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Abstract

For regular test functions h, we prove that all the derivatives of the solution of the Stein equation for the generalized inverse Gaussian and Kummer distributions are bounded. In order to complete the results in [6] where the authors have established an upper bound of the solution and its first and second derivatives of the Stein equation for the generalized inverse Gaussian and Kummer distributions, we provide an explicit upper bound of the third derivative of these solutions. Furthermore, for $k \geq 4$, by doing a stepwise calculation, we express the bound of the k-th derivative as a function of the bounds of the derivatives of lower order.

Keywords: Generalized Inverse Gaussian distribution, Kummer distribution, Stein characterization, Stein equation.

1 Introduction

The generalized inverse Gaussian (hereafter GIG) distribution with parameters $p \in \mathbb{R}, a > 0, b > 0$ has density

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0$$

where K_p is the modified Bessel function of the third kind.

For $a > 0$, $b \in \mathbb{R}$, $c > 0$, the Kummer distribution K(a,b,c) with parameters a, b, c has density

$$d_{a,b,c}(x) = \frac{1}{\Gamma(a)\psi(a, a-b+1; c)} x^{a-1}(1+x)^{-a-b} e^{-cx}, \quad x > 0$$

where ψ is the confluent hypergeometric function of the second kind.

For details on GIG and Kummer distributions see for example [5, 7, 8, 11].

In this paper, these two distributions are considered in the context of Stein's method introduced by Stein in [14]. A first instance of the method was found in [14], where Stein showed that a random variable X has a standard normal distribution if and only if, for all real-valued absolutely continuous function f such that $\mathbb{E}\,|f'(Z)| < \infty$, with Z a random variable following the standard normal distribution,

$$\mathbb{E}\,[f'(X) - Xf(X)] = 0.$$

The corresponding Stein equation is

$$f'(x) - xf(x) = h(x) - \mathbb{E}h(Z) \tag{1}$$

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where h is a bounded function. The corresponding Stein operator is $f \mapsto (T_f)(x) = f'(x) - xf(x)$.
If f_h solves equation (1), then for any random variable X, we have

$$|\mathbb{E}[f_h'(X) - Xf_h(X)]| = |\mathbb{E}h(X) - \mathbb{E}h(Z)|.$$

Thus, we can bound $|\mathbb{E}h(X) - \mathbb{E}h(Z)|$ given h, by finding a solution f_h of the Stein equation (1) and bounding the left-hand side of the previous equation. For more details on Stein's method, see [3, 12].
Solving the Stein equation for other distributions than the standard normal distribution, and bounding the solution and its derivatives has become a challenge often addressed in the literature (see for example [2] for the Poisson distribution, [10] for the gamma distribution, [1] for the exponential distribution). A bound of the solution of Stein equation and its first and second derivatives for GIG and Kummer distributions is derived in [6] by using the density Stein approach [3, 9, 13].
The purpose of this paper is to complete the results of [6] by providing upper bound of the third derivative of these two solutions. We also express a bound of the k-th derivative for $k \geq 4$ of these solutions by doing a stepwise calculation. Sections 2 and 3 present the main results obtained for GIG and Kummer distributions respectively. Section 4 contains the proofs.

2 About the bounds of the derivatives of the solution of the Stein equation for the generalized inverse Gaussian distribution

Let

$$s(x) = x^2 \quad \text{and} \quad \tau_{p,a,b}(x) = \frac{b}{2} + (p+1)x - \frac{a}{2}x^2. \tag{2}$$

Then, as observed in [4], the GIG density $g_{p,a,b}$ satisfies

$$(s(x)g_{p,a,b}(x))' = \tau_{p,a,b}(x)g_{p,a,b}(x). \tag{3}$$

This enables us to apply Theorem 2.1 in [6] to retrieve the following Stein characterization of the GIG distribution (Proposition 3.1. in [6]) given in [7, 4]:

Proposition 2.1 *A random variable X follows the GIG distribution with density $g_{p,a,b}$ if and only if, for all real-valued and differentiable function f such that $\lim_{x\to\infty} g_{p,a,b}(x)f(x) = \lim_{x\to 0} g_{p,a,b}(x)f(x) = 0$, and such that the following expectation exists, we have:*

$$\mathbb{E}\left[X^2 f'(X) + \left(\frac{b}{2} + (p+1)X - \frac{a}{2}X^2\right) f(X)\right] = 0.$$

The corresponding Stein equation is

$$x^2 f'(x) + \left(\frac{b}{2} + (p+1)x - \frac{a}{2}x^2\right) f(x) = h(x) - \mathbb{E}h(W) \tag{4}$$

where h is a bounded function and W a random variable following the GIG distribution with parameters p, a, b.
Theorem 3.2 in [6] shows that the bounded solution of Stein's equation (4) is given by

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g_{p,a,b}(x)} \int_0^x g_{p,a,b}(t)\left[h(t) - \mathbb{E}h(W)\right] dt \\ =& \frac{-1}{s(x)g_{p,a,b}(x)} \int_x^{+\infty} g_{p,a,b}(t)\left[h(t) - \mathbb{E}h(W)\right] dt. \end{aligned} \tag{5}$$

Previous work in [6] provided explicit bounds of f_h, f_h' and f_h''. We complete this result by providing an upper bound of the third derivative $f_h^{(3)}$ of f_h and propose a bound of k-th derivative for $k \geq 4$ by doing a stepwise calculation. The following proposition established in [6] will be useful to prove the results obtained.

Proposition 2.2 *Consider a density function $g > 0$ on $(0,\infty)$ satisfying the equation $(s(x)g(x))' = \tau(x)g(x)$, where s and τ are polynomial functions such that $s > 0$ on $(0,\infty)$. Assume that $\lim_{x\to 0} s(x)g(x) = \lim_{x\to\infty} s(x)g(x) = 0$. Assume also that $\tau(0) > 0$ and, for some $\beta > 0$, τ is increasing on $(0,\beta]$, decreasing on $[\beta,\infty)$ and has a unique zero α on $[\beta,\infty)$. Let h be a bounded continuous function. Then*

$$||f_h|| \le N_0 \, ||h(.) - \mathbb{E}h(W)|| \,, \tag{6}$$

where

$$N_0 = \max\left(\frac{1}{\tau(0)};\ \frac{1}{s(\beta)g(\beta)}\right)$$

and $\|f\| = \sup_{x>0} |f(x)|$.

In the remainder of this section, we define

$$\alpha_k = \frac{p+2k+1+\sqrt{(p+2k+1)^2+ab}}{a}, \quad \tau_k(x) = \frac{b}{2} + (p+2k+1)x - \frac{a}{2}x^2, \quad \forall k \ge 1;$$

$$h_k(x) = h^{(k)}(x) - k(p+k-ax)f^{(k-1)}(x) + \left(3 + \frac{(k-3)(k+2)}{2}\right) af^{(k-2)}(x) \quad \forall k \ge 3$$

and we denote by $g_k := g_{p+2k,a,b}$ the density of the GIG$(p+2k,a,b)$ distribution. We have

$$(s(x)g_k(x))' = \tau_k(x)g_k(x) \ \text{ and } \ \lim_{x\to 0} s(x)g_k(x) = \lim_{x\to\infty} s(x)g_k(x) = 0 \ \ \forall k \ge 1. \tag{7}$$

We have the following results.

Theorem 2.1 *Let $h : (0,\infty) \to \mathbb{R}$ be a bounded function, three times differentiable such that h', h'' and $h^{(3)}$ are bounded. Then the third derivative $f_h^{(3)}$ of the solution f_h of Stein equation (4) is such that*

$$\left\| f_h^{(3)} \right\| \le M_3 = \begin{cases} \max(G_1^3, G_2^3) & \text{if} \quad p \le -7 \\ \max(G_3^3, G_4^3, G_2^3) & \text{if} \quad p > -7 \end{cases} \tag{8}$$

where

$$G_1^3 = \left(\left\|h^{(3)}\right\| + 3\max\left(|p+3|, (4+\sqrt{(p+7)^2+ab}\right) M'' + 3aM'\right) \frac{1}{\alpha_3^2 g_3(\alpha_3)} \int_0^{\alpha_3} g_3(t)dt,$$

$$G_2^3 = \frac{5a + a\sqrt{(p+7)^2+ab}}{\alpha_3^2\left(p+7+\sqrt{(p+7)^2+ab}\right)} \left(||h'|| + \frac{6a + a\sqrt{(p+7)^2+ab}}{6p+42+6\sqrt{(p+7)^2+ab}} \|h(.) - \mathbb{E}h(W)\|\right)$$
$$+ \frac{2}{\alpha_3^2}\left(\|h''\| + aM\,\|h(.) - \mathbb{E}h(W)\|\right),$$

$$G_3^3 = \frac{2}{b}\left(\left\|h^{(3)}\right\| + 3\max\left(4, |p+3|\right) M'' + 3aM'\right),$$

$$G_4^3 = \left(\left\|h^{(3)}\right\| + 3\max\left(|p+3|, (4+\sqrt{(p+7)^2+ab}\right) M'' + 3aM'\right) \frac{a^2}{(p+7)^2 g_3\left(\frac{p+7}{a}\right)},$$

W is a random variable following GIG distribution with parameters p, a, b, the constants M, M' and M'' are given in [6] and are such that $\|f_h\| \le M$, $\|f_h'\| \le M'$ and $\|f_h''\| \le M''$.

The proof of Theorem 2.1 will be given in Section 4.

Theorem 2.2 *Let $h : (0,\infty) \to \mathbb{R}$ be a bounded function, k times differentiable such that $h^{(k)}$ is bounded for all $k \geq 1$. The k-th derivative $f_h^{(k)}$ of the solution f_h of Stein equation (4) is such that*

$$\left\| f_h^{(k)} \right\| \leq M_k = \begin{cases} \max(G_1^k, G_2^k) & \text{if} \quad p \leq -2k-1 \\ \max(G_3^k, G_4^k, G_2^k) & \text{if} \quad p > -2k-1 \end{cases} \tag{9}$$

where

$$G_1^k = H \times \frac{1}{\alpha_k^2 g_k(\alpha_k)} \int_0^{\alpha_k} g_k(t)dt,$$

$$G_2^k = \frac{2}{\alpha_k^2} \|h_{k-1}\|,$$

$$\|h_{k-1}\| \quad \leq \quad \left\|h^{(k-1)}\right\| \quad + \quad \frac{a(k-1)\left(k+2+\sqrt{(p+2k+1)^2+ab}\right)}{4p+8k+4+4\sqrt{(p+2k+1)^2+ab}} \|h_{k-2}\| \;+\; a\left(3+\frac{(k-4)(k+1)}{2}\right) M_{k-3},$$

$$G_3^k = \frac{2}{b}\left(\left\|h^{(k)}\right\| + k\max\left(k+1, |p+k|\right) M_{k-1} + a\left(3+\frac{(k-3)(k+2)}{2}\right) M_{k-2}\right),$$

$$G_4^k = H \times \frac{a^2}{(p+2k+1)^2 \, g_k\left(\frac{p+2k+1}{a}\right)},$$

$$H \quad = \quad \left\|h^{(k)}\right\| \quad + \quad k\max\left(|p+k|, k+1+\sqrt{(p+2k+1)^2+ab}\right) M_{k-1} \quad + \quad a\left(3+\frac{(k-3)(k+2)}{2}\right) M_{k-2},$$

and M_i is such that, $\left\|f_k^{(i)}\right\| \leq M_i$ for all $i \geq 1$.

The proof of Theorem 2.2 will be given in Section 4.

3 About the bounds of the derivatives of the solution of the Stein equation for the Kummer distribution

Let

$$s(x) = x(1+x) \qquad \text{and} \qquad \tau(x) = (1-b)x - cx(1+x) + a. \tag{10}$$

Then the density $d_{a,b,c}$ of Kummer distribution satisfies

$$(s(x)d_{a,b,c}(x))' = \tau(x)d_{a,b,c}(x).$$

Theorem 4.1 in [6] give the following Stein characterization of the Kummer distribution.

Theorem 3.1 *A random variable X follows the Kummer distribution with density $d_{a,b,c}$ if and only if, for all differentiable function f such that the expectation exists,*

$$\mathbb{E}\left[X(1+X)f'(X) + \left[(1-b)X - cX(1+X) + a\right] f(X)\right] = 0.$$

The corresponding Stein equation is

$$x(x+1)f'(x) + \left[(1-b)x - cx(1+x) + a\right] f(x) = h(x) - \mathbb{E}h(W) \tag{11}$$

where h is a bounded function and W has K(a,b,c) distribution.
The bounded solution of equation (11) is

$$
\begin{aligned}
f_h(x) =& \frac{1}{x(1+x)d_{a,b,c}(x)} \int_0^x d_{a,b,c}(t)\left[h(t)-\mathbb{E}h(W)\right]dt \\
=& \frac{-1}{x(1+x)d_{a,b,c}(x)} \int_x^{+\infty} d_{a,b,c}(t)\left[h(t)-\mathbb{E}h(W)\right]dt.
\end{aligned}
\tag{12}
$$

In [6], the bounds of f_h, f_h' and f_h'' have been proposed. We complete this result by providing a bound of the third derivative $f_h^{(3)}$ of f_h and we propose a bound of k-*th* derivative for $k \geq 4$ as in the case of GIG distribution.

In the remainder of this section, we define

$$
\alpha_k = \frac{2k+1-b-c+\sqrt{(2k+1-b-c)^2+4c(a+k)}}{2c}, \quad \forall k \geq 1;
$$

$$
\tau_k(x) = (2k+1-b)x - cx(1+x) + a + k, \quad \forall k \geq 1;
$$

$$
h_k(x) = h^{(k)}(x) - k(k-b-c-2cx)f^{(k-1)}(x) + k(k-1)cf^{(k-2)}(x) \quad \forall k \geq 3
$$

and we denote by $d_k := d_{a+k,b-2k,c}$ the density of the K$(a+k, b-2k, c)$ distribution. We have

$$
(s(x)d_k(x))' = \tau_k(x)d_k(x) \text{ and } \lim_{x\to 0} s(x)d_k(x) = \lim_{x\to\infty} s(x)d_k(x) = 0 \quad \forall k \geq 1. \tag{13}
$$

Using similar arguments as in the case of GIG distribution, we obtain the following results

Theorem 3.2 *Let $h : (0,\infty) \to \mathbb{R}$ be a bounded, three times differentiable function such that h', h'' and $h^{(3)}$ are bounded. Then the third derivative $f_h^{(3)}$ of the solution f_h given by equation* (12) *is such that*

$$
\left\| f_h^{(3)} \right\| \leq N_3 = \begin{cases} \max(K_1^3, K_2^3) & \text{if} \quad 7-b-c \leq 0 \\ \max(K_3^3, K_4^3, K_2^3) & \text{if} \quad 7-b-c > 0 \end{cases} \tag{14}
$$

where

$$
K_1^3 = \left(\left\| h^{(3)} \right\| + 3\max\left(|3-b-c|, 4+\sqrt{(7-b-c)^2+4c(a+3)}\right) K'' + 6cK' \right) \times \frac{1}{\alpha_3^2 d_3(\alpha_3)} \int_0^{\alpha_3} d_3(t)dt,
$$

$$
\begin{aligned}
K_2^3 = \frac{2}{\alpha_3(1+\alpha_3)}\left(\|h''\| + 2cK\|h(.)-\mathbb{E}h(W)\|\right) + \frac{10c+2c\sqrt{(7-b-c)^2+4c(a+3)}}{\alpha_3(1+\alpha_3)\left(7-b+\sqrt{(7-b-c)^2+4c(a+3)}\right)} \times \\
\left(\|h'\| + \frac{6c+c\sqrt{(7-b-c)^2+4(a+3)}}{21-3b+3\sqrt{(7-b-c)^2+4(a+3)}} \|h(.)-\mathbb{E}h(W)\| \right),
\end{aligned}
$$

$$
K_3^3 = \frac{1}{a+3}\left(\left\| h^{(3)} \right\| + 3\max\left(4, |3-b-c|\right) K'' + 6cK' \right),
$$

$$
\begin{aligned}
K_4^3 = \left(\left\| h^{(3)} \right\| + 3\max\left(|3-b-c|, 4+\sqrt{(7-b-c)^2+4c(a+3)}\right) K'' + 6cK' \right) \times \\
\frac{4c^2}{(7-b-c)(7-b+c)d_3\left(\frac{7-b-c}{2c}\right)},
\end{aligned}
$$

$W \sim K(a,b,c)$, K, K' and K'' are given in [6] and are such that $\|f_h\| \leq K$, $\|f_h'\| \leq K'$ and $\|f_h''\| \leq K''$.

Noticing that $f_h^{(k)}$ satisfies the differential equation

$$
x(1+x)f^{(k+1)}(x) + \left[(2k+1-b)x - cx(1+x) + a + k\right] f^{(k)}(x) = h_k(x), \tag{15}
$$

we have

Theorem 3.3 *Let $h : (0,\infty) \to \mathbb{R}$ be a bounded, k times differentiable function such that $h^{(k)}$ is bounded for all $k \geq 1$. The k-th derivative $f_h^{(k)}$ of the solution f_h of Stein equation (11) is such that*

$$\left\| f_h^{(k)} \right\| \leq N_k = \begin{cases} \max(K_1^k, K_2^k) & \textit{if} \quad 2k+1-b-c \leq 0 \\ \max(K_2^k, K_3^k, K_4^k) & \textit{if} \quad 2k+1-b-c > 0 \end{cases} \tag{16}$$

where

$$K_1^k = T \times \frac{1}{\alpha_k(1+\alpha_k)d_k(\alpha_k)} \int_0^{\alpha_k} d_k(t)dt,$$
$$K_2^k = \frac{2}{\alpha_k(\alpha_k+1)} \left\| h_{k-1} \right\|,$$

$$\left\| h_{k-1} \right\| \leq \left\| h^{(k-1)} \right\| + \frac{c(k-1)\left(k+2+\sqrt{(2k+1-b-c)^2+4c(a+k)}\right)}{4k+2-2b+2\sqrt{(2k+1-b-c)^2+4c(a+k)}} \left\| h_{k-2} \right\| + (k-1)(k-2)cN_{k-3},$$

$$K_3^k = \frac{1}{a+k}\left(\left\| h^{(k)} \right\| + k\max\left(k+1, |k-b-c|\right) N_{k-1} + k(k-1)cN_{k-2} \right),$$
$$K_4^k = T \times \frac{4c^2}{(2k+1-b-c)(2k+1-b+c)d_k\left(\frac{2k+1-b-c}{2c}\right)},$$
$$T = \left\| h^{(k)} \right\| + k\max\left(|k-b-c|, k+1+\sqrt{(2k+1-b-c)^2+4c(a+k)}\right) N_{k-1} + k(k-1)cN_{k-2},$$

and N_j is such that, $\left\| f_h^{(j)} \right\| \leq N_j$ for all $j \geq 1$.

4 Proofs

4.1 Proof of Theorem 2.1

First, assume that $p \leq -7$.
$f_h^{(3)}$ satisfies the differential equation

$$x^2 f^{(4)}(x) + \left(\frac{b}{2} + (p+7)x - \frac{a}{2}x^2\right) f^{(3)}(x) = h^{(3)}(x) - 3(p+3-ax)f''(x) + 3af'(x). \tag{17}$$

Let $h_3(x) = h^{(3)}(x) - 3(p+3-ax)f''(x) + 3af'(x)$. Then, by equation (17), $f_h^{(3)}$ solves the differential equation

$$x^2 f^{(4)}(x) + \left(\frac{b}{2} + (p+7)x - \frac{a}{2}x^2\right) f^{(3)}(x) = h_3(x). \tag{18}$$

Since $\mathbb{E}h_3(Y) = \mathbb{E}\left[Y^2 f^{(4)}(Y) + \left(\frac{b}{2} + (p+7)Y - \frac{a}{2}Y^2\right) f^{(3)}(Y)\right] = 0$ if $Y \sim$GIG$(p+6,a,b)$ by the Stein characterization of GIG$(p+6,a,b)$ distribution, equation (18) is the Stein equation for GIG$(p+6,a,b)$ distribution. We have

$$\begin{aligned} f_h^{(3)}(x) =& \frac{1}{s(x)g_3(x)} \int_0^x g_3(t)h_3(t)dt \\ =& \frac{-1}{s(x)g_3(x)} \int_x^{+\infty} g_3(t)h_3(t)dt. \end{aligned} \tag{19}$$

For $p \leq -7$, τ_3 is decreasing on $(0,\infty)$ and its only zero is $\alpha_3 = \frac{p+7+\sqrt{(p+7)^2+ab}}{a}$. If $x \leq \alpha_3$ and if h_3 is bounded, then

$$\left| f_h^{(3)}(x) \right| \leq \left\| h_3 \right\| \frac{1}{s(x)g_3(x)} \int_0^x g_3(t)dt.$$

Let $\ell(x)=\dfrac{1}{s(x)g_3(x)}\displaystyle\int_0^x g_3(t)dt$. ℓ is differentiable on $(0,\infty)$ and

$$
\begin{aligned}
\ell'(x) &= \frac{-(s(x)g_3(x))'}{(s(x)g_3(x))^2}\int_0^x g_3(t)dt+\frac{1}{s(x)}\\
&\geq -\frac{(s(x)g_3(x))'}{(s(x)g_3(x))^2}\frac{1}{\tau_3(x)}\int_0^x \tau_3(t)g_3(t)dt+\frac{1}{s(x)} \qquad \left(1\leq \frac{\tau_3(t)}{\tau_3(x)}\quad \forall t\leq x\right)\\
&= -\frac{1}{s(x)}+\frac{1}{s(x)}=0 \quad \left((s(x)g_3(x))'=\tau_3(x)g_3(x) \text{ and } \lim_{t\to 0}s(t)g_3(t)=0\right).
\end{aligned}
$$

Then for $x\leq\alpha_3$, $\ell(x)\leq\ell(\alpha_3)$.

$$
\begin{aligned}
|h_3(x)| &= \left|h^{(3)}(x)-3(p+3-ax)f_h''(x)+3af_h'(x)\right|\\
&\leq \left\|h^{(3)}\right\|+3|p+3-ax|\,\|f_h''\|+3a\,\|f_h'\|\\
&\leq \left\|h^{(3)}\right\|+3\max\left(|p+3|,4+\sqrt{(p+7)^2+ab}\right)\|f_h''\|+3a\,\|f_h'\| \quad (0<x<\alpha_3)\\
&\leq \left\|h^{(3)}\right\|+3\max\left(|p+3|,4+\sqrt{(p+7)^2+ab}\right)M''+3aM'
\end{aligned}
$$

Hence for $p\leq -7$ and $x\leq\alpha_3$,

$$
\left\|f_h^{(3)}\right\|\leq\left(\left\|h^{(3)}\right\|+3\max\left(|p+3|,4+\sqrt{(p+7)^2+ab}\right)M''+3aM'\right)\frac{1}{\alpha_3^2 g_3(\alpha_3)}\int_0^{\alpha_3}g_3(t)dt.
$$

For $p\leq -7$ and $x>\alpha_3$, consider the following Stein equation solving by f_h'':

$$
s(x)f^{(3)}(x)+\tau_2(x)f''(x)=h_2(x)=h''(x)-2(p+2-ax)f'(x)+af(x). \tag{20}
$$

By equation (20), one obtains

$$
\begin{aligned}
f_h^{(3)}(x) &= \frac{h_2(x)}{s(x)}-\frac{\tau_2(x)f_h''(x)}{s(x)}\\
\left|f_h^{(3)}(x)\right| &\leq \frac{\|h_2\|}{s(x)}+\frac{|\tau_2(x)f_h''(x)|}{s(x)}.
\end{aligned}
$$

For $p\leq -7$ and $x>\alpha_3$, τ_2 is decreasing and negative. We have

$$
f_h''(x)=\frac{-1}{s(x)g_2(x)}\int_x^{+\infty}g_2(t)h_2(t)dt
$$

$$
\begin{aligned}
|\tau_2(x)f_h''(x)| &= \left|\frac{-\tau_2(x)}{s(x)g_2(x)}\int_x^{+\infty}g_2(t)h_2(t)dt\right|\\
&\leq \|h_2\|\frac{-\tau_2(x)}{s(x)g_2(x)}\int_x^{+\infty}g_2(t)dt \quad (\tau_2 \text{ is negative})\\
&\leq \|h_2\|\frac{-\tau_2(x)}{s(x)g_2(x)}\int_x^{+\infty}\frac{\tau_2(t)}{\tau_2(x)}g_2(t)dt \qquad \left(\frac{\tau_2(t)}{\tau_2(x)}\geq 1\quad\forall x\leq t\right)\\
&= \|h_2\|\frac{-1}{s(x)g_2(x)}\int_x^{+\infty}(s(t)g_2(t))'dt\\
&= \|h_2\| \quad \left(\lim_{t\to 0}s(t)g_2(t)=0\right).
\end{aligned}
$$

$$
|h_2(x)|=|h''(x)-2(p+2-ax)f_h'(x)+af_h(x)|\leq\|h''\|+2\,|(p+2-ax)f_h'(x)|+a\,\|f_h\|;
$$

$$
f_h'(x)=\frac{-1}{s(x)g_1(x)}\int_x^{+\infty}g_1(t)h_1(t)dt
$$

$$
|(p+2-ax)f_h'(x)|\leq\|h_1\|\frac{-(p+2-ax)}{s(x)g_1(x)}\int_x^{+\infty}g_1(t)dt\leq\|h_1\|\frac{p+2-ax}{\tau_1(x)}.
$$

Let $\kappa(x) = \dfrac{p+2-ax}{b/2+(p+3)x-ax^2/2} = \dfrac{p+2-ax}{\tau_1(x)}$. κ is differentiable on (α_3, ∞) and

$$\kappa'(x) = -\frac{ab+2(p+2)(p+3)-2a(p+2)x+a^2x^2}{2\tau_1^2(x)} = -\frac{ab+(p+2)^2+2(p+2)+(p+2-ax)^2}{2\tau_1^2(x)}.$$

Thus for $p \le -3$ or $p \ge -2$, $\kappa'(x) \le 0$. For $-3 < p < -2$, if $1-ab-(p+3)^2 > 0$, then the only possible positive zero of $x \mapsto ab+2(p+2)(p+3)-2a(p+2)x+a^2x^2$ is $\dfrac{p+2+\sqrt{1-ab-(p+3)^2}}{a}$, which is smaller than α_3. Hence for all $p \in \mathbb{R}$, κ is decreasing on (α_3, ∞) and

$$\kappa(x) \le \kappa(\alpha_3) = \frac{5a+a\sqrt{(p+7)^2+ab}}{4p+28+4\sqrt{(p+7)^2+ab}}.$$

The function $h_1(x) = h'(x)-(p+1-ax)f_h(x)$ is such that

$$\begin{aligned}|h_1(x)| &\le \|h'\| + \left|\frac{-(p+1-ax)}{s(x)g_{p,a,b}(x)}\int_x^{+\infty} g_{p,a,b}(t)(h(t)-\mathbb{E}h(W))dt\right| \\ &\le \|h'\| + \|h(.)-\mathbb{E}h(W)\|\,\frac{p+1-ax}{\tau_{p,a,b}(x)}.\end{aligned}$$

$v : x \mapsto \dfrac{p+1-ax}{\tau_{p,a,b}(x)}$ is decreasing on (α_3, ∞), as a consequence, for all $x > \alpha_3$

$$v(x) \le v(\alpha_3) = \frac{6a+a\sqrt{(p+7)^2+ab}}{6p+42+6\sqrt{(p+7)^2+ab}}.$$

We obtain

$$\|h_2\| \;\le\; \|h''\| \;+\; \frac{5a+a\sqrt{(p+7)^2+ab}}{2p+14+2\sqrt{(p+7)^2+ab}}\left(\|h'\| + \frac{6a+a\sqrt{(p+7)^2+ab}}{6p+42+6\sqrt{(p+7)^2+ab}}\,\|h(.)-\mathbb{E}h(W)\|\right) \;+\; a\,\|f_h\|$$

and for $p \le -7$, $x > \alpha_3$

$$\begin{aligned}\left\|f_h^{(3)}\right\| \le \frac{5a+a\sqrt{(p+7)^2+ab}}{\alpha_3^2\left(p+7+\sqrt{(p+7)^2+ab}\right)}&\left(\|h'\| + \frac{6a+a\sqrt{(p+7)^2+ab}}{6p+42+6\sqrt{(p+7)^2+ab}}\,\|h(.)-\mathbb{E}h(W)\|\right) \\ &+ \frac{2}{\alpha_3^2}\left(\|h''\| + aM\,\|h(.)-\mathbb{E}h(W)\|\right).\end{aligned}$$

Now assume that $p > -7$

– On $\left(0, \frac{p+7}{a}\right)$, τ_3 is such that $\frac{\tau_3(t)}{\tau_3(0)} \ge 1$, $\forall 0 \le t \le \frac{p+7}{a}$. Since

$$\begin{aligned}|h_3(x)| &\le \left\|h^{(3)}\right\| + 3|p+3-ax|\,\|f_h''\| + 3a\,\|f_h'\| \\ &\le \left\|h^{(3)}\right\| + 3\max(4, |p+3|)\,\|f_h''\| + 3a\,\|f_h'\| \\ &\le \left\|h^{(3)}\right\| + 3\max(4, |p+3|)\,M'' + 3aM',\end{aligned}$$

we have

$$\begin{aligned}|f_h^{(3)}(x)| &\le \|h_3\|\,\frac{1}{s(x)g_3(x)}\int_0^x g_3(t)dt \\ &\le \|h_3\|\,\frac{1}{\tau_3(0)s(x)g_3(x)}\int_0^x \tau_3(t)g_3(t)dt \\ &= \frac{1}{\tau_3(0)}\,\|h_3\| \\ &= \frac{2}{b}\,\|h_3\|\end{aligned}$$

and

$$\left\|f_h^{(3)}\right\| \leq \frac{2}{b}\left(\left\|h^{(3)}\right\| + 3\max\left(4, |p+3|\right) M_2 + 3aM_1\right).$$

– On $\left(\frac{p+7}{a}, \infty\right)$, the similar arguments as in the case $p \leq -7$ and Proposition 2.2 lead to the result.

□

4.2 Proof of Theorem 2.2

Assume $p \leq -2k-1$. The function $f_h^{(k)}$ satisfies the following differential equation

$$x^2 f^{(k+1)}(x) + \left(\frac{b}{2} + (p+1+2k)x - \frac{a}{2}x^2\right) f^{(k)}(x) = h_k(x) \tag{21}$$

where

$$h_k(x) = h^{(k)}(x) - k(p+k-ax)f^{(k-1)}(x) + \left(3 + \frac{(k-3)(k+2)}{2}\right) af^{(k-2)}(x). \tag{22}$$

Equation (21) shows that $f_h^{(k)}$ solves Stein equation for GIG$(p+2k, a, b)$ distribution with second member h_k given by (22). Let g_k be the density of the GIG$(p+2k, a, b)$ distribution. Then

$$\begin{aligned} f_h^{(k)}(x) =& \frac{1}{s(x)g_k(x)} \int_0^x g_k(t)h_k(t)dt \\ =& \frac{-1}{s(x)g_k(x)} \int_x^{+\infty} g_k(t)h_k(t)dt. \end{aligned} \tag{23}$$

For $p \leq -2k-1$, τ_k is decreasing on $(0, \infty)$ and has unique zero α_k. If $x \leq \alpha_k$ and if h_k is bounded, we have

$$\left|f_h^{(k)}(x)\right| \leq \|h_k\| \frac{1}{\alpha_k^2 g_k(\alpha_k)} \int_0^{\alpha_k} g_k(t)dt.$$

h_k is such that

$$\begin{aligned} |h_k(x)| =& \left|h^{(k)}(x) - k(p+k-ax)f_h^{(k-1)}(x) + \left(3 + \frac{(k-3)(k+2)}{2}\right) af_h^{(k-2)}(x)\right| \\ \leq& \left\|h^{(k)}\right\| + k|p+k-ax| \left\|f_h^{(k-1)}\right\| + a\left(3 + \frac{(k-3)(k+2)}{2}\right)\left\|f_h^{(k-2)}\right\| \\ \leq& \left\|h^{(k)}\right\| + k\max\left(|p+k|, k+1+\sqrt{(p+2k+1)^2+ab}\right)\left\|f_h^{(k-1)}\right\| \\ &+ a\left(3 + \frac{(k-3)(k+2)}{2}\right)\left\|f_h^{(k-2)}\right\|. \end{aligned}$$

As consequence, for $p \leq -2k-1$ and $x \leq \alpha_k$,

$$\left\|f_h^{(k)}\right\| \leq H \times \frac{1}{\alpha_k^2 g_k(\alpha_k)} \int_0^{\alpha_k} g_k(t)dt$$

where

$$H = \left\|h^{(k)}\right\| + k\max\left(|p+k|, k+1+\sqrt{(p+2k+1)^2+ab}\right) M_{k-1} + a\left(3 + \frac{(k-3)(k+2)}{2}\right) M_{k-2}.$$

For $p \leq -2k-1$ and $x > \alpha_k$, τ_{k-1} is decreasing and negative, hence $\dfrac{\tau_{k-1}(t)}{\tau_{k-1}(x)} \geq 1 \ \ \forall x \leq t$.
Replacing k by $k-1$ in equation (21), we have

$$f_h^{(k)}(x) = \frac{h_{k-1}(x)}{s(x)} - \frac{\tau_{k-1}(x)f_h^{k-1}(x)}{s(x)}$$

$$\left|f_h^{(k)}(x)\right| \leq \frac{\|h_{k-1}\|}{s(x)} + \frac{\left|\tau_{k-1}(x) f_h^{k-1}(x)\right|}{s(x)}.$$

$$\begin{aligned}
\left|\tau_{k-1}(x) f_h^{k-1}(x)\right| &= \left|\frac{-\tau_{k-1}(x)}{s(x)g_{k-1}(x)} \int_x^{+\infty} g_{k-1}(t)h_{k-1}(t)dt\right| \\
&\leq \|h_{k-1}\| \frac{-\tau_{k-1}(x)}{s(x)g_{k-1}(x)} \int_x^{+\infty} g_{k-1}(t)dt \\
&\leq \|h_{k-1}\| \frac{-\tau_{k-1}(x)}{s(x)g_{k-1}(x)} \int_x^{+\infty} \frac{\tau_{k-1}(t)}{\tau_{k-1}(x)} g_{k-1}(t)dt \\
&= \|h_{k-1}\| \frac{-1}{s(x)g_{k-1}(x)} \int_x^{+\infty} (s(t)g_{k-1}(t))' dt \\
&= \|h_{k-1}\|.
\end{aligned}$$

$$|h_{k-1}(x)| \leq \left\|h^{(k-1)}\right\| + (k-1)\left|(p+k-1-ax)f^{(k-2)}(x)\right| + \left(3 + \frac{(k-4)(k+1)}{2}\right) a \left\|f^{(k-3)}\right\|.$$

$$\begin{aligned}
\left|(p+k-1-ax)f^{(k-2)}(x)\right| &= \left|\frac{-(p+k-1-ax)}{s(x)g_{k-2}} \int_x^{\infty} g_{k-2}(t)h_{k-2}(t)dt\right| \\
&\leq \|h_{k-2}\| \frac{-(p+k-1-ax)}{s(x)g_{k-2}} \int_x^{\infty} g_{k-2}(t)(t)dt \\
&\leq \|h_{k-2}\| \frac{-(p+k-1-ax)}{s(x)g_{k-2}\tau_{k-2(x)}} \int_x^{\infty} \tau_{k-2}(t)g_{k-2}(t)(t)dt \\
&= \|h_{k-2}\| \frac{-(p+k-1-ax)}{s(x)g_{k-2}\tau_{k-2(x)}} \int_x^{\infty} \left(s(t)g_{k-2}(t)(t)\right)' dt \\
&= \|h_{k-2}\| \frac{p+k-1-ax}{\tau_{k-2(x)}}.
\end{aligned}$$

Let $\varphi(x) = \dfrac{p+k-1-ax}{\tau_{k-2(x)}} = \dfrac{p+k-1-ax}{\frac{b}{2} + (p+2k-3)x - \frac{a}{2}x^2}$. φ is differentiable on (α_k, ∞) and

$$\begin{aligned}
\varphi'(x) &= -\frac{ab + 2(p+k-1)(p+2k-3) - 2a(p+k-1)x + a^2x^2}{2\tau_{k-2}^2(x)} \\
&= -\frac{ab + (p+k-1)^2 + 2(p+k-1)(k-2) + (p+k-1-ax)^2}{2\tau_{k-2}^2(x)}.
\end{aligned}$$

For $p \leq -2k+3$ or $p \geq -k+1$, $\varphi'(x) \leq 0$ and $\varphi(x) \leq \varphi(\alpha_k)$ for all $x > \alpha_k$.
For $-2k+3 < p < -k+1$, if $(k-2)^2 - (p+2k-3)^2 - ab > 0$, then the only possible positive zero of $x \mapsto ab + 2(p+k-1)(p+2k-3) - 2a(p+k-1)x + a^2x^2$ is $\dfrac{p+k-1+\sqrt{(k-2)^2-(p+2k-3)^2-ab}}{a} < \alpha_k$. Hence for all $x > \alpha_k$ and for all $p \in \mathbb{R}$,

$$\varphi(x) \leq \varphi(\alpha_k) = \frac{(k+2)a + a\sqrt{(p+2k+1)^2+ab}}{4p+8k+4+4\sqrt{(p+2k+1)^2+ab}}.$$

As a consequence for $x > \alpha_k$,

$$\|h_{k-1}\| \quad \leq \quad \left\|h^{(k-1)}\right\| \ + \ \frac{a(k-1)\left(k+2+\sqrt{(p+2k+1)^2+ab}\right)}{4p+8k+4+4\sqrt{(p+2k+1)^2+ab}} \|h_{k-2}\| \ + \ a\left(3 + \frac{(k-4)(k+1)}{2}\right) M_{k-3}.$$

Now, assume $p > -2k-1$.

– On $\left(0, \frac{p+2k+1}{a}\right)$, τ_k is such that $\frac{\tau_k(t)}{\tau_k(0)} \geq 1$, $\forall 0 \leq t \leq \frac{p+2k+1}{a}$, since

$$\begin{aligned} |h_k(x)| &\leq \left\|h^{(k)}\right\| + k|p+k-ax| \left\|f_h^{k-1}\right\| + a\left(3 + \frac{(k-3)(k+2)}{2}\right) \left\|f_h^{k-2}\right\| \\ &\leq \left\|h^{(k)}\right\| + k\max\left(|p+k|, k+1\right) \left\|f_h^{k-1}\right\| + a\left(3 + \frac{(k-3)(k+2)}{2}\right) \left\|f_h^{k-2}\right\| \\ &\leq \left\|h^{(k)}\right\| + k\max\left(|p+k|, k+1\right) M_{k-1} + a\left(3 + \frac{(k-3)(k+2)}{2}\right) M_{k-2}, \end{aligned}$$

we have

$$\begin{aligned} |f_h^{(k)}(x)| &\leq \|h_k\| \frac{1}{s(x)g_k(x)} \int_0^x g_k(t)dt \\ &\leq \|h_k\| \frac{1}{\tau_k(0)s(x)g_k(x)} \int_0^x \tau_k(t)g_k(t)dt \\ &= \frac{1}{\tau_k(0)} \|h_k\| \\ &= \frac{2}{b} \|h_k\| . \end{aligned}$$

As a consequence,

$$\left\|f_h^{(k)}\right\| \leq \frac{2}{b}\left(\left\|h^{(k)}\right\| + k\max\left(|p+k|, k+1\right) M_{k-1} + a\left(3 + \frac{(k-3)(k+2)}{2}\right) M_{k-2}\right).$$

– On $\left(\frac{p+2k+1}{a}, \infty\right)$, similar arguments as in the previous cases lead to the result.

□

***Remark*:** These results can be used to estimate the rate of convergence via Stein's approach of:

- a continuous fraction with entries gammas to the generalized inverse Gaussian or Kummer distributions;
- the total resistance of an infinite tree, whose edges are equipped with independent resistances that are inverse Gaussian or reciprocal inverse Gaussian distributions to the reciprocal inverse Gaussian distribution (the inverse Gaussian and reciprocal inverse Gaussian distributions are respectively the GIG distribution with $p = -1/2$ and $p = 1/2$);
- the generalized inverse Gaussian distribution to the inverse gamma distribution.

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