely, there exists $\eta < \eta_0$ such that γ is η-hyperbolic whenever γ is (Θ, ε)-Anosov. Let C_0 be such that the number of η-hyperbolic elements in a conjugacy class $[\gamma]$ is bounded above by $C_0 l(\gamma)$.

Let $[\gamma]$ be the conjugacy class of an infinite order element. By Lemma 2.38, $[\gamma]$ contains an η_0-hyperbolic element, which is thus (Θ, ε)-proximal.

Since Γ is quasi-isometrically embedded in G, there exists a constant C_1 such that

$$\lim_{n\to+\infty} \frac{1}{n} d(x, \gamma^n x) \leq C_1 \lim_{n\to+\infty} \frac{1}{n} \|\mu(\gamma^n)\|_{eucl} = C_1 \|\lambda(\gamma)\|_{eucl} \ .$$

Moreover, there is a constant C_2 such that

$$\lim_{n\to+\infty} \frac{1}{n} d(x, \gamma x) \geq l(\gamma) - C_2$$

(see [CDP90, Page 119]).

Since every (Θ, ε)-proximal element of Γ is η-hyperbolic, we obtain

$$\mathrm{Card}\{\gamma' \in [\gamma] \mid \gamma' \ \eta\text{-hyperbolic}\} \leq C_0 l([\gamma]) \leq C_0 C_1 \|\lambda(\gamma)\|_{eucl} + C_0 C_2 \ .$$

□

We now have all the tools to prove Theorem 2.31.

Proof of Theorem 2.31. We first prove the inequality $h_\varphi \leq \delta_\varphi$, then the reverse inequality.

$h_\varphi \leq \delta_\varphi$. Let φ be an element of $D^*(\Theta) \cap \mathfrak{a}^*_+(\Theta)$. Recall first that, in a finitely generated hyperbolic group, there are only finitely many conjugacy classes of elements of torsion. Thus we only need to bound the number of conjugacy classes of elements of infinite order.

By Corollary 2.39, there exists $\varepsilon > 0$ such that every conjugacy class in Γ of infinite order contains a (Θ, ε)-proximal element.

Define

$$\Gamma_\varphi(R) = \{\gamma \in \Gamma \mid \varphi(\mu(\gamma)) \leq R\} \ .$$

For every $R > 0$ and every conjugacy class $[\gamma] \in [\Gamma]$ of infinite order such that $\varphi(\lambda(\gamma)) \leq R$, we can find $\gamma' \in [\gamma]$ which is (Θ, ε)-proximal by Corollary

2.39. By Lemma 2.35 and by definition of φ, there exists a constant C (independent of R) such that $\gamma' \in \Gamma_\varphi(R)$. We deduce that

$$\operatorname{Card}\Gamma_\varphi(R+C) \geq \operatorname{Card}\{[\gamma] \in [\Gamma] \text{ of infinite order } \mid \varphi(\lambda(\gamma)) \leq R\} \ .$$

The inequality $\delta_\varphi \geq h_\varphi$ easily follows.

$\delta_\varphi \leq h_\varphi$. Assume first that Γ is Zariski dense. Let φ be any element of $\mathfrak{a}_+^*$. By Abels–Margulis–Soifer's Lemma 2.36, there exists $\varepsilon > 0$ and a finite subset $F \subset \Gamma$ such that for any $\gamma \in \Gamma$, there exists $f \in F$ such that $f\gamma$ is ε-loxodromic. By Proposition 2.4 and Lemma 2.35, there is a constant C depending on F and ε such that $|\varphi(\lambda(f\gamma)) - \varphi(\mu(\gamma))| \leq C$. We thus have

$$\operatorname{Card}\Gamma_\varphi(R) \leq \operatorname{Card}(F) \cdot \operatorname{Card}\{\gamma \in \Gamma \mid \varphi(\lambda(\gamma)) \leq R+C,\ \gamma \text{ is } \varepsilon\text{-loxodromic}\}.$$

Now, by Corollary 2.39, every conjugacy class $[\gamma]$ such that $\varphi(\lambda(\gamma)) \leq R+C$ contains at most $C'(R+C)$ elements that are ε-loxodromic.

We thus obtain

$$\operatorname{Card}\Gamma_\varphi(R) \leq C'(R+C) \operatorname{Card} F \cdot \operatorname{Card}\{[\gamma] \in [\Gamma] \mid \varphi(\lambda(\gamma)) \leq R+C\} \ ,$$

from which the inequality $\delta_\varphi \leq h_\varphi$ easily follows.

If Γ is not assumed Zariski dense, we can still apply Lemma 2.37 to make every element $\gamma \in \Gamma$ (Θ, ε)-proximal by multiplying it by an element f chosen in a finite set. By Lemma 2.35, we do have $|\varphi(\lambda(f\gamma)) - \varphi(\mu(\gamma))| \leq C$ provided that φ belongs to $W(\Theta)$. The rest of the proof works in a similar way and we eventually obtain the inequality $\delta_\varphi \leq h_\varphi$ for $\varphi \in W(\Theta)$. □

Recall that, if ρ_Θ denotes the tensor product of the fundamental representations $\{\rho_\theta, \theta \in \Theta\}$, the for any $\gamma \in \Gamma$ we have

$$(\mu_1 - \mu_2)(\rho_\Theta(\gamma)) = \inf_{\theta \in \Theta} \theta(\mu(\gamma)) \ .$$

The following lemma describes the corresponding relation between critical exponents.

Lemma 2.40. *Let Γ be a Θ-Anosov subgroup of G. Then*

$$\delta_{1,2}(\rho_\Theta(\Gamma)) = \sup_{\theta \in \Theta} \delta_\theta(\Gamma) \ .$$

Proof. For every $\theta \in \Theta$, we have

$$\operatorname{Card}\{\gamma \in \Gamma \mid \theta(\gamma) \leq R\} \leq \operatorname{Card}\{\gamma \in \Gamma \mid (\mu_1 - \mu_2)(\rho_\Theta(\gamma)) \leq R\}$$

from which we deduce

$$\delta_\theta(\Gamma) \leq \delta_{1,2}(\rho_\Theta(\Gamma))$$

for all $\theta \in \Theta$. Moreover

$$\operatorname{Card}\{\gamma \in \Gamma \mid (\mu_1 - \mu_2)(\rho_\Theta(\gamma)) \leq R\} \leq \sum_{\theta \in \Theta} \operatorname{Card}\{\gamma \in \Gamma \mid \theta(\mu(\gamma)) \leq R\} \ ,$$

from which we get

$$\delta_{1,2}(\rho_\Theta(\Gamma)) \leq \sup_{\theta \in \Theta} \delta_\theta(\Gamma) \ .$$

□

2.4. **Projectively convex cocompact representations.**

2.4.1. *Hilbert geometries.* We recall in this section some classical facts on convex subsets of $\mathbf{P}(\mathbb{R}^n)$ and their Hilbert geometry. The main references for this are Benoist [Ben01], Crampon [Cra09, Cra11], Dancinger–Guéritaud–Kassel [DGK17].

An open domain Ω of $\mathbf{P}(\mathbb{R}^n)$ is said to be *properly convex* if it is convex and bounded in some affine chart. Hilbert constructed a natural projective invariant distance on a properly convex domain in $\mathbf{P}(\mathbb{R}^n)$. To define it, let us choose an affine chart in which Ω is bounded. Given u and v two points in this affine chart, we denote by uv the length of the segment $[u, v]$.

Definition 2.41. *Let x, y be two points in Ω. Let a and b denote respectively the intersections of the half lines $[y, x)$ and $[x, y)$ with $\partial\Omega$. Then the Hilbert distance between x and y is given by*

$$d_H(x, y) = \frac{1}{2} \log \left(\frac{xb \cdot ay}{ax \cdot yb} \right) .$$

This distance actually does not depend on the chosen affine chart (it is essentially the logarithm of a projective cross-ratio), and if a projective transformation maps Ω to Ω', then it induces an isometry between the Hilbert distances of Ω and Ω'. In particular, the group of projective transformations preserving Ω acts by isometries for the Hilbert distance.

The Hilbert distance is induced by a Finsler metric for which straight lines are geodesic.

Example 2.42. *Let $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ be the set of lines in $\mathbb{R}^n$ in restriction to which the quadratic form*

$$\mathbf{q}(x_1, \ldots, x_n) \mapsto x_1^2 + \ldots + x_{n-1}^2 - x_n^2$$

is negative. Ω identifies with the symmetric space of the group $\mathrm{SO}(n-1, 1)$ of linear transformations preserving $\mathbf{q}$, that is, the hyperbolic space of dimension $n-1$. In that case, the Hilbert distance on Ω is induced by the $\mathrm{SO}(n-1, 1)$-invariant Riemannian metric of constant curvature -1. In particular, Ω is a hyperbolic convex domain.

Example 2.43. *Let $E \simeq \mathbb{R}^{k(k+1)/2}$ be the space of quadratic forms on $\mathbb{R}^k$ and let $\Omega \subset \mathbf{P}(\mathbb{R}^{k(k+1)/2})$ be the projectivization of the cone of positive definite quadratic forms. Then the group $\mathrm{SL}(k, \mathbb{R})$ acts transitively on Ω, and Ω identifies with the symmetric space $\mathrm{SL}(k, \mathbb{R})/\mathrm{SO}(k)$. In that case, the Hilbert distance on Ω is related to the Cartan projection in the following way:*

$$d_H(x, y) = \varepsilon_1(\mu(x, y)) - \varepsilon_n(\mu(x, y)) .$$

Note that, for $k \geq 3$, Ω is not hyperbolic.

Definition 2.44. [DGK17] *If $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ is a proper convex domain with strictly convex and $\mathcal{C}^1$ boundary, and a discrete group $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ acts on Ω, we denote by $\Lambda_\Omega^{\mathrm{orb}}(\Gamma) \subset \partial\Omega$ the set of accumulation points of the orbit of any point in Ω. We say that a discrete subgroup Γ of $\mathrm{SL}(n, \mathbb{R})$ acts* convex-cocompactly *on Ω if Γ preserves Ω and acts cocompactly on the convex hull of $\Lambda_\Omega^{\mathrm{orb}}(\Gamma)$.*

We say that Γ is strongly projectively convex-cocompact *it acts convex-cocompactly on a properly convex domain Ω which is strictly convex with $\mathcal{C}^1$ boundary.*

Example 2.45. *If* $\Gamma \subset \mathrm{SO}(n-1,1)$ *is a convex-cocompact group of hyperbolic isometries, then it preserves the convex domain* $\Omega \simeq \mathbb{H}^{n-1}$ *introduced in Example 2.42 and acts properly discontinuously on its convex core* $C \subset \Omega$*. It is thus projectively convex-cocompact.*

Theorem 2.46. [DGK17] *Let* Γ *be a discrete subgroup of* $\mathrm{SL}(n,\mathbb{R})$*. If* Γ *is strongly projectively convex-cocompact, then* Γ *is projective Anosov.*

More precisely, we have the following description of the boundary maps ξ and ξ^* associated to Γ:

Theorem 2.47. [DGK17] *Let* Γ *be a discrete subgroup of* $\mathrm{SL}(n,\mathbb{R})$ *preserving a Gromov hyperbolic convex domain* Ω *and acting properly discontinuously and cocompactly on a non-emply convex set* $C \subset \Omega$*. Let* $\overline{C}$ *denote the closure of* C *in* $\mathbf{P}(\mathbb{R}^n)$*. Then* Γ *is projective Anosov and*

- *the boundary map* ξ *is a homeomorphism from* $\partial_\infty\Gamma$ *to* $\overline{C} \cap \partial\Omega$*,*
- *for every* $x \in \partial_\infty\Omega$*,* $\xi^*(x)$ *is the hyperplane tangent to* $\partial\Omega$ *at* $\xi(x)$*.*

Conversely, Danciger–Guéritaud–Kassel prove that a projective Anosov subgroup of $\mathrm{SL}(n,\mathbb{R})$ is strongly projectively convex-cocompact as soon as it preserves a proper convex domain. In particular, we have the following:

Theorem 2.48. [DGK17] *Let* $\iota : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(\frac{n(n+1)}{2},\mathbb{R})$ *be the representation given by the action of* $\mathrm{SL}(n,\mathbb{R})$ *on the space of quadratic forms on* $\mathbb{R}^n$*. If* $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ *is projective Anosov, then* $\iota(\Gamma) \subset \mathrm{SL}(\frac{n(n+1)}{2},\mathbb{R})$ *is strongly projetively convex cocompact.*

Remark 2.49. *The adjective "strongly" is here to distinguish the notion from a weaker notion of convex-cocompactness that includes discrete subgroups that are not hyperbolic. We omit it from now on.*

2.4.2. *Hilbert entropy and critical exponent.* Let Γ be a discrete subgroup of $\mathrm{SL}(n,\mathbb{R})$, preserving a proper convex subset $\Omega \subset \mathbf{P}(\mathbb{R}^n)$. Then Γ is a subgroup of isometries of (Ω, d_H) where d_H is the Hilbert distance on Ω. We denote the critical exponent associated to this metric by δ_H:

$$\delta_H = \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{Card}\{\gamma \in \Gamma \,|\, d_H(x,\gamma x) \leq R\}.$$

Remark 2.50. *Any projective Anosov representation in* $\mathrm{SL}(n,\mathbb{R})$ *preserves a proper, strictly convex subset in* $\mathbf{P}(\mathbb{R}^{n(n-1)/2})$ *as explained by Example 2.45 and Theorem 2.48, and we can look at this critical exponent. For a subgroup* $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ *which is strongly convex cocompact, we can look at the proper convex subset preserved in* $\mathbf{P}(\mathbb{R}^n)$ *or the one* $\mathbf{P}(\mathbb{R}^{n(n-1)/2})$*. The two choices give two Hilbert critical exponents, which only differ by a factor 2.*

Let Γ be a strongly convex cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$ and $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ be a proper convex subset preserved by Γ. There is a one-to-one correspond between the set of conjugacy classes $[\Gamma]$ and the closed geodesics of Ω/Γ. For a conjugacy class $[\gamma] \in [\Gamma]$ we denote by $\ell_H(\gamma)$ be the length of the corresponding closed geodesic for the Hilbert metric on Ω/Γ. The exponential growth of the number of closed geodesics is denoted by h_H: $h_H := \limsup_{R\to\infty} \frac{1}{R} \log \mathrm{Card}\{[\gamma] \in [\Gamma] \,|\, \ell_H(\gamma) \leq R\}$.

The work of Coornaert–Knieper on growth rate of conjugacy classes in Gromov hyperbolic groups has the following consequence:

Theorem 2.51 (Coornaert – Knieper, [CK02])**.** *Let Γ be a strongly convex cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$ and $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ be a proper convex subset preserved by Γ, then:*

$$\delta_H(\Gamma) = h_H(\Gamma).$$

For any element $\gamma \in \Gamma$ we can compute the length of the closed geodesic corresponding to $[\gamma]$ in the quotient manifold. A direct computation shows that γ acts by translation on the geodesic joining γ^- to γ^+ and the translation distance is given by $\ell_H(\gamma) := \frac{1}{2}(\varepsilon_1((\lambda)(\gamma)) - \varepsilon_n((\lambda)(\gamma)))$. Therefore one has:

Corollary 2.52. *Let Γ be a strongly convex cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$ then;*

$$\delta_H(\Gamma) = 2\delta_{1,n}.$$

Remark 2.53. *As mentioned previously, when Γ is only supposed to be a projective Anosov subgroup of $\mathrm{SL}(n,\mathbb{R})$ then it can be seen as a strongly convex cocompact subgroup in $\mathrm{SL}(n(n+1)/2,\mathbb{R})$. In this case, the length of the closed geodesic corresponding to $\gamma \in \mathrm{SL}(n(n+1)/2,\mathbb{R})$ is given by $(\varepsilon_1 - \varepsilon_n)(\lambda(\gamma))$, and therefore $\delta_H(\Gamma) = \delta_{1,n}$.*

Let Γ be a discrete subgroup of $\mathrm{SL}(n,\mathbb{R})$ acting convex-cocompactly on a Gromov hyperbolic convex domain Ω.

Corollary 2.54. $\delta_H = 2\delta_{1n}$.

3. Lower bound

This section is devoted to the proof of the following lower bound on the Hausdorff dimension. For a projective Anosov subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$, we write

$$\Lambda_\Gamma = \xi(\partial_\infty(\Gamma)) \subset \mathbf{P}(\mathbb{R}^n)\ ,$$

$$\Lambda_\Gamma^* = \xi^*(\partial_\infty(\Gamma)) \subset \mathbf{P}(\mathbb{R}^{n*})$$

and

$$\Lambda^{\mathrm{sym}}(\Gamma) = (\xi,\xi^*)(\partial_\infty\Gamma) \subset \mathbf{P}(\mathbb{R}^n) \times \mathbf{P}(\mathbb{R}^{n*})\ .$$

Theorem 3.1. *Let Γ be a strongly projectively convex-cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$. Then we have*

$$2\delta_{1,n}(\Gamma) \leq \mathrm{DimH}(\Lambda_\Gamma^{\mathrm{Sym}})$$

The proof is divided into two parts. First, we use the Hilbert distance on a Γ-invariant proper convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ to establish the equality between $2\delta_{\alpha_{1,n}}(\Gamma)$ and the Hausdorff dimension of $\Lambda_\Gamma \subset \partial\Omega$ for Gromov's "quasi-distance" on $\partial\Omega$. Then, we compare this quasi-distance with a Riemannian distance on $\mathbf{P}(\mathbb{R}^n) \times \mathbf{P}(\mathbb{R}^{n*})$.

3.1. **Gromov metric on the boundary.** Let Γ be a strongly projectively convex-cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$. Let Ω be a Γ-invariant Gromov hyperbolic convex domain $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ and $\mathcal{C} \subset \Omega$ a closed Γ-invariant subset of Ω on which Γ acts cocompactly. Let d_Ω denote the Hilbert distance on Ω.

Recall that Theorem 2.47 states that Λ_Γ is the intersection of the closure of $\mathcal{C}$ in $\mathbf{P}(\mathbb{R}^n)$ with $\partial\Omega$ and that Λ_Γ^* is the set of hyperplanes tangent to Ω at a point in Λ_Γ.

Given $x \in \Omega$ and $\xi, \eta \in \partial\Omega$, we define the Gromov product $(\xi|\eta)_x$ by

$$(\xi|\eta)_x = \lim_{k\to+\infty} \frac{1}{2}[d_\Omega(x_k, x) + d_\Omega(y_k, x) - d_\Omega(x_k, y_k)],$$

where (x_k) and (y_k) are sequences in Ω such that $x_k \to \xi$ and $y_k \to \eta$. Then the Gromov distance between ξ and η is

$$d_x(\xi, \eta) = e^{-(\xi|\eta)_x}.$$

This definition makes sense for all $x \in \Omega$ and $\xi, \eta \in \partial\Omega$. However, in the rest of this discussion, we will only consider it for $x \in \mathcal{C}$ and $\xi, \eta \in \Lambda_\Gamma = \partial_\infty\mathcal{C}$ in order to apply the theory of Gromov hyperbolic spaces.

There is a small caveat to this definition, that d_x is not in general a distance. However, d_x^ε for $\varepsilon > 0$ small enough is. We can apply here the main result of [Coo93]:

Theorem 3.2 (Coornaert, [Coo93])**.** *Let (X, d) be a complete Gromov hyperbolic space, and let Γ be a discrete group of isometries that acts cocompactly on X. Fix $x \in X$, and consider the Gromov distance d_x on the visual boundary $\partial_\infty X$. For $\varepsilon > 0$ small enough, one has $\mathrm{DimH}((X, d_x^\varepsilon)) = \varepsilon\delta(\Gamma)$, where $\delta(\Gamma)$ is the critical exponent of the action of Γ on (X, d).*

Applying this result to $(X, d) = (\mathcal{C}, d_\Omega)$, we get that $\frac{1}{\varepsilon}\mathrm{DimH}(\Lambda_\Gamma, d_x^\varepsilon) = \delta_H(\Gamma)$. By Corollary 2.54, we get $\frac{1}{\varepsilon}\mathrm{DimH}(\Lambda_\Gamma, d_x^\varepsilon) = 2\delta_{1,n}(\Gamma)$.

We need to consider small enough powers of d_x to get a distance, but the theorem states in particular that $\frac{1}{\varepsilon}\mathrm{DimH}((X, d_x^\varepsilon))$ does not depend on ε (which also followed from Proposition 2.1). Alternatively, one could consider coverings of Λ_Γ by "balls" for the function d_x, where we call balls the sets $B_x(\xi, r) := \{\eta \in \partial\Omega,\ d_x(\xi, \eta) \leq r\}$. Mimicking the definition of Hausdorff dimension, we would get a non negative real number $\mathrm{DimH}(\Lambda_\Gamma, d_x)$ such that $\mathrm{DimH}(\Lambda_\Gamma, d_x) = \frac{1}{\varepsilon}\mathrm{DimH}(\Lambda_\Gamma, d_x^\varepsilon)$ for all $\varepsilon > 0$. We could then rephrase Coornaert's theorem in our case in the following more synthetic way:

Proposition 3.3. *Let Γ be a strongly projectively convex-cocompact subgroup of $\mathrm{SL}(n,\mathbb{R})$, and let $\Omega \subset \mathbf{P}(\mathbb{R}^n)$ be a Γ-invariant Gromov hyperbolic convex domain on which Γ acts convex-cocompactly. For any $x \in \Omega$, we have:*

$$\mathrm{DimH}(\Lambda_\Gamma, d_x) = 2\delta_{1,n}(\Gamma).$$

3.2. **Gromov distance VS Euclidean distance.** We keep the same notations as in the previous subsection. We now wish to show that $\mathrm{DimH}(\Lambda_\Gamma, d_x) \leq \mathrm{DimH}(\Lambda_\Gamma^{\mathrm{sym}})$. For $p \in \partial\Omega$, let $p^* = T_p\partial\Omega \in \mathbf{P}(\mathbb{R}^{n*})$. The required inequality will easily follow from the following comparison lemma:

Lemma 3.4. *Given d and d^* Riemannian distances on $\mathbf{P}(\mathbb{R}^n)$ and $\mathbf{P}(\mathbb{R}^{n*})$, there is a constant $C > 0$ such that:*

$$\forall p, q \in \partial\Omega\ d_x(p, q) \leq C\sqrt{d(p, q)d^*(p^*, q^*)}$$

Proof. First of all, since Ω and Ω^* are proper convex sets, we can assume without loss of generality that d and d^* are Euclidean distances in affine charts in which Ω and Ω^* are bounded.

Consider sequences $p_n \in [xp)$, $q_n \in [xq)$ that converge to p and q respectively. Consider p^- (resp. q^-) the other intersection point between $\partial\Omega$ and the projective line (xp) (resp. (xq)). Finally, consider $a_n, b_n \in \partial\Omega$ the endpoints of the geodesic joining p_n and q_n (see Figure 1).

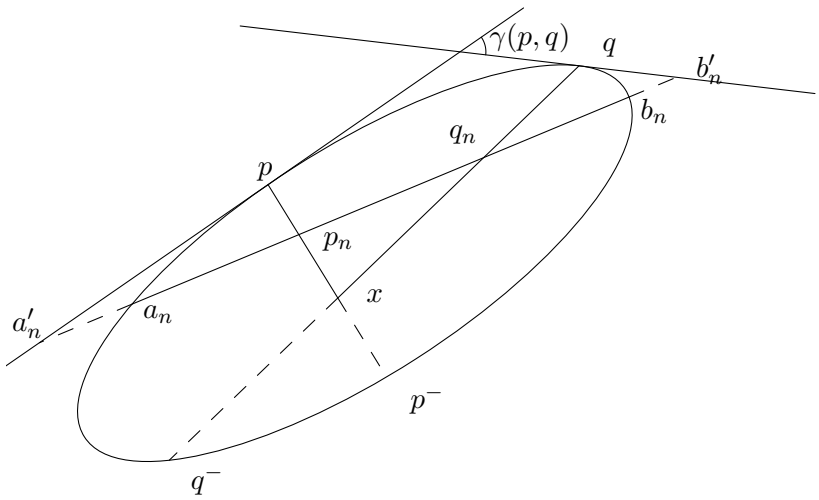

FIGURE 1. Computing the Gromov product

We have that:

$$\begin{aligned}(p_n|q_n)_x &= \frac{1}{2}\left(d_\Omega(x,p_n) + d_\Omega(x,q_n) - d_\Omega(p_n,q_n)\right)\\ &= \frac{1}{4}\mathrm{Log}\left(\frac{p^-p_n\cdot px}{p^-x\cdot pp_n}\cdot\frac{q^-q_n\cdot qx}{q^-x\cdot qq_n}\cdot\frac{a_np_n\cdot b_nq_n}{a_nq_n\cdot b_np_n}\right)\\ &= \frac{1}{4}\mathrm{Log}\left(\frac{a_np_n\cdot b_nq_n}{pp_n\cdot qq_n}\right) + \underbrace{\frac{1}{4}\mathrm{Log}\frac{1}{a_nq_n\cdot b_np_n}}_{\to -\mathrm{Log}\sqrt{d(p,q)}} + \underbrace{\frac{1}{4}\mathrm{Log}\left(\frac{p^-p_n\cdot px\cdot q^-q_n\cdot qx}{p^-x\cdot q^-x}\right)}_{\text{bounded}}\end{aligned}$$

This gives us a constant $C_1 > 0$ such that:

$$e^{-(p_n|q_n)_x} \leq C_1\sqrt{pq}\left(\frac{pp_n}{a_np_n}\cdot\frac{qq_n}{b_nq_n}\right)^{\frac{1}{4}}$$

In order to deal with the terms $\frac{pp_n}{a_np_n}$ and $\frac{qq_n}{b_nq_n}$, we consider the affine plane $\mathcal{P}_{p,q}$ containing x, p, q. Note that it also contains all the points defined above.

Denote by a'_n (resp. b'_n) the intersection of the line (p_na_n) with the tangent space to $\partial\Omega$ at p. Note that $\frac{a_np_n}{a'_np_n} \to 1$ as $n \to +\infty$, so that we can work

with $\frac{pp_n}{a'_n p_n}$ instead of $\frac{pp_n}{a_n p_n}$.

Now look at the triangle $a'_n p_n p$, denote by α_n the angle at a'_n, and $\theta(p)$ the angle at p, $\theta(p)$, does not depend on n as it is the angle between the line (xp) and the tangent line $T_p\partial\Omega \cap \mathcal{P}_{p,q}$ (see Figure 2). The law of sines gives us $\frac{pp_n}{a'_n p_n} = \frac{\sin\alpha_n}{\sin\theta(p)}$.

We now consider the triangle $b'_n q_n q$, and denote by β_n the angle at b'_n, and $\varphi(q)$ the angle at q. Just as in the previous case, we get $\frac{qq_n}{b'_n q_n} = \frac{\sin\beta_n}{\sin\varphi(q)}$.
We now find:

$$e^{-(p_n|q_n)_x} \leq \frac{C_1}{(\sin\theta(p)\sin\varphi(q))^{\frac{1}{4}}}\sqrt{pq}\,(\sin\alpha_n \sin\beta_n)^{\frac{1}{4}}$$

Notice that the sequence (α_n) (resp. (β_n)) has a limit $\alpha(p,q)$ (resp. $\beta(p,q)$) which is the angle at p (resp. at q) between the line (pq) and $T_p\partial\Omega \cap \mathcal{P}_{p,q}$ (resp. $T_q\partial\Omega \cap \mathcal{P}_{p,q}$).

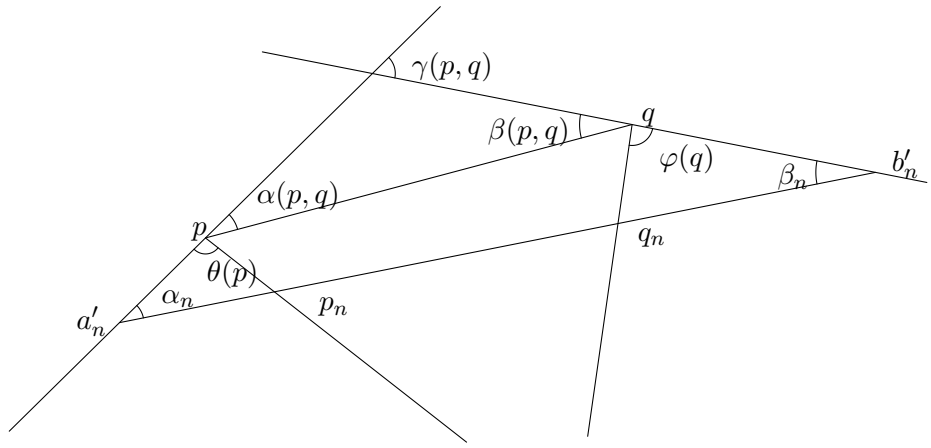

FIGURE 2.

We thus obtain:

$$d_x(p,q) \leq \frac{C_1}{(\sin\theta(p)\sin\varphi(q))^{\frac{1}{4}}}\sqrt{pq}\,(\alpha(p,q)\beta(p,q))^{\frac{1}{4}}$$

The function θ is continuous on the compact set $\partial\Omega$ (because Ω has $\mathcal{C}^1$ boundary), and never vanishes (because x is in the interior of Ω), hence is bounded away from 0. The same goes for φ (notice that $\varphi(q) = \pi - \theta(q)$), and we can thus find a constant $C_2 > 0$ such that:

$$d_x(p,q) \leq C_2\sqrt{pq}(\alpha(p,q)\beta(p,q))^{\frac{1}{4}}\ .$$

Consider now the exterior angle $\gamma(p,q)$ between the lines $T_p\partial\Omega \cap \mathcal{P}_{p,q}$ and $T_q\partial\Omega \cap \mathcal{P}_{p,q}$.

Notice that we have $\alpha(p,q)+\beta(p,q)=\gamma(p,q)$, so we can find an inequality involving only the angle $\gamma(p,q)$:

$$d_x(p,q) \leq C_2\sqrt{pq}\sqrt{\gamma(p,q)}$$

Finally, there is a constant C_3 such that the angle $\gamma(p,q)$ between the lines $p^* \cap \mathcal{P}_{p,q}$ and $q^* \cap \mathcal{P}_{p,q}$ is smaller than $C_3 d^*(p^*,q^*)$. This gives the desired inequality. □

Let us now conclude the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.4, we have

$$d_x(p,q) \leq \frac{1}{\sqrt{2}}\sqrt{d(p,q)^2+d^*(p^*,q^*)^2}$$

for all $p,q \in \Lambda_\Gamma$. Since $\sqrt{d(p,q)^2+d^*(p^*,q^*)^2}$ is a Riemannian distance on a neigbourhood of $\Lambda^{\mathrm{sym}}(\Gamma) \subset \mathbf{P}(\mathbb{R}^n) \times \mathbf{P}(\mathbb{R}^{n*})$, we deduce that

$$\mathrm{DimH}(\Lambda_\Gamma, d_x) \leq \mathrm{DimH}(\Lambda^{\mathrm{sym}}(\Gamma)) \ .$$

Since

$$2\delta_{1,n}(\Gamma) = \mathrm{DimH}(\Lambda_\Gamma, d_x)$$

by Proposition 3.3, the theorem follows. □

Let us finally recall the following consequence for every projective Anosov group:

Corollary 3.5. *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a projective Anosov group. Then*

$$\delta_{1,n}(\Gamma) \leq \mathrm{DimH}(\Lambda^{\mathrm{sym}}(\Gamma)) \ .$$

Proof. Let ρ be the representation of $\mathrm{SL}(n,\mathbb{R})$ into $\mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^n))$. Then $\rho(\Gamma)$ is projective Anosov and preserves the proper convex cone of positive definite quadratic forms in $\mathrm{Sym}^2(\mathbb{R}^n)$. Thus $\rho(\Gamma)$ is projectively convex cocompact by Theorem 2.48. Applying Theorem 3.1, we obtain

$$\delta_{1,n}(\Gamma) = 2\delta_{1,n(n+1)/2}(\rho(\Gamma)) \leq \mathrm{DimH}(\Lambda^{\mathrm{sym}}(\Gamma)) \ .$$

□

4. Upper bound

In this section we prove the upper inequality for the Hausdorff dimension of the limit set of a general projective Anosov subgroup:

Theorem 4.1. *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a projective Anosov subgroup. Then*

$$\mathrm{DimH}(\Lambda_\Gamma) \leq \delta_{1,2}(\Gamma) \ .$$

Applying this theorem to $\rho(\Gamma)$ where $\rho : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(\mathbb{R}^n \otimes \mathbb{R}^{n*})$ is the tensor product of the standard representation with its dual, we obtain the *a priori* stronger inequality:

Corollary 4.2. *Let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a projective Anosov subgroup. Then*

$$\mathrm{DimH}(\Lambda_\Gamma^{\mathrm{sym}}) \leq \delta_{1,2}(\Gamma) \ .$$

Proof of Corollary 4.2 assuming Theorem 4.1. By Theorem 4.1, we have

$$\mathrm{DimH}(\Lambda_\Gamma^{\mathrm{sym}}) \leq \delta_{1,2}(\rho(\Gamma)) \ .$$

By Proposition 2.21, we have

$$(\mu_1 - \mu_2)(\rho(\gamma)) = \inf\{\mu_1 - \mu_2)(\gamma), (\mu_{n-1} - \mu_n)(\gamma)\}$$

for all $\gamma \in \Gamma$, hence

$$\delta_{1,2}(\rho(\Gamma)) = \sup\{\delta_{1,2}(\Gamma), \delta_{n-1,n}(\Gamma)\}$$

by Lemma 2.40. Finally, since $(\mu_{n-1} - \mu_n)(\gamma) = (\mu_1 - \mu_2)(\gamma^{-1})$, we have

$$\delta_{1,2}(\Gamma) = \delta_{n-1,n}(\Gamma) \ .$$

We conclude that

$$\delta_{1,2}(\rho(\Gamma)) = \delta_{1,2}(\Gamma)$$

and the corollary follows. □

Let us now turn to the proof of Theorem 4.1. The main technical tool for the proof is Lemma 4.4 that quantifies the distortion of balls by proximal elements. The second part of the proof presents a covering of the limit set by images of a fixed ball in order to obtain the upper bound.

4.1. **Distortion of balls by loxodromic elements.** For a proximal element $g \in \mathrm{SL}(n, \mathbb{R})$ we denote by $g^+ \in \mathbf{P}(\mathbb{R}^n)$ its attractive line and by $H^-(g) \in \mathbf{P}(\mathbb{R}^{n*})$ its repelling hyperplane.

We endow $\mathbf{P}(\mathbb{R}^n)$ with the Riemannian metric that lifts to the round metric on $\mathbb{S}^{n-1}$.

Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$. If $g \in \mathrm{SL}(n, \mathbb{R})$ is proximal and $g^+ = [e_1]$ and $H^-(g) = \mathrm{span}(e_2, ..., e_n)$, then the norm of the restriction of g to $H^-(g)$ is $e^{\mu_2(g)}$. A simple computation then gives the following estimate for the distortion of balls in $\mathbf{P}(\mathbb{R}^n)$ under the action of g.

Lemma 4.3. *There are $r_0 > 0$ and $L > 0$ such that, for all $g \in \mathrm{SL}(n, \mathbb{R})$ proximal satisfying $g^+ = [e_1]$ and $H^-(g) = \mathrm{span}(e_2, ..., e_n)$, and for all $r \in (0, r_0)$, we have*

$$g \cdot B([e_1], r) \subset B([e_1], Lre^{(\mu_2 - \mu_1)(g)}).$$

Fix $\varepsilon > 0$ smaller than the r_0 given by Lemma 4.3. Let $B(x, \varepsilon)$ be the ball of radius ε about $x \in \mathbf{P}(\mathbb{R}^n)$.

Let $\Gamma_{x,\varepsilon}$ be the set of elements $\gamma \in \Gamma$ that are ε-proximal and such that $\gamma^+ \in B(x, \varepsilon)$. The following lemma controls the distortion of balls by elements of $\Gamma_{x,\varepsilon}$.

Lemma 4.4. *$\exists C > 0$ such that for all $x \in \mathbf{P}(\mathbb{R}^n)$, all $\varepsilon \in (0, r_0)$ and all $\gamma \in \Gamma_{x,\varepsilon}$, the diameter of $\gamma \cdot B(x, \varepsilon)$ is at most $C\varepsilon e^{\mu_2(\gamma) - \mu_1(\gamma)}$.*

Proof. Since the metric on $\mathbf{P}(\mathbb{R}^n)$ is $\mathrm{SO}(n, \mathbb{R})$ invariant and multiplication by $\mathrm{SO}(n, \mathbb{R})$ does not change the Cartan projections, we can assume that $x = [e_1]$.

The subset of $\mathbf{P}(\mathbb{R}^n) \times \mathbf{P}(\mathbb{R}^{n*})$ consisting of pairs of a line ℓ and a hyperplane H such that $d(\ell, H) \geq \varepsilon$ and $d(\ell, [e_1]) \leq \varepsilon$ is a compact subset of the set of pairs (ℓ, H) which are in general position. Since $\mathrm{SL}(n, \mathbb{R})$ is locally compact and acts transitively on this latter set, there exists a compact set

$K' \subset \mathrm{SL}(n,\mathbb{R})$ such that for all $\gamma \in \Gamma_{[e_1],\varepsilon}$, there exists $k \in K'$ such that
$\begin{cases} k \cdot \gamma^+ = (k\gamma k^{-1})^+ = [e_1] \\ k \cdot H^-(\gamma) = H^-(k\gamma k^{-1}) = \mathrm{span}(e_2, ..., e_n) \end{cases}$

Take $\gamma \in \Gamma_{[e_1],\varepsilon}$ and let $k \in K'$ be as above. We have

$$\gamma \cdot B(e_1, \varepsilon) \subset \gamma \cdot B(\gamma^+, 2\varepsilon) = k^{-1}(k\gamma k^{-1})k \cdot (B(\gamma^+, 2\varepsilon)).$$

By compactness of K' there exists $C_1 > 1$ such that for all $k \in K'$, all $\xi \in B(e_1, \varepsilon)$ and all $r > 0$, we have $k \cdot B(\xi, r) \subset B(k\xi, C_1 r)$. Therefore:

$$\gamma \cdot B(e_1, \varepsilon) \subset k^{-1}(k\gamma k^{-1})B(e_1, 2C_1\varepsilon).$$

By Lemma 4.3:

$$(k\gamma k^{-1})B(e_1, 2C_1\varepsilon) \subset B(e_1, 2C_1 L\varepsilon e^{(\mu_2-\mu_1)(k\gamma k^{-1})}).$$

By Proposition 2.4 , there exists $C_2 > 1$ such that for all $k \in K'$:

$$|(\mu_2 - \mu_1)(k\gamma k^{-1}) - (\mu_2 - \mu_1)(\gamma)| \leq \log(C_2).$$

In particular, $\gamma \cdot B(e_1, \varepsilon) \subset k^{-1}B(e_1, 2C_1 L C_2 \varepsilon e^{(\mu_2-\mu_1)(\gamma)})$

By using compactness of K' another time we have:

$$\gamma \cdot B(e_1, \varepsilon) \subset B(\gamma^+, 2C_1^2 L C_2 \varepsilon e^{(\mu_2-\mu_1)(\gamma)}),$$

this concludes the proof of Lemma 4.4.

□

4.2. **Proof of the upper bound.** To prove an upper bound on Hausdorff dimension it is sufficient to find a good cover of the sets. We show in the lemma that Λ can be covered by translate of a ball by some particular proximal elements

Let B and B' be two open balls of $\mathbf{P}(\mathbb{R}^n)$ with disjoint closure and intersecting Λ. For every $\varepsilon > 0$, define Γ_0^ε to be the set of elements $g \in \Gamma$ such that

- $g({}^cB') \subset B$,
- $g(B)$ has diameter less than ε.

One easily verifies that Γ_0^ε is a semi-group.

Lemma 4.5. *For any $\varepsilon > 0$, $\cup_{\gamma \in \Gamma_0^\varepsilon} \gamma \cdot B$ is a covering of $\Lambda \cap B$.*

Proof. Set $O^\varepsilon = \cup_{\gamma \in \Gamma_0^\varepsilon} \gamma \cdot B$. By definition it is a Γ_0^ε-invariant open subset of B such that $\Lambda_\Gamma \cap O^\varepsilon \neq \emptyset$. The set $C^\varepsilon = \Lambda_\Gamma \cap (B \setminus O^\varepsilon)$ is closed in B and Γ_0^ε-invariant. We want to prove that C^ε is empty. Assume by contradiction that it is not the case, and pick $c \in C$.

Let $\gamma \in \Gamma$ be a proximal element such that $\gamma_- \in B'$ and $\gamma_+ \in B$. Then for k large enough, γ^k belongs to Γ_0^ε. Since $c \neq \gamma_-$, $\gamma^k c$ converges to γ_+ as k goes to $+\infty$. Since C^ε is Γ_0^ε-invariant and closed, we obtain that $\gamma_+ \in C^\varepsilon$.

By Corollary 8.2.G of [Gro87], the set of pairs (γ^+, γ^-) of elements $\gamma \in \Gamma$ is dense in $\Lambda_\Gamma \times \Lambda_\Gamma$. We thus obtain that $C^\varepsilon = \Lambda_\Gamma \cap B$, contradicting the fact that $\Lambda_\Gamma \cap O^\varepsilon$ is non-empty. □

We now have all the tools to prove Theorem 4.1.

Proof of Theorem 4.1. By compactness of Λ_Γ it is sufficient to prove that the Hausdorff dimension of $\Lambda \cap B$ is less than $\delta_{1,2}(\Gamma)$ for any ball B of radius less than r_0 given by Lemma 4.3.

Fix such a ball B, and let $r > 0$ be its radius. Consider a ball $B' \subset \mathbf{P}(\mathbb{R}^n)$ whose closure is disjoint from $\overline{B}$. Therefore there exists $\eta > 0$ such that every element $\gamma \in \Gamma$ with $\gamma_- \in B'$ and $\gamma_+ \in B$ is η-proximal. Let $C > 0$ be given by Lemma 4.4.

By Lemma 4.5, we have $\Lambda_\Gamma \cap B \subset \cup_{\gamma \in \Gamma_0^\varepsilon} \gamma \cdot B$. By definition of Γ_0^ε, this gives a covering of $\Lambda_\Gamma \cap B$ by balls of radius less than ε.

Let $s > 0$. By definition of s-dimensional Hausdorff measure, we have :

$$H_\varepsilon^s(\Lambda_\Gamma \cap B) \leq \sum_{\gamma \in \Gamma_0^\varepsilon} \operatorname{diam}(\gamma \cdot B(r))^s.$$

Hence by Lemma 4.4,

$$\begin{aligned} H_\varepsilon^s(\Lambda_\Gamma \cap B(r)) &\leq C^s r^s \sum_{\gamma \in \Gamma_0 \setminus F_\varepsilon} e^{s(\mu_2(\gamma) - \mu_1(\gamma))} \\ &\leq C^s r^s \sum_{\gamma \in \Gamma} e^{-s(\mu_1 - \mu_2)(\gamma)} \end{aligned}$$

Since ε can be taken arbitrarily small, we obtain :

$$H^s(\Lambda_\Gamma \cap B) \leq C^s r^s \sum_{\gamma \in \Gamma} e^{-s(\mu_1 - \mu_2)(\gamma)} \ .$$

Therefore, for all $s > \delta_{\alpha_{1,2}}$, $H^s(\Lambda_\Gamma \cap B) < +\infty$ which in turns implies that $\mathrm{DimH}(\Lambda_\Gamma \cap B) \leq \delta_{\alpha_{1,2}}$.

□

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