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# Recent Progress in Unifying the Time- and Frequency-Domain Methods

Zhizhang (David) Chen and Michel M. Ney

**Abstract** Along with other numerical methods such as finite-difference methods, Method of Moments (MoM) or Method of Weighted Residual (MWR) has traditionally been applied in the frequency domain and has been shown to be very effective and efficient in computing open structure problems. It is generally considered to be a numerical method that is different from other numerical methods. In this paper, we summarize our recent progress in using the MoM as a general framework to unify most of numerical methods developed so far, either in frequency-domain or in time-domain. As the result, numerical issues can now be understood and derived with a common procedure. The significance of such unification is that new grid-based numerical methods, particularly effective and efficient for specific structures, can now be developed with the MoM procedure using new expansion and testing functions.

**Keywords** Method of moments (MoM) · or method of weighted residuals (MWR) · frequency-domain method · time-domain methods · mode matching and finite-different time-domain method

## 1 Introduction

Many numerical methods have been developed for simulation of electromagnetic structures. There are mainly two types of numerical methods: frequency-domain and time-domain methods. The frequency-domain methods solve Maxwell's equations in the temporal spectral domain while the time-domain methods in the original time domain in which Maxwell's equations were originally formulated. The frequency-domain methods [1] include finite-difference methods, finite-element

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methods, method of lines and mode matching. The time-domain methods include the finite-difference time-domain methods [2], transmission-line-matrix methods [3], time-domain finite-element methods [4], and time-domain integral equation methods [5]. These methods have been applied widely in solving electromagnetic structure problems. And they have been shown to be derived and developed independently based on different mathematical bases.

In this paper, we present our recent progress in unifying these methods with the method of moments (MoM) [6], or method of weighted residuals (MWR). We will show that the frequency-domain and time-domain methods can be derived with the Method of Moment (MoM). More specifically, solutions of numerical methods can be obtained by expanding solutions in terms of sets of basis functions and by minimizing the errors with sets of testing functions. Differences between different methods are simply the uses of different expansion and testing (or weighting) functions.

The work presented in this paper is the summary of our recent work. The intention is not to reduce the significance of other ways of deriving numerical methods but to provide an alternative means to understand the existing methods and to possibly create new methods, particularly effective to certain classes of electromagnetic problems.

## 2 The Method of Moments

Although Maxwell's equations for electromagnetic fields involve vectorized electric and magnetic field quantities  $\mathbf{E}$  and  $\mathbf{H}$ , they can be de-vectorized and expressed in a system of scalar equations such as:

$$L\phi - f = 0 \quad (1)$$

where  $L$  is a mathematical operator that can be either differential, integral or mixed differential and integral operators.  $\phi$  is a component of  $\mathbf{E}$  or  $\mathbf{H}$ , or a one-column vector containing all the field components.  $f$  is the known source function.

With the Method of Moments (MoM) [6], (1) was solved with two computing steps: solution expansion and error minimization.

In the solution expansion step, a pre-selected set of known basis functions in both space and time is first chosen and then used to expand  $\phi$ . Suppose that the basis functions in time and space for  $\phi$  are  $\Phi_m(\mathbf{r})$  and  $T_n(t)$ , respectively.  $n$  and  $m$  are the indices for each basis function which is independent of each other. Then,

$$\phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \Phi_m(\mathbf{r}) T_n(t) \quad (2)$$

Here  $A_{mn}$  are the expansion coefficients that are to be found.  $T_n(t)$  is expansion basis function in time. For the frequency-domain methods, it does not exist as the frequency-domain methods deal with a temporal frequency rather than the time factor.

In the error minimization step, the residual error of placing (2) in (1) is:

$$R = L \left[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \Phi_m(\mathbf{r}) T_n(t) \right] - f \quad (3)$$

Making  $R$  to be completely zero in the whole solution domain is difficult to achieve. By the way of MoM, a preselected known testing or weighting functions, denoted as  $W_k(\mathbf{r})$  and  $P_l(t)$ , is chosen. It is then used to make  $R$  null in terms of its inner product with the testing functions:

$$\langle R, W_k(\mathbf{r}) P_l(t) \rangle = 0 \quad (4)$$

More specifically, substitution of (3) into (4) leads to a system of equations that contain the expansion coefficient  $A_{mn}$ :

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \langle L \Phi_m(\mathbf{r}) T_n(t), W_k(\mathbf{r}) P_l(t) \rangle = \langle f, W_k(\mathbf{r}) P_l(t) \rangle \quad (5)$$

The above equation is normally solvable, in some cases in a recursive fashion. As a result, the approximated solutions are obtained with (2).

It should be noted that the expansion basis function and testing functions have to be chosen carefully to avoid solution divergence. Readers are referred to [7, 8] for more details.

### 3 Derivations of Frequency-Domain Numerical Methods

Many frequency numerical methods have been developed for solving frequency-domain Maxwell's equations [1]. Due to limitation of space, we will consider two of them: the spectral domain and mode matching methods.

#### 3.1 Spectral Domain Method

The spectral domain method was developed specifically for planar structures types, such as microstrip lines ([1] and references therein). Based on the frequency-domain Maxwell's equations, an integral equation is first developed where current densities or charges on the metal strips are the unknown functions to be solved for. Then the method of moment is applied and a system of linear equations is obtained for the expansion coefficients. The key is that the elements of the coefficient matrix of the system of linear equations are efficiently found through the use of the Green's functions in the spectral (or spatial frequency) domain rather than directly in the spatial domain [9]. Therefore, the spectral domain method falls within the framework of MoM.

### 3.2 Mode Matching Method

Mode matching is one of the most frequently used methods for solving boundary-value problems of waveguide structures ([1] and references therein).

The first step of the mode matching is to expand the unknown fields in the individual regions in terms of their respective modes. The expanded field components are then matched at the interfaces of the two adjacent regions. By utilizing the orthogonality property of the mode functions, a set of linear simultaneous equations can be established for the unknown modal expansion coefficients. Such a process can be considered exactly the same as that of the MoM. More specifically, it can be described as follows.

Suppose that the problem to be solved is:

$$\begin{array}{ll} \text{Region \#1} & \text{Region \#2} \\ L\phi_1 - f_1 = 0 & L\phi_2 - f_2 = 0 \end{array} \quad (6)$$

$$B_1(\phi_1) - B_2(\phi_2) = 0 \quad \text{at interface} \quad \mathbf{r} = \mathbf{r}_{\text{interface}} \quad (7)$$

where  $B_1$  and  $B_2$  are the linear operators for the interface conditions.

Then,  $\phi_1$  and  $\phi_2$  are expanded in terms of the known mode functions  $\phi_{1j}$  and  $\phi_{2j}$  that satisfy (6) and the boundary conditions in region #1 and region #2, respectively:

$$\phi_1 = \sum_j a_j \phi_{1j} \quad (8)$$

$$\phi_2 = \sum_j b_j \phi_{2j} \quad (9)$$

Substitution of them into the interface conditions (7) leads to the residual:

$$R = \sum_j a_j B_1(\phi_{1j}) - \sum_j b_j B_2(\phi_{2j}) \quad (10)$$

Minimization of  $R$  by integration of (10) with the mode functions reads:

$$\begin{aligned} \sum_j a_j \int B_1(\phi_{1j}) \phi_{1i} ds &= \sum_j b_j \int B_2(\phi_{2j}) \phi_{1i} ds \\ \sum_j a_j \int B_1(\phi_{1j}) \phi_{2i} ds &= \sum_j b_j \int B_2(\phi_{2j}) \phi_{2i} ds \end{aligned} \quad (11)$$

The above equations allow the solutions of the expansion coefficients  $a_j$  and  $b_j$ .

### 3.3 Other Frequency-Domain Methods

Other frequency-domain numerical methods can be derived in a similar way. Readers are referred to [10].

## 4 Derivations of Time-Domain Numerical Methods

Like its frequency-domain counterparts, many time-domain methods have been developed. In the following paragraphs, we present our recent results in deriving these time-domain methods with MoM.

### 4.1 Finite-Difference Based Time-Domain (FDTD) Methods

The finite-differenced based time domain methods widely used for solving electromagnetic structure problems. They include finite-difference time-domain (FDTD) method, transmission-line-matrix (TLM) method, multi-resolution time-domain (MRTD) method, pseudo-spectral time-domain (PSTD) method, Crank-Nicolson FDTD, alternating-direction-implicit (ADI) FDTD and unconditionally stable FDTD using weighted Laguerre polynomials. They all can be derived from MoM. However, because of space limitations here, only the conventional FDTD of Yee's scheme and the PSTD are derived with MoM. For details on the derivations of other finite-difference based methods, readers are referred to [11].

Before the derivation, the following rooftop function  $T$  is introduced:

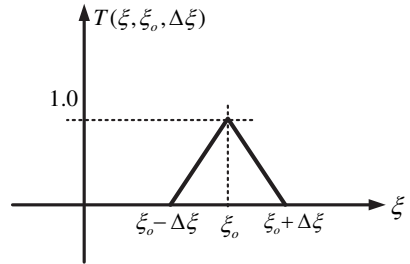
$$T(\xi, \xi_o, \Delta\xi) = \begin{cases} 1 - \frac{|\xi - \xi_o|}{\Delta\xi} & \text{when } |\xi - \xi_o| \leq \Delta\xi \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

Its graphical presentation is shown in Fig. 1:

In the finite-difference time-domain method, the equations to be solved are time-domain Maxwell's equations with six field components in the  $x$ ,  $y$ , and  $z$ , directions, respectively. For instance, one of the equations can be expressed as:

$$\epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \quad (13)$$

By the way of MoM, the field components are expanded as follows:



**Fig. 1** Illustration of the triangle function  $T$

$$\begin{aligned}
E_x &= \sum_{i_x, i_y, i_z, n} E_x \Big|_{i_x + \frac{1}{2}, i_y, i_z}^n T \left[ x, \left( i_x + \frac{1}{2} \right) \Delta x, \Delta x \right] \\
&\quad T(y, i_y \Delta y, \Delta y) T(z, i_z \Delta z, \Delta z) T(t, n \Delta t, \Delta t) \\
H_y &= \sum_{i_x, i_y, i_z, n} H_y \Big|_{i_x + \frac{1}{2}, i_y, i_z + \frac{1}{2}}^{n + \frac{1}{2}} T \left[ x, \left( i_x + \frac{1}{2} \right) \Delta x, \Delta x \right] \\
&\quad T(y, i_y \Delta y, \Delta y) T \left[ z, \left( i_z + \frac{1}{2} \right) \Delta z, \Delta z \right] T \left[ t, \left( n + \frac{1}{2} \right) \Delta t, \Delta t \right] \\
H_z &= \sum_{i_x, i_y, i_z, n} H_z \Big|_{i_x + \frac{1}{2}, i_y + \frac{1}{2}, i_z}^{n + \frac{1}{2}} T \left[ x, \left( i_x + \frac{1}{2} \right) \Delta x, \Delta x \right] \\
&\quad T \left[ y, \left( i_y + \frac{1}{2} \right) \Delta y, \Delta y \right] T(z, i_z \Delta z, \Delta z) T \left[ t, \left( n + \frac{1}{2} \right) \Delta t, \Delta t \right]
\end{aligned} \tag{14}$$

where  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  are the spatial steps and  $\Delta t$  is the time step.  $i_x$ ,  $i_y$ ,  $i_z$  and  $n$  are the spatial and temporal indices.  $E_x|_{i_x + \frac{1}{2}, i_y, i_z}^n$ ,  $H_y|_{i_x + \frac{1}{2}, i_y, i_z + \frac{1}{2}}^{n + \frac{1}{2}}$  and  $H_z|_{i_x + \frac{1}{2}, i_y + \frac{1}{2}, i_z}^{n + \frac{1}{2}}$  are expansion coefficients. Note that because of the use of the rooftop function, expansion coefficients happen to be the field values at the grid points. For instance,  $E_x|_{i_x + \frac{1}{2}, i_y, i_z}^n = E_x(t = n \Delta t, x = (i_x + \frac{1}{2}) \Delta x, y = i_y \Delta y, z = i_z \Delta z)$

Substitution of (14) into (13) leads to

$$\begin{aligned}
\varepsilon \frac{E_x|_{i_x + \frac{1}{2}, i_y, i_z}^{n+1} - E_x|_{i_x + \frac{1}{2}, i_y, i_z}^n}{\Delta t} &= \frac{H_z|_{i_x + \frac{1}{2}, i_y + \frac{1}{2}, i_z}^{n + \frac{1}{2}} - H_z|_{i_x + \frac{1}{2}, i_y - \frac{1}{2}, i_z}^{n + \frac{1}{2}}}{\Delta y} \\
&\quad - \frac{H_y|_{i_x + \frac{1}{2}, i_y, i_z + \frac{1}{2}}^{n + \frac{1}{2}} - H_y|_{i_x + \frac{1}{2}, i_y, i_z - \frac{1}{2}}^{n + \frac{1}{2}}}{\Delta z} = 0
\end{aligned} \tag{15}$$

This is exactly the same as the FDTD equation derived by replacing differentials with their finite-difference counterparts [2].

For derivations of other finite-difference based time-domain methods with Method of Moments (MoM) including the derivations of PSTD and Crank Nicolson methods, readers are referred to [11].

## 4.2 Derivations of the Time-Domain Finite-Element Methods

The time-domain finite-element methods have gained much attention recently due to demands for wideband and transient simulations of modern communication devices

and components. In the derivations of the time-domain FEM, MoM procedure was applied for expansion and error testing in the spatial domain [4, 12]. In the time domain, the finite-differences were used to replace the temporal derivatives [12]. Since a finite-difference approach can be derived from the MoM as shown above for the derivation of the FDTD method, the time-domain FEM is then derivable from the MoM. More details can be found in [12].

### ***4.3 Derivations of the Time-Domain Integral Equation (TDIE) Methods***

TDIE methods are another type of numerical methods where solutions can be obtained from field integration over interfaces and boundaries with appropriate Green's functions [13]. They have the advantage of reducing problem complexity by one dimension, i.e. three-dimensional structures are solved with two-dimensional equations and two-dimensional structures with one-dimensional equation. However, computational inefficiency and late-time instability have prevented them from becoming effective simulation tools. Much of the recent research efforts have been focused on resolving the instability issue.

There are many different integral equations derived from Maxwell's equations such as the electric field integral equation (EFIE). They have been solved by following the solution steps of MoM, expansion of field quantities and minimization of residual errors [12].

## **5 Numerical Instability**

In a physically realizable problem, field quantities are of finite values. Therefore, expansion (2) should be finite at any spatial location and at any time. In a normal situation, expansion basis functions  $\Phi_m(\mathbf{r})$  and  $T_n(t)$  are selected to be bounded. Therefore, to ensure that (2) is finite, the expansion coefficient  $A_{mn}$  should be finite in its value in particular when  $n \rightarrow \infty$  and  $m \rightarrow \infty$ . This has translated into the well-known CFL stability condition in the FDTD method; it ensures that the expansion coefficients do not grow with time. More information for the FDTD case can be found in [11]. For other methods, appropriate conditions can also be developed based on the MoM perspective.

## **6 Numerical Dispersion**

Since numerical methods are shown to provide expanded approximate solutions, it is very desirable to assess the errors of such approximations and impacts of discretization parameters  $\Delta x$ ,  $\Delta y$ ,  $\Delta z$  and  $\Delta t$  on these errors.



Direct evaluation of residual  $R$  represented by (3) is normally difficult. An alternative way is to examine the approximate solution (2) in the spectral domain. In other words, because the solutions are approximate, relationship between the spatial frequencies and temporal frequency of the approximate solutions will be different from the one for the exact solutions. This leads to the so-called numerical dispersion errors. In the FDTD case, such a numerical dispersion can be found in a simple analytical form as described in [11]. In other cases, they can be found only numerically.

## 7 Discussions and Conclusions

In this paper, we summarize our recent results in unifying time- and frequency-domain numerical methods for computing electromagnetic structures with a common MoM framework. It has been shown that numerical methods are derivable with the MoM procedure. The differences among the different methods are the use of different expansion and testing functions. Therefore, not only numerical methods are unified under the framework of MoM, but also new methods, particularly effective and efficient for specific structures, can now be developed with a common procedure. In addition, numerical instability and dispersions can be explained relatively easily within the framework.

It should be mentioned that the work presented is not intended to exclude existing or other ways of developing numerical methods. It is meant to provide another perspective of understanding numerical methods and their associate physical interpretations. It is hoped that it may give another dimension in advance of computational electromagnetics.

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