



General uniqueness results for large solutions

Julián López-Gómez, Luis Maire, Laurent Veron

► To cite this version:

Julián López-Gómez, Luis Maire, Laurent Veron. General uniqueness results for large solutions. *Zeitschrift für Angewandte Mathematik und Physik*, 2020, 71:109, pp.1-14. 10.1007/s00033-020-01325-5 . hal-02419307

HAL Id: hal-02419307

<https://hal.science/hal-02419307>

Submitted on 19 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

General uniqueness results for large solutions

Julián López-Gómez* Luis Maire† Laurent Véron‡

Abstract We give a series of very general sufficient conditions in order to ensure the uniqueness of large solutions for $-\Delta u + f(x, u) = 0$ in a bounded domain Ω where $f : \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}_+$ is a continuous function, such that $f(x, 0) = 0$ for $x \in \overline{\Omega}$, and $f(x, r) > 0$ for x in a neighborhood of $\partial\Omega$ and all $r > 0$.

2010 Mathematics Subject Classification. 35 J 61; 31 B 15; 28 C 05 .

Key words: Keller-Osserman condition; local graph condition .

Contents

1	Introduction	1
2	Proof of Theorem 1.1	7
3	Proof of Theorem 1.2	11
4	Appendix	13
4.1	On the Keller-Osserman condition	13
4.2	On the strong barrier property	13

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}_+$ a continuous function such that $f(x, 0) = 0$ and $r \mapsto f(x, r)$ is nondecreasing for $x \in \overline{\Omega}$, and $f(x, r) > 0$ for x in a neighborhood of $\partial\Omega$ and all $r > 0$. This paper deals with the uniqueness question of the solution of the equation

$$-\Delta u + f(x, u) = 0 \quad \text{in } \Omega, \quad (1.1)$$

*Instituto de Matemática Interdisciplinar (IMI), Departamento de Analisis Matemático y Matemática Aplicada, Universidad Complutense de Madrid, 28040-Madrid, Spain.

Email: Lopez-Gomez@mat.ucm.es

†Departamento de Matemática Aplicada, Ciencia e Ingeniería de los Materiales y Tecnología Electrónica, Spain.

Email: luis.maire@urjc.es

‡Département de Mathématiques, Université François Rabelais, Tours, France.

Email: veronl@lmpt.univ-tours.fr

satisfying the blow-up condition

$$\lim_{d(x) \rightarrow 0} u(x) = \infty, \quad (1.2)$$

where $d(x) := \text{dist}(x, \partial\Omega)$. Whenever a solution to (1.1)-(1.2) exists it is called a *large solution* or an *explosive solution*. Although, thanks to [21, Corollary 3.3], in the one-dimensional case $N = 1$ with $f(x, u) \equiv f(u)$ the above problem admits a unique solution, the question of ascertaining whether or not (1.1)-(1.2) possesses a unique solution received only partial answers even in the autonomous case when $f(x, u) \equiv f(u)$ is independent of $x \in \Omega$. Astonishingly, when $N = 1$ the large solution can be unique even when $f(u)$ is somewhere decreasing (see [21] and [20]), which measures the real level of difficulty of the problem of characterizing the set of $f(x, u)$ for which (1.1)-(1.2) has a unique positive solution; it is an extremely challenging problem.

Existence of large solutions is associated to the *Keller–Osserman condition*. When f is independent of x , this condition was introduced in [11] and [27] for proving the first existence results of large solutions in a smooth bounded domain. It reads

$$\int_a^\infty \frac{ds}{\sqrt{F(s) - F(a)}} < \infty \quad \text{for some } a > 0 \quad \text{where } F(s) = \int_0^s f(t)dt. \quad (1.3)$$

When $f = f(x, r)$ a more general version called in this paper (KO-loc) is introduced in [13] and in [32]. It asserts that, for any compact subset K of Ω , there exists a continuous nondecreasing function $h_K : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$f(x, r) \geq h_K(r) \geq 0 \quad \text{for all } x \in K \text{ and } r \geq 0 \quad (1.4)$$

where h_K satisfies

$$\int_a^\infty \frac{ds}{\sqrt{H_K(s) - H_K(a)}} < \infty \quad \text{for some } a > 0 \quad \text{where } H_K(s) = \int_0^s h_K(t)dt. \quad (1.5)$$

The condition (KO-loc) guarantees the existence of a *maximal solution*, u^{max} , to equation (1.1). It is obtained as the limit of a decreasing sequence of large solutions $\{u_n\}_{n \in \mathbb{N}}$ in an increasing sequence of smooth domains $\{\Omega_n\}_{n \in \mathbb{N}}$ such that $\overline{\Omega_n} \subset \Omega$ and $\cup_{n \geq 1} \Omega_n = \Omega$ (see e.g. [13], [32], [24]). However, it is not always true that the maximal solution is a large solution. This property depends essentially of the regularity of the domain. If $f(x, u) = u^p$ with $p > 1$, the necessary and sufficient condition for such a property to hold is given by [12, 26]. The existence of a *minimal large solution* necessitates a minimum of assumptions, either on the regularity of Ω or on the function $f(x, r)$ (see [32]). Actually, if Ω is the interior of its closure there exists a decreasing sequence of smooth domains Ω'_n such that

$$\cap_{n \geq 1} \Omega'_n = \overline{\Omega}.$$

If f is defined in $\Omega' \times \mathbb{R}$ where Ω' is a neighborhood of $\overline{\Omega}$ with the same monotonicity and (KO-loc) properties therein as in $\overline{\Omega} \times \mathbb{R}$, and if $f(x, r)|_{\partial\Omega} > 0$, then a sequence of large solutions $\{u'_n\}_{n \in \mathbb{N}}$ can be constructed in Ω'_n and the limit, u' , of the $\{u'_n\}_{n \in \mathbb{N}}$ is a candidate for being the minimal large solution, u^{min} , since it remains smaller than any large solution in Ω . If $f(x, r)|_{\partial\Omega} = 0$, then the construction of the minimal solution is possible as soon as for any $n > 0$ (1.1) admits a solution with value n on $\partial\Omega$. For this a minimal regularity condition on $\partial\Omega$ is needed, the Wiener condition [10, p. 206]. Furthermore, because of the maximum principle and the fact that $f(x, 0) = 0$ and

$f(x, r) > 0$ for all $r > 0$ when x belongs to some neighborhood \mathcal{V} of $\partial\Omega$ the above (KO-loc) assumption can be weakened in the sense that the function $h_K : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (1.4) and (1.5) has to exist only when K is a compact subset of \mathcal{V} .

The main property of u^{max} and u^{min} is that any solution u of (1.1)-(1.2), should it exists, satisfies

$$u^{min}(x) \leq u(x) \leq u^{max}(x) \quad \text{for all } x \in \Omega.$$

The problem of uniqueness reduces to prove that $u^{max} = u^{min}$. The first results in this direction dealing with $f(x, u) = u^p$ for some $p > 1$, using the asymptotic expansion of any large solution, are proved in [1]. In this approach, the regularity of the boundary is a crucial assumption. The key point is to prove that

$$\lim_{d(x) \rightarrow 0} \frac{u^{max}(x)}{u^{min}(x)} = 1.$$

After this relation is obtained the uniqueness follows from the fact that there holds

$$((1 + \epsilon)r)^p \geq (1 + \epsilon)f(r) \quad \text{for all } \epsilon, r \geq 0.$$

For regular domains Ω , this technique was substantially refined in [15] and [17] to cover the non-autonomous case when $f(x, u) = a(x)u^p$ for some non-negative function $a(x)$ such that $a(x) > 0$ for sufficiently small $d(x)$ (see also [3], [4], [5] and [6]). The asymptotic expansion of a large solution near the boundary requiring so many assumptions, both on the nonlinearity f and on the regularity of $\partial\Omega$, that a new method was introduced in [22] in order to bypass this step. To apply that method the boundary has to satisfy the *local graph condition*, an assumption which is used also in this article. According to it, for every $P \in \partial\Omega$, there exist a neighborhood Q_P of P , a positive oriented basis, $\{\vec{v}_1, \dots, \vec{v}_N\}$, obtained from the canonical one by a rotation, and a function $F \in C(\mathbb{R}^{N-1}; \mathbb{R})$ such that

$$\begin{aligned} F(0, \dots, 0) &= 0, \\ Q_P \cap \Omega &= Q_P \cap (\{x \in \mathbb{R}^N : x_N < F(x_1, \dots, x_{N-1})\} + P), \end{aligned} \tag{1.6}$$

where the coordinates (x_1, \dots, x_N) in (1.6) are expressed with respect to the basis $\{\vec{v}_1, \dots, \vec{v}_N\}$ (see Figure 1.1). Naturally, $\partial\Omega$ satisfies the local graph property if it is Lipschitz continuous.

Similarly, in order to avoid the use of the asymptotic expansions of the large solutions near the boundary in the proof of the uniqueness, another technique was introduced in [16], and later refined in [2] and [19], in a radially symmetric context, based on the strong maximum principle. This technique, which works out even in the context of cooperative systems, [18], will be combined in this paper with the technique of [22] in order to get the new findings of this paper.

As far as concerns the nonlinearity $f(\cdot, r)$, in most of the previous papers it is imposed that its rate of decay (or blow-up) near $\partial\Omega$ is a precise function of $d(x)$ (see, e.g., [7, 9, 14, 15, 28, 29, 32, 33, 34, 35]). Throughout this paper it is assumed that $x \mapsto f(x, r)$ *decays completely nearby* $\partial\Omega$ in the sense that, for every $z \in \partial\Omega$, there exists $\delta > 0$ such that $|x - z| < \delta$ and $x \in \Omega$ imply

$$\begin{aligned} f(x, \ell + r) - f(x + \epsilon \vec{v}_N, \ell + r) &\geq f(x, \ell) - f(x + \epsilon \vec{v}_N, \ell) \geq 0 \\ \text{for all } r, \ell \geq 0, \text{ if } x + \epsilon \vec{v}_N &\in \overline{\Omega} \text{ and } \epsilon \in (0, \delta). \end{aligned} \tag{1.7}$$

where $\vec{v}_N = (0, \dots, 0, 1)$ if (1.6) holds. Note that this assumption is not intrinsic to the domain since it depends of the choice of the neighborhood Q_P and the frame $\{\vec{v}_1, \dots, \vec{v}_N\}$. In the special case

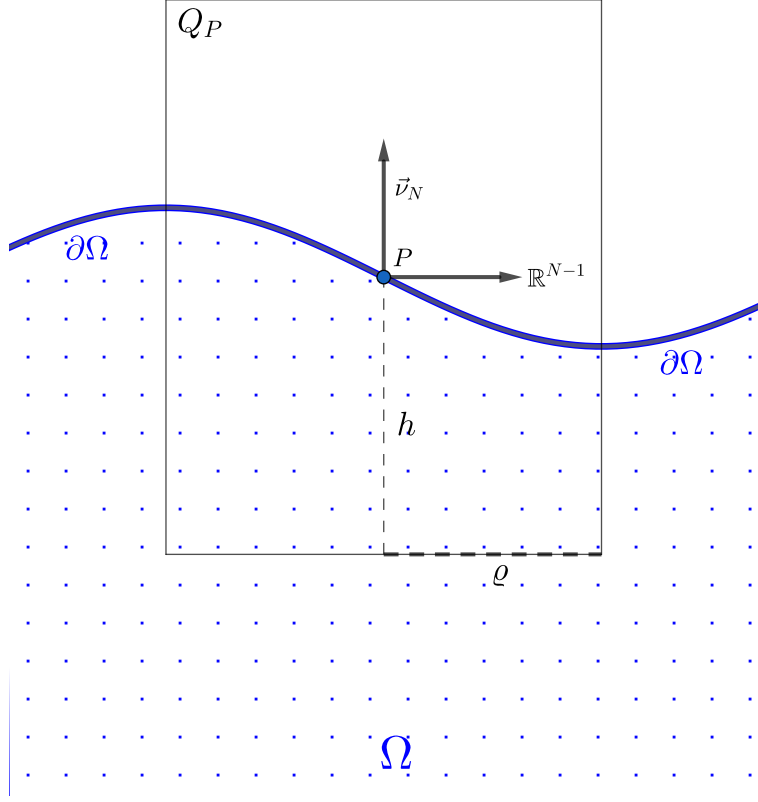


Figure 1.1: The neighborhood Q_P

when

$$f(x, r) = \mathfrak{a}(x) \tilde{f}(r)$$

where $\tilde{f} : \mathbb{R} \mapsto \mathbb{R}$ is monotone nondecreasing, positive on $(0, \infty)$ and vanishes at 0, and $\mathfrak{a} \in C(\overline{\Omega})$ is nonnegative and positive in a neighborhood of $\partial\Omega$, the assumption (1.7) holds if and only if $x \mapsto \mathfrak{a}(x)$ decays nearby $\partial\Omega$ in the sense that

$$0 \leq \mathfrak{a}(x + \epsilon \vec{\nu}_N) \leq \mathfrak{a}(x) \text{ if } x + \epsilon \vec{\nu}_N \in \overline{\Omega} \text{ and } \epsilon \in (0, \delta). \quad (1.8)$$

If Ω has a Lipschitz boundary, then there is a truncated circular cone $C_\gamma = C \cap B_\delta$ such that any point $P \in \partial\Omega$ is the vertex of the image $\mathcal{I}_P(C_\gamma)$ of C_γ by an isometry \mathcal{I}_P of \mathbb{R}^N and $\mathcal{I}_P(C_\gamma) \subset \Omega^c$. In such case, ν_N can be chosen to be the axis of rotational symmetry of C_γ .

In this paper, associated to $f(x, u)$, we consider the function g defined on $\overline{\Omega} \times \mathbb{R}_+$ by

$$g(x, \ell) := \inf \{ f(x, \ell + u) - f(x, u) : u \geq 0 \}, \quad \text{for all } (x, \ell) \in \overline{\Omega} \times \mathbb{R}_+. \quad (1.9)$$

There always holds $g \leq f$ and $g(x, \cdot)$ is monotone nondecreasing as $f(x, \cdot)$ is. Thus, if g satisfies (KO-loc), so does f , but the converse is not true in general as it is shown in the Appendix. Moreover,

if $f(x, \cdot)$ is convex for all $x \in \overline{\Omega}$, then $f = g$. This is due to the fact that

$$\begin{aligned} f(x, u + \ell) - f(x, u) &= \int_0^\ell \partial_u^r f(x, u + s) ds \\ &\geq \int_0^\ell \partial_u^r f(x, u' + s) ds = f(x, u' + \ell) - f(x, u') \end{aligned}$$

for all $u \geq u' \geq 0$, $\ell > 0$ and $x \in \overline{\Omega}$, since the right partial derivative $\partial_u^r f(x, u)$ of $u \mapsto f(x, u)$ is nondecreasing with u . Hence, the minimum of

$$u \mapsto f(x, u + \ell) - f(x, u)$$

is achieved at $u = 0$ and therefore, $f = g$. Finally, if f decays completely nearby $\partial\Omega$, then g also decays in the sense that

$$0 \leq g(x + \epsilon \vec{\nu}_N, r) \leq g(x, r) \text{ for all } r \geq 0, \text{ if } x + \epsilon \vec{\nu}_N \in \overline{\Omega} \text{ and } \epsilon \in (0, \delta). \quad (1.10)$$

Furthermore, taking $\ell = 0$ in (1.7), it becomes apparent that f satisfies the same inequality (1.10) as g .

The following equation

$$-\Delta u + g(x, u) = 0 \quad \text{in } \Omega, \quad (1.11)$$

closely related to (1.1) plays a fundamental role in our study. Following [23, Def. 2.6], we introduce the following concept.

Definition Let $z \in \partial\Omega$. We say that equation (1.11) possesses a strong barrier at z if there exists a number $r_z > 0$ such that, for every $r \in (0, r_z]$, there exists a positive supersolution $u = u_{r,z}$ of (1.11) in $\Omega \cap B_r(z)$ with

$$u_{r,z} \in C(\overline{\Omega} \cap B_r(z)) \quad \text{and} \quad \lim_{\substack{y \rightarrow x \\ y \in \Omega \cap B_r(z)}} u_{r,z}(y) = \infty \quad \text{for all } x \in \Omega \cap \partial B_r(z). \quad (1.12)$$

Notice that the local supersolution $u_{r,z}$ of (1.11) is also a supersolution of (1.1) since $g \leq f$. Our first result is the following.

Theorem 1.1 Suppose that Ω is Lipschitz continuous and $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfies $f(x, 0) = 0$, $r \mapsto f(x, r)$ is nondecreasing for all $x \in \overline{\Omega}$, and $f(\cdot, r)$ decays completely nearby $\partial\Omega$ as it is formulated in (1.7). Assume, in addition, that the function $g \in C(\overline{\Omega} \times \mathbb{R})$ defined from f by (1.9) is positive on a neighborhood \mathcal{V} of $\partial\Omega$ and satisfies (KO-loc); that is, for any compact subset $K \subset \mathcal{V}$ there exists a continuous nondecreasing function $h_K : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$g(x, r) \geq h_K(r) \geq 0 \quad \text{for all } x \in K \text{ and } r \geq 0, \quad (1.13)$$

where h_K satisfies (1.5). If the equation (1.1) possesses a strong barrier at any $z \in \partial\Omega$, then the problem (1.1)-(1.2) possesses a unique solution, i.e. $u^{\min} = u^{\max}$.

The assumption that g satisfies (KO-loc) is actually an assumption on f . Indeed, f must grow sufficiently fast at ∞ so that g still satisfies (KO-loc). This assumption is weaker than the superadditivity with constant C introduced in [24], according with it

$$f(x, u + \ell) \geq f(x, u) + f(x, \ell) - C \quad \text{for all } x \in \bar{\Omega} \text{ and } u, \ell \geq 0. \quad (1.14)$$

Under the superadditivity assumption, there holds, for any $\ell, u \geq 0$, that

$$g(x, \ell) \geq f(x, \ell + u) - f(x, u) \geq f(x, \ell) - C.$$

Therefore, if f satisfies (KO-loc), so does g . Our second result, valid under a weaker assumption on Ω , requires a new assumption on f .

Theorem 1.2 *Assume that Ω satisfies the local graph property and that the assumptions on f and g in Theorem 1.1 hold. Suppose, moreover, that there exists $\phi \in C^2(\mathbb{R}_+)$ such that $\phi(0) = 0$, $\phi(r) > 0$ for $r > 0$, and*

$$\phi'(r) \geq 0 \quad \text{and} \quad \phi''(r) \leq 0 \quad \text{for all } r \geq 0,$$

for which the function f verifies the inequality

$$\frac{f(x, r + \epsilon\phi(r))}{f(x, r)} \geq 1 + \epsilon\phi'(r) \quad \text{for all } r \geq 0 \text{ and } x \in \bar{\Omega}, \quad (1.15)$$

for some sufficiently small $\epsilon > 0$. Then, the problem (1.1)-(1.2) possesses at most one solution.

Although the assumption on (f, ϕ) may look unusual, it turns out that when $\phi(r) = r$ it is equivalent to

$$r \mapsto \frac{f(x, r)}{r} \quad \text{is nondecreasing on } (0, \infty), \quad (1.16)$$

which is the assumption used in [22]. Since (1.16) implies that

$$\frac{f(x, r + \epsilon \ln(1 + r))}{r + \epsilon \ln(1 + r)} \geq \frac{f(x, r)}{r}$$

or, equivalently,

$$\frac{f(x, r + \epsilon \ln(1 + r))}{f(x, r)} \geq 1 + \epsilon \frac{\ln(1 + r)}{r} \quad \text{for all } r \geq 0, \quad (1.17)$$

which entails

$$\frac{f(x, r + \epsilon \ln(1 + r))}{f(x, r)} \geq 1 + \frac{\epsilon}{1 + r} \quad \text{for all } r \geq 0, \quad (1.18)$$

and (1.18) is (1.15) for the special choice $\phi(r) = \ln(1 + r)$, it becomes apparent that (1.15) is substantially weaker than (1.16).

2 Proof of Theorem 1.1

Since Ω is a Lipschitz continuous bounded domain, it satisfies the local graph property at each point of the boundary. Let $P \in \partial\Omega$ and consider a basis $\{\vec{\nu}_1, \dots, \vec{\nu}_N\}$ and a neighborhood Q_P satisfying (1.6). Throughout this proof, it is assumed that any point of \mathbb{R}^N is expressed in coordinates with respect to $\{\vec{\nu}_1, \dots, \vec{\nu}_N\}$. Setting

$$\hat{x} := (x_1, \dots, x_{N-1}) \quad \text{for every } x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N,$$

and denoting by $\hat{B}_\varrho(P)$ the ball of \mathbb{R}^{N-1} with center $P = (\hat{P}, 0)$ and radius ϱ , we can assume that

$$Q_P = \{x \in \mathbb{R}^N : |\hat{x} - \hat{P}| < \varrho, |x_N - P_N| < h\} = \hat{B}_\varrho(P) \times (P_N - h, P_N + h)$$

for some $\varrho > 0, h > 0$ such that $\partial\Omega$ is bounded away from the “top” and the “bottom” of Q_P and

$$\partial\Omega \cap \overline{Q_P} = \overline{\partial\Omega \cap Q_P}$$

(see Figure 1.1). Thus, setting

$$\Theta_\epsilon := (Q_P \cap \Omega) - \epsilon \vec{\nu}_N, \quad \epsilon \geq 0, \quad (2.1)$$

the existence of $\epsilon_1 > 0$ such that

$$\overline{\Theta}_\epsilon \subset \Omega \quad \text{for all } 0 < \epsilon < \epsilon_1 \quad (2.2)$$

is guaranteed. Subsequently, we denote

$$\Gamma_{0,\epsilon} := (Q_P \cap \partial\Omega) - \epsilon \vec{\nu}_N, \quad \Gamma_{\infty,\epsilon} := (\partial Q_P \cap \Omega) - \epsilon \vec{\nu}_N, \quad \text{for all } \epsilon \geq 0, \quad (2.3)$$

(see Figure 2.2) and consider

$$\epsilon_0 := \min\{\epsilon_1, \delta\},$$

where δ is the one of (1.7). Then, the following lemma of technical nature holds.

Lemma 2.1 *Under the assumptions of Theorem 1.1, the problem*

$$\begin{cases} -\Delta \ell + g(x, \ell) = 0 & \text{in } \Theta_0 \\ \ell = 0 & \text{in } \Gamma_{0,0} \\ \ell = +\infty & \text{in } \Gamma_{\infty,0}, \end{cases} \quad (2.4)$$

admits, at least, a positive solution, ℓ .

Proof. As we are imposing that $\ell = 0$ on $\Gamma_{0,0}$, our singular boundary condition is reminiscent of those considered previously in [22, 24]. To construct ℓ one can argue as follows. First, consider any increasing sequence of nonnegative functions, $\{b_n\}_{n \in \mathbb{N}} \subset C^{0,1}(\partial\Theta_0)$, satisfying

$$\begin{cases} b_n(x) = 0 & \text{for all } x \in \Gamma_{0,0} \\ \lim_{n \rightarrow \infty} b_n(x) = +\infty & \text{for all } x \in \Gamma_{\infty,0}, \end{cases}$$

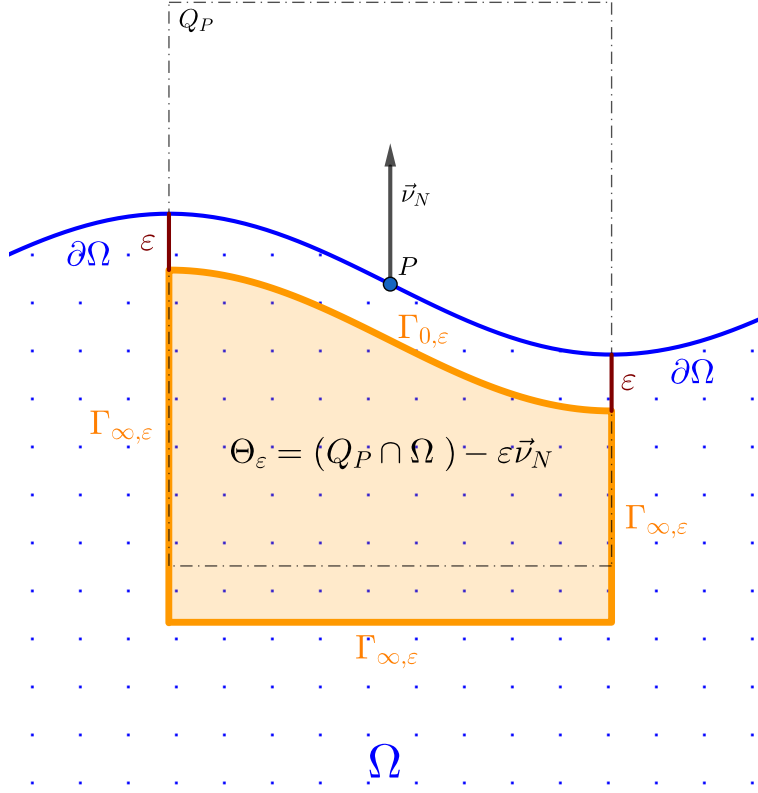


Figure 2.2: The domain Θ_ϵ

and let L_n be the unique positive solution of the (non-singular) boundary value problem

$$\begin{cases} -\Delta L + g(x, L) = 0 & \text{in } \Theta_0 \\ L = b_n & \text{on } \partial\Theta_0. \end{cases} \quad (2.5)$$

The solution is the unique minimizer of the lower semicontinuous convex functional

$$J(L) = \int_{\Theta_0} \left(\frac{1}{2} |\nabla L|^2 + G(x, L) \right) dx \quad \text{with } G(x, L) = \int_0^L g(x, s) ds,$$

defined over the affine space of functions in $H^1(\Theta_0)$ with trace b_n on $\partial\Theta_0$. Since $\{b_n\}_{n \in \mathbb{N}}$ is increasing, it follows from the maximum principle that

$$L_n \leq L_{n+1} \quad \text{for all } n \geq 1.$$

Since g satisfies the strong barrier property there exists $r_P > 0$ such that, for any $r \in (0, r_P]$, there exists a supersolution $u_{r,P}$ of (1.11) in $\Omega \cap B_r(P)$ which is continuous in $\bar{\Omega} \cap B_r(P)$. Up to changing

Q_P we can assume that for some $r \in (0, r_P]$,

$$B_r(P) \cap \partial\Omega = Q_P \cap \partial\Omega = \Gamma_{0,0} \quad \text{and} \quad \Omega \cap B_r(P) \subset Q_P \cap \Omega = \Theta_0.$$

By the maximum principle, $L_n \leq u_{r,P}$ in $B_r(P) \cap \partial\Omega$. Since $\partial\Omega$ is Lipschitz and the L_n remain locally bounded in a neighborhood of $Q_P \cap \partial\Omega$, it follows by [10, Th. 8.29] that they are locally Hölder continuous near $Q_P \cap \partial\Omega$ and hence the sequence $\{L_n\}_{n \in \mathbb{N}}$ is locally uniformly continuous near $Q_P \cap \partial\Omega$. Therefore, the pointwise limit

$$\ell := \lim_{n \rightarrow \infty} L_n$$

is well defined in Θ_0 and achieves finite values in $\bar{\Omega} \cap B_r(P)$ since it is dominated by $u_{r,P}$. In what follows we prove that ℓ is continuous in $\bar{\Omega} \cap Q_P$, vanishes on $\Gamma_{0,0}$ and satisfies (2.4). For every $\zeta \in \Theta_0$ consider $\tilde{r} > r > 0$ so that $\bar{B}_{\tilde{r}}(\zeta) \subset \Theta_0$. Obviously, there exists an integer n_0 such that $L_n|_{B_{\tilde{r}}(\zeta)}$ is well defined for all $n \geq n_0$. Let m denote the maximal positive large solution of

$$-\Delta m + g(x, m) = 0 \quad \text{in } B_{\tilde{r}}(z).$$

Then, we have that

$$0 \leq L_n(x) < \|m\|_{C(\bar{B}_r(z))} \quad \text{for all } x \in \bar{B}_r(z) \text{ and } n \geq n_0.$$

Thus, combining a rather standard compactness argument together with the interior Schauder estimates there exists a subsequence, $\{L_{n_k}\}_{k \in \mathbb{N}}$, which converges locally uniformly to ℓ in Θ_0 . Clearly ℓ satisfies (2.4), and since the sequence $\{L_n\}_{n \in \mathbb{N}}$ is locally Hölder continuous up to $Q_P \cap \partial\Omega$, ℓ vanishes on $\Gamma_{0,0}$. \square

The next result provides us with a supersolution of (1.1) in Θ_ϵ .

Proposition 2.2 *For every $\epsilon \in (0, \epsilon_0)$, the function*

$$\bar{u}_\epsilon(x) = u^{\min}(x + \epsilon \vec{v}_N) + \ell(x + \epsilon \vec{v}_N), \quad x \in \Theta_\epsilon,$$

provides us with a supersolution of (1.1) in Θ_ϵ such that

$$\bar{u}_\epsilon = +\infty \quad \text{on } \partial\Theta_\epsilon.$$

Proof. The fact that $\bar{u}_\epsilon = +\infty$ on $\partial\Theta_\epsilon$ follows readily from the definition. Indeed, by (2.1) and (2.3), we have that $x + \epsilon \vec{v}_N \in \partial\Omega$ if $x \in \partial\Theta_\epsilon \setminus \Gamma_{\infty,\epsilon}$. Thus,

$$\bar{u}_\epsilon(x) \geq u^{\min}(x + \epsilon \vec{v}_N) = +\infty \quad \text{for all } x \in \partial\Theta_\epsilon \setminus \Gamma_{\infty,\epsilon}.$$

On the other hand, by (2.4), we have that, for every $x \in \Gamma_{\infty,\epsilon}$,

$$\bar{u}_\epsilon(x) \geq \ell(x + \epsilon \vec{v}_N) = +\infty.$$

Therefore, $\bar{u}_\epsilon = +\infty$ on $\partial\Theta_\epsilon$. Now, restricting ourselves to Θ_ϵ , it follows from (2.4) that

$$\begin{aligned} -\Delta \bar{u}_\epsilon(x) &= -\Delta u^{\min}(x + \epsilon \vec{v}_N) - \Delta \ell(x + \epsilon \vec{v}_N) \\ &= -f(x + \epsilon \vec{v}_N, u^{\min}(x + \epsilon \vec{v}_N)) - g(x + \epsilon \vec{v}_N, \ell(x + \epsilon \vec{v}_N)). \end{aligned}$$

Thus, owing to (1.7)-(1.10), it becomes apparent that

$$-\Delta \bar{u}_\epsilon(x) \geq -f(x, u^{\min}(x + \epsilon \vec{v}_N)) - g(x, \ell(x + \epsilon \vec{v}_N))$$

for every $x \in \Theta_\epsilon$. Finally, by the definition of $g(x, u)$ (see (1.9)), we find that

$$\begin{aligned} & -f(x, u^{\min}(x + \epsilon \vec{v}_N)) - g(x, \ell(x + \epsilon \vec{v}_N)) \\ & \geq -[f(x, u^{\min}(x + \epsilon \vec{v}_N)) + f(x, u^{\min}(x + \epsilon \vec{v}_N) + \ell(x + \epsilon \vec{v}_N)) - f(x, u^{\min}(x + \epsilon \vec{v}_N))] \\ & = -f(x, \bar{u}_\epsilon(x)). \end{aligned}$$

Therefore, \bar{u}_ϵ is a supersolution of (1.1) in Θ_ϵ , which ends the proof. \square

We can complete now the proof of Theorem 1.1. By (2.2), u^{\max} is bounded on $\partial\Theta_\epsilon$ for all $0 < \epsilon \leq \epsilon_0$. Thus, it follows from the strong maximum principle that

$$\bar{u}_\epsilon(x) = u^{\min}(x + \epsilon \vec{v}_N) + \ell(x + \epsilon \vec{v}_N) \geq u^{\max}(x), \quad \text{for all } 0 < \epsilon \leq \epsilon_0, x \in \Theta_\epsilon. \quad (2.6)$$

To prove (2.6) we argue by contradiction. Since

$$\bar{u}_\epsilon(x) = +\infty > u^{\max}(x), \quad \text{for all } 0 < \epsilon \leq \epsilon_0, x \in \partial\Theta_\epsilon,$$

if (2.6) fails, then, for some $\epsilon \in (0, \epsilon_0)$, there exists an open subset, $D = D(\epsilon)$, with $\bar{D} \subset \Theta_\epsilon$, such that

$$\begin{cases} \bar{u}_\epsilon = u^{\min}(\cdot + \epsilon \vec{v}_N) + \ell(\cdot + \epsilon \vec{v}_N) \not\leq u^{\max} & \text{in } D \\ \bar{u}_\epsilon = u^{\max} & \text{on } \partial D. \end{cases} \quad (2.7)$$

Thus, setting

$$v := u^{\max} - \bar{u}_\epsilon,$$

we find from Proposition 2.2 and assumption (2.7) that

$$-\Delta v = -\Delta u^{\max} + \Delta \bar{u}_\epsilon \leq -[f(x, u^{\max}) - f(x, \bar{u}_\epsilon)] < 0 \quad \text{in } D,$$

while $v = 0$ on ∂D . Consequently, $v < 0$ in D , which implies $u^{\max} < \bar{u}_\epsilon$ in D and contradicts the assumption (2.7). This contradiction shows the above claim.

Now, letting $\epsilon \downarrow 0$ in (2.6) yields

$$u^{\min}(x) + \ell(x) \geq u^{\max}(x) \quad \text{for all } x \in \Theta_0.$$

Therefore, it becomes clear that

$$0 \geq \limsup_{d(x) \downarrow 0} (u^{\max}(x) - u^{\min}(x)) \geq 0,$$

which entails

$$\lim_{d(x) \downarrow 0} (u^{\max}(x) - u^{\min}(x)) = 0.$$

Finally, setting

$$L := u^{\min} - u^{\max} \leq 0,$$

by the monotonicity of f we find that

$$\begin{cases} -\Delta L = f(x, u^{max}) - f(x, u^{min}) \geq 0 & \text{in } \Omega \\ L = 0 & \text{on } \partial\Omega, \end{cases}$$

and, consequently, applying the maximum principle, we can infer that $L = 0$. This ends the proof of Theorem 1.1.

3 Proof of Theorem 1.2

We assume that u^{max} is a large solution, i.e. satisfies (1.1)-(1.2). The next result which has the same expression as Lemma 2.1 needs actually a slightly different proof due to the fact that the boundary may not be regular at all.

Lemma 3.1 *Under the assumptions of Theorem 1.2, there exists a nonnegative function $\ell \in C^1(\Theta_0)$, bounded on any compact subset of $\Omega \cap Q_P$, satisfying*

$$\begin{cases} -\Delta \ell + g(x, \ell) = 0 & \text{in } \Theta_0 \\ \ell = +\infty & \text{on } \Gamma_{\infty,0}. \end{cases} \quad (3.1)$$

Proof. Since the equation (1.11) admits a strong barrier at P , we can assume that there admits a supersolution in $B_\rho(P) \subset Q_P$, where Q_P is the cylinder of diameter ρ . Hence, $B_\rho(P) \subset Q_P$ and $B_\rho(P) \cap \Omega \subset Q_P \cap \Omega = \Theta_0$. We denote the barrier by $u_{\rho,P}$. For $\sigma > 0$ small compared to ρ we consider a domain Θ'_σ such that

$$\Omega \cap \Theta_\sigma \subset \Theta'_\sigma \subset \Omega \cap \Theta_{\frac{\sigma}{2}},$$

and we denote by $\Gamma'_{0,\sigma}$ its upper boundary and by $\Gamma'_{\infty,\sigma}$ its lateral and lower boundaries. We can assume that $\Gamma'_{0,\sigma}$ is Lipschitz continuous. Let $\ell = \ell_{n,\sigma}$ be the solution, obtained by minimization, of

$$\begin{cases} -\Delta \ell + g(x, \ell) = 0 & \text{in } \Theta'_\sigma \\ \ell = 0 & \text{on } \Gamma'_{0,\sigma} \\ \ell = n & \text{on } \Gamma'_{\infty,\sigma}. \end{cases}$$

Since $u_{\rho,P} \in C(\overline{\Omega} \cap B_\rho(P))$ is positive in $\Omega \cap B_\rho(P)$, for sufficiently small σ we have that $\ell_{n,\sigma} \leq u_{\rho,P}$ in $\Theta'_\sigma \cap B_\rho(P)$. Thus, By the maximum principle

$$\ell_{n,\sigma} \leq \ell_{n',\sigma'} \quad \text{in } \Theta'_\sigma \text{ if } n' > n \text{ and } \sigma' < \sigma.$$

When $\sigma \downarrow 0$, $\ell_{n,\sigma}$ increases and converges to a function $\ell := \ell_n$ which satisfies

$$\begin{cases} -\Delta \ell + g(x, \ell) = 0 & \text{in } \Theta_0 \\ \ell \leq u_{\rho,P} & \text{on } \Gamma_{0,0} \\ \ell = n & \text{on } \Gamma_{\infty,0}. \end{cases}$$

As g satisfies (KO-loc), ℓ_n remains locally bounded in Θ_0 . Therefore, $\ell_n \uparrow \ell$ as $n \rightarrow \infty$. Clearly, ℓ is bounded on any compact set $K \subset \overline{\Omega} \cap Q_P$, it belongs to $C^1(\Theta_0)$, by standard elliptic regularity theory, and satisfies (3.1). \square

Now, suppose that $u(x)$ is any positive solution of (1.1)-(1.2) and consider

$$\bar{u}_\varepsilon(x) := u(x + \varepsilon \vec{\nu}_N) + \ell(x + \varepsilon \vec{\nu}_N), \quad x \in \Theta_\varepsilon, \quad (3.2)$$

for sufficiently small $\varepsilon > 0$. The argument of the proof of Proposition 2.2 works out *mutatis mutandis* to show that \bar{u}_ε is a supersolution of (1.1) in Θ_ε . Moreover, by (2.2), u is bounded on $\partial\Theta_\varepsilon$ for sufficiently small $\varepsilon > 0$. Thus, arguing as in the last step of the proof of Theorem 1.1, it follows from the strong maximum principle that

$$\bar{u}_\varepsilon(x) = u(x + \varepsilon \vec{\nu}_N) + \ell(x + \varepsilon \vec{\nu}_N) \geq u^{max}(x), \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0, x \in \Theta_\varepsilon. \quad (3.3)$$

As there exists a decreasing sequence $\varepsilon_n \downarrow 0$ as $n \uparrow +\infty$ such that the function

$$\ell = \lim_{n \rightarrow \infty} \ell(\cdot + \varepsilon_n \vec{\nu}_N)$$

solves (3.1), particularizing (3.3) at $\varepsilon = \varepsilon_n$ and letting $n \uparrow +\infty$ yields

$$u(x) + \ell(x) \geq u^{max}(x) \quad \text{for all } x \in \Theta_0. \quad (3.4)$$

On the other hand, by the definition of u^{max} there holds

$$u^{max}(x) + \ell(x) \geq u(x) \quad \text{for all } x \in \Theta_0.$$

Therefore, for every $x \in \Theta_0$, we have that

$$\ell(x) \geq u^{max}(x) - u(x) \geq 0. \quad (3.5)$$

Finally, in order to infer from (3.5) that $u(x) = u^{max}(x)$ for all $x \in \Theta_0$, we will use the next result of technical nature.

Lemma 3.2 *Let $u_1(x)$ and $u_2(x)$ be positive solutions of (1.1)-(1.2) such that*

$$\lim_{d(x) \downarrow 0} \frac{u_2(x) - u_1(x)}{\varphi(u_1(x))} = 0. \quad (3.6)$$

Then, $u_1 = u_2$ in Ω .

Proof. For sufficiently small $\varepsilon > 0$, consider the function v defined by

$$v := u_1 + \varepsilon \varphi(u_1), \quad (3.7)$$

where φ is the function introduced in the statement of Theorem 1.2. We claim that $v \geq u_2$ in a neighborhood of $\partial\Omega$. Indeed, by (3.6), for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x) < \delta$, then

$$\frac{u_2(x) - u_1(x)}{\varphi(u_1(x))} \leq \varepsilon = \frac{v(x) - u_1(x)}{\varphi(u_1(x))}.$$

Thus, $v(x) \geq u_2(x)$ provided $d(x) \leq \delta$. On the other hand, since $\varphi'' \leq 0$, we have that

$$\begin{aligned} -\Delta v &= -\Delta u_1 - \varepsilon \varphi'(u_1) \Delta u_1 - \varepsilon \varphi''(u_1) |\nabla u_1|^2 \\ &\geq -(1 + \varepsilon \varphi'(u_1)) \Delta u_1 \\ &= -(1 + \varepsilon \varphi'(u_1)) f(x, u_1(x)). \end{aligned}$$

Hence,

$$-\Delta v + f(x, v) \geq f(x, v) - (1 + \varepsilon \varphi'(u_1))f(x, u_1).$$

Consequently, thanks to (3.7) and (1.15), it is clear that

$$-\Delta v + f(x, v) \geq 0$$

in Ω . So, v is a supersolution of (1.1) and hence, $v \geq u_2$ in Ω for sufficiently small $\varepsilon > 0$. Thus, letting $\varepsilon \downarrow 0$ yields $u_1 \geq u_2$ in Ω . By symmetry, $u_1 = u_2$ holds, which ends the proof. \square

Dividing (3.5) by $\varphi(u(x))$ and letting $d(x) \downarrow 0$, yields

$$\lim_{d(x) \downarrow 0} \frac{u^{max}(x) - u(x)}{\varphi(u(x))} = 0.$$

Consequently, by Lemma 3.2, we find that $u = u^{max}$. This ends the proof of Theorem 1.2.

4 Appendix

4.1 On the Keller-Osserman condition

The next result shows how imposing the Keller–Osserman condition on the associated function g is stronger than imposing it on f .

Proposition 4.1 *There are increasing functions f that satisfy (KO) and such that the corresponding function g does not.*

Proof. To construct such an example, one can consider any function f such that

$$u^2 \leq f(u) \leq u^3 \quad \text{and} \quad f(u) = f(\min I_n) \quad \text{for all } u \in I_n,$$

where I_n , $n \geq 1$, is an arbitrary sequence of intervals such that

$$\lim_{n \rightarrow +\infty} (\max I_n - \min I_n) = +\infty \quad \text{and} \quad \max I_n < \min I_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

By the properties of u^2 and u^3 , such a sequence of intervals exists. For this choice we have that, for any given $\ell > 0$ and $u > 0$, $[u, \ell + u] \subset I_n$ for sufficiently large $n > 0$ and hence,

$$f(\ell + u) - f(u) = 0.$$

Thus, $g(\ell) = 0$. Therefore, $g \equiv 0$, which does not satisfy (KO).

4.2 On the strong barrier property

The general problem of finding conditions so that the strong barrier property occurs is open. We give below some cases where it holds and a case where it does not. They all deal with nonlinearity of the form

$$f(x, r) = \alpha(x) \tilde{f}(r) \tag{4.1}$$

where $\alpha \in C(\overline{\Omega})$ is nonnegative and positive near $\partial\Omega$ and $\tilde{f} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and nondecreasing, vanishes at 0 and satisfies (1.3).

1- If $\alpha > 0$ on $\partial\Omega$, then the Keller–Osseman condition holds in $\overline{\mathcal{V}}$, where \mathcal{V} is a neighborhood of $\partial\Omega$, because the function α can be extended to Ω^c as a continuous and positive function by Whitney embedding theorem (see e.g. [8]). It is a completely open problem to find [out](#) sufficient conditions in the case where $\alpha > 0$ vanishes on the boundary.

2- If $\partial\Omega$ is C^2 and, for some $\alpha > 0$,

$$g(x, r) \geq d^\alpha(x) u^p$$

it is proved in [23] that the strong barrier property holds. When $\partial\Omega$ is Lipschitz the distance function loses its intrinsic interest and has often to be replaced by the first eigenfunction ϕ_1 of $-\Delta$ in $H_0^1(\Omega)$. In such case, we conjecture that the strong barrier property holds if

$$g(x, r) \geq \phi_1^\alpha(x) u^p$$

for some $\alpha > 0$.

3- If $\partial\Omega$ is C^2 and

$$g(x, r) \leq e^{-\frac{\kappa}{d(x)}} r^p$$

with $\kappa > 0$ and $p > 1$, then the strong barrier property does not hold. Indeed, it is proved in [25] that, for every $a \in \partial\Omega$ and $k > 0$, the problem

$$\begin{aligned} -\Delta u + e^{-\frac{\kappa}{d(x)}} u^p &= 0 & \text{in } \Omega, \\ u &= k\delta_a & \text{on } \partial\Omega, \end{aligned} \quad (4.2)$$

admits a unique positive solution, $v_{a,k}$. Furthermore, the nonlinearity $r \mapsto r^p$ satisfies the Keller–Osseman condition. Hence, the equation

$$-\Delta u + e^{-\frac{\kappa}{d(x)}} u^p = 0 \quad \text{in } \Omega \quad (4.3)$$

admits a minimal, u^{min} , and a maximal, u^{max} , large solution (probably they are equal). However, $v_{a,k} \uparrow u^{min}$ when $k \rightarrow \infty$. Arguing by contradiction, assume that the equation satisfies the strong barrier property at $z \in \partial\Omega$. Then, there exists $r > 0$ such that the solution $u := u_n$ of the problem

$$\begin{aligned} -\Delta u + e^{-\frac{\kappa}{d(x)}} u^p &= 0 & \text{in } B_r(z) \cap \Omega, \\ u &= n & \text{on } \Omega \cap \partial B_r(z), \\ u &= 0 & \text{on } \partial\Omega \cap B_r(z), \end{aligned}$$

converges, as $n \rightarrow \infty$, to a barrier function $u_{r,z} \in C(\overline{\Omega} \cap B_r(z))$ satisfying

$$\begin{aligned} -\Delta u + e^{-\frac{\kappa}{d(x)}} u^p &= 0 & \text{in } B_r(z) \cap \Omega, \\ u &= \infty & \text{on } \Omega \cap \partial B_r(z). \end{aligned}$$

Taking a point $a \in \partial\Omega \cap B_{2r}^c(z)$, for any $k > 0$ there exists $n = n(k)$ such that $v_{a,k} \leq n(k)$ on $\Omega \cap \partial B_r(z)$. Since $v_{a,k} = 0$ on $\partial\Omega \cap B_r(z)$, it follows that $v_{a,k} \leq u_n$. Thus, letting $k \rightarrow \infty$, yields $u^{min} \leq u_{r,z}$, which is a contradiction.

4- If $\partial\Omega$ is C^2 and

$$g(x, r) = e^{-\frac{1}{d^\alpha(x)}} r^p,$$

with $0 < \alpha < 1$ and $p > 1$, it is proved in [30] that the limit when $k \rightarrow \infty$ of the solutions $v_{a,k}$ of

$$\begin{aligned} -\Delta u + e^{-\frac{1}{d^\alpha(x)}} u^p &= 0 & \text{in } \Omega \\ u &= k\delta_a & \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

is a solution of

$$-\Delta u + e^{-\frac{1}{d^\alpha(x)}} u^p = 0 \quad \text{in } \Omega$$

which vanishes on $\partial\Omega \setminus \{a\}$ and blows up at a . We conjecture that the strong barrier property holds if

$$g(x, r) \geq e^{-\frac{1}{d^\alpha(x)}} r^p.$$

References

- [1] C. Bandle and M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior, *J. Anal. Math.* **58** (1992), 9–24.
- [2] S. Cano-Casanova and J. López-Gómez, Blow-up rates of radially symmetric large solutions, *J. Math. Anal. Appns.* **352** (2009), 166–174.
- [3] F. C. Cirstea and V. Radulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, *Comm. Contemp. Math.* **4** (2002), 559–586.
- [4] F. C. Cirstea and V. Radulescu, Solutions with boundary blow-up for a class of nonlinear elliptic problems, *Houston J. Math.* **29** (2003), 821–829.
- [5] O. Costin and L. Dupaigne, Boundary blow-up solutions in the unit ball: Asymptotics, uniqueness and symmetry, *J. Diff. Eqns.* **249** (2010), 931–964.
- [6] O. Costin, L. Dupaigne and O. Goubet, Uniqueness of large solutions, *J. Math. Anal. Appl.* **395** (2012), 806–812.
- [7] Y. Du and Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic equations, *SIAM J. Math. Anal.* **31** (1999), 1–18.
- [8] L. C. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992.
- [9] J. García-Melián, R. Letelier and J. C. Sabina de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, *Proc. Amer. Math. Soc.* **129** (2001), 3593–3602.
- [10] D. Gilbarg and N. S. Trudinger *Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, London-Berlin-Heidelberg-New York, 1983.
- [11] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure and Appl. Maths.* **X** (1957), 503–510.
- [12] D. Labutin, Wiener regularity for large solutions of nonlinear equations, *Ark. Mat.* **41** (2003), 307–339.

- [13] J. López-Gómez, Large solutions, metasolutions, and asymptotic behavior of the regular positive solutions of a class of sublinear parabolic problems, *El. J. Diff. Eqns. Conf.* **05** (2000), 135–171.
- [14] J. López-Gómez, The boundary blow-up rate of large solutions, *J. Diff. Eqns.* **195** (2003), 25–45.
- [15] J. López-Gómez, Optimal uniqueness theorems and exact blow-up rates of large solutions, *J. Diff. Eqns.* **224** (2006), 385–439.
- [16] J. López-Gómez, Uniqueness of radially symmetric large solutions, *Disc. Cont. Dyn. Systems Supplement 2007* (2007), 677–686.
- [17] J. López-Gómez, *Metasolutions of Parabolic Problems in Population Dynamics*, CRC Press, Boca Raton, 2015.
- [18] J. López-Gómez and L. Maire, Uniqueness of large positive solutions for a class of radially symmetric cooperative systems, *J. Math. Anal. Appl.* **435** (2016), 1738–1752.
- [19] J. López-Gómez and L. Maire, Uniqueness of large positive solutions, *Z. Angew. Math. Phys.* (2017) 68:86.
- [20] J. López-Gómez and L. Maire, Multiplicity of large solutions for quasi-monotone pulse-type nonlinearities, *J. Math. Anal. Appl.* **459** (2018), 490–505.
- [21] J. López-Gómez and L. Maire, Uniqueness of large solutions for non-monotone nonlinearities, *Nonl. Anal. RWA* **47** (2019), 291–305.
- [22] M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. Henri Poincaré* **14** (1997), 237–274.
- [23] M. Marcus and L. Véron, The Boundary Trace and Generalized Boundary Value Problem for Semilinear Elliptic Equations with Coercive Absorption, *Comm. Pure and Appl. Maths.* **LVI** (2003), 0689–0731.
- [24] M. Marcus and L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equations, *J. Evol. Eqns.* **3** (2004), 637–652.
- [25] M. Marcus and L. Véron, Boundary trace of positive solutions of nonlinear elliptic inequalities, *Ann. Scuola Norm. Sup Pisa CL. Sci* **V** (2004), 453–533.
- [26] M. Marcus and L. Véron, Maximal solutions for $-\Delta u + u^q = 0$ in open and finely open sets, *J. Math. Pures App.* **91** (2009), 256–295.
- [27] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. of Maths.* **7** (1957), 1641–1647.
- [28] T. Ouyang and Z. Xie, The uniqueness of blow-up for radially symmetric semilinear elliptic equations, *Nonl. Anal.* **64** (2006), 2129–2142.
- [29] T. Ouyang and Z. Xie, The exact boundary blow-up rate of large solutions for semilinear elliptic problems, *Nonl. Anal.* **68** (2008), 2791–2800.
- [30] A. Shishkov and L. Véron, Diffusion versus absorption in semilinear elliptic equations, *J. Math. Anal. Appl.* **352** (2009), 206–217.

- [31] L. Véron, Semilinear elliptic equations with uniform blow up on the boundary, *J. D'Analyse Math.* **59** (1992), 231–250.
- [32] L. Véron, Large Solutions of Elliptic Equations with Strong Absorption, in *Progress in Non-linear Differential Equations and Their Applications* **Vol. 63**, 453–464, Birkhauser Verlag Basel/Switzerland (2005).
- [33] Z. Xie, Uniqueness and blow-up rate of large solutions for elliptic equation $-\Delta u = \lambda u - b(x)h(u)$, *J. Diff. Eqns.* **247** (2009), 344–363.
- [34] Z. Zhang, Y. Ma, L. Mi and X. Li, Blow-up rates of large solutions for elliptic equations, *J. Diff. Eqns.* **249** (2010), 180–199.
- [35] Z. Zhang and L. Mi, Blow-up rates of large solutions for semilinear elliptic equations, *Commun. Pure Appl. Anal.* **10** (2011), 1733–1745.