



HAL
open science

Revisiting the asymptotics of the flow for some dynamical systems on the torus

Marc Briane, Loïc Hervé

► **To cite this version:**

Marc Briane, Loïc Hervé. Revisiting the asymptotics of the flow for some dynamical systems on the torus. 2019. hal-02418449

HAL Id: hal-02418449

<https://hal.science/hal-02418449>

Preprint submitted on 18 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Revisiting the asymptotics of the flow for some dynamical systems on the torus

Marc Briane & Loïc Hervé

Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
mbriane@insa-rennes.fr & loic.herve@insa-rennes.fr

Wednesday 18th December, 2019

Abstract

In this paper we study the large time asymptotics of the flow of a dynamical system $X' = b(X)$ posed in the d -dimensional torus. Rather than using the classical unique ergodicity condition which is not fulfilled if b vanishes at different points, we only assume that the set of the averages of b with respect to the invariant probability measures for the flow is reduced to a singleton. We also rewrite the Liouville theorem which holds for any invariant probability measure μ , namely μb is divergence free, as a divergence-curl formula satisfied by any regular periodic function. The combination of these two tools turns out to be a new approach to get the asymptotics for some flows. This allows us to obtain the desired asymptotics in any dimension when $b = a \xi$ with a a possibly vanishing periodic nonnegative function and ξ a nonzero vector in \mathbb{R}^d , or when $b = A \nabla v$ with A a periodic nonnegative symmetric matrix-valued function and v a periodic function.

Keywords: dynamical system, flow, torus, \mathbb{Z}^d -periodic, asymptotics, invariant measure, ergodic theorem, divergence-curl result

Mathematics Subject Classification: 37C10, 37C40, 34E10

1 Introduction

This paper is devoted to the large time asymptotics of the solution $X(t, x)$ to the dynamical system

$$\begin{cases} \frac{\partial X}{\partial t}(t, x) = b(X(t, x)), & t \in \mathbb{R} \\ X(0, x) = x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

where b is a C^1 -regular \mathbb{Z}^d -periodic vector field in \mathbb{R}^d . More precisely, we focus on the existence of the limit of $X(t, x)/t$ as $t \rightarrow \infty$ for $x \in \mathbb{R}^d$. This problem is strongly connected to the asymptotic behavior of the transport equation with an oscillating velocity

$$\frac{\partial u_\varepsilon}{\partial t}(t, x) + b\left(\frac{x}{\varepsilon}\right) \cdot \nabla_x u_\varepsilon(t, x) = 0 \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^d, \quad (1.2)$$

which is dealt with in [1, 9, 6, 7, 13, 3]. Otherwise, this question naturally arises in ergodic theory since it involves the flow T_t defined by

$$(T_t\varphi)(x) := \varphi(X(t, x)) \quad \text{for } \varphi \text{ continuous and } \mathbb{Z}^d\text{-periodic in } \mathbb{R}^d, \quad x \in \mathbb{R}^d, \quad (1.3)$$

which is associated with system (1.1), and the Borel measures μ on the torus $Y_d := \mathbb{R}^d/\mathbb{Z}^d$ which are *invariant for the flow*, i.e.

$$\forall t \in \mathbb{R}, \quad \mu \circ T_t = \mu. \quad (1.4)$$

A strengthened variant of the famous Birkhoff ergodic theorem [5, Theorem 2, Section 1.8] claims that if the flow is *uniquely ergodic*, i.e. there exists a unique probability measure μ on Y_d invariant for the flow, then any continuous \mathbb{Z}^d -periodic function f satisfies

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \left[\frac{1}{t} \int_0^t f(X(s, x)) ds \right] = \int_{Y_d} f(y) d\mu(y), \quad (1.5)$$

and the converse actually holds true. In the particular case where $f := b$, limit (1.5) yields

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \int_{Y_d} b(y) d\mu(y). \quad (1.6)$$

The unique ergodicity condition seems to be a rather restrictive condition on the flows of type (1.1). For example, if the vector field b vanishes at two points $x \neq y$ in the torus, then any convex combination of the Dirac masses δ_x and δ_y are invariant for the flow. When b does not vanish in \mathbb{R}^d , the unique ergodicity of the flow is not clear. This is true in dimension one (see Proposition 3.1) but the situation is more complicated in higher dimension. As best we know the asymptotics of the flow is completely known only in dimension two for a non-vanishing vector field b . Specifically in dimension two, Peirone [11] obtained the asymptotics of the flow through the following alternative:

- Under the existence of a periodic trajectory $X(\cdot, y)$ of (1.1) with respect to the torus (see (1.14)), the limit of (1.6) does exist but may depend on x .
- In the absence of periodic trajectory with respect to the torus, the limit of (1.6) exists independently of x . To this end, Peirone used an one-dimensional ergodicity result transversally to Siegel's curve, but his approach has nothing to do with the unique ergodicity of the flow.

On the other hand, when the two-dimensional flow associated with a non-vanishing vector field $b = (b_1, b_2)$ admits an invariant measure with a positive \mathbb{Z}^2 -periodic density σ with respect to the Lebesgue measure, or equivalently, by virtue of Liouville's theorem σb is divergence free in \mathbb{R}^2 , then Kolmogorov's theorem [10] implies the existence of a diffeomorphism on the torus which transforms the dynamical system (1.1) into the dynamical system

$$\begin{cases} \frac{\partial Y}{\partial t}(t, y) = a(Y(t, y)) \xi, & t \in \mathbb{R} \\ Y(0, x) = y \in \mathbb{R}^2, \end{cases} \quad (1.7)$$

where a is a regular \mathbb{Z}^2 -periodic vector field in \mathbb{R}^2 and $\xi = (\xi_1, \xi_2)$ is a nonzero constant vector in \mathbb{R}^2 . In this two-dimensional setting with the additional assumption that b_1 is nonvanishing, Tassa [13] also obtained the asymptotics of the flow through the following alternative which is strongly connected to the above Peirone's alternative:

- If ξ_1 and ξ_2 are rationally dependent, the limit of (1.6) exists but does depend on x .
- If ξ_1 and ξ_2 are rationally independent, or equivalently the so-called *rotation number* is irrational (see, *e.g.*, [12, Chapter I, Section 4.1]), the limit of (1.6) exists independently of x . The more general result where the vector field b (rather than b_1) is nonvanishing with an invariant density $\sigma \in C^5(Y_2)$ is due to Kolmogorov [10].

Note that in [11, 13] the nonvanishing condition of the vector field b is essential to obtain the asymptotics of the flow.

Actually, the unique ergodicity of the flow (1.3) associated with b is not needed to get the desired asymptotics (1.6) which is much less restrictive than (1.5). Indeed, our approach consists in replacing the unique ergodicity assumption by the weaker uniqueness condition

$$\# \left\{ \int_{Y_d} b(y) d\mu(y) : \mu \text{ is an invariant probability measure for the flow} \right\} = 1. \quad (1.8)$$

This new condition is obtained by observing that the Birkhoff time average in (1.6) is only addressed (contrary to (1.5)) to the function b . Then, we revisit (see Proposition 2.1) the proof of the existence of an invariant probability measure for the flow defined on a compact space (the torus here). We conclude by proving (see Theorem 2.1) that condition (1.8) turns out to be equivalent to asymptotics (1.6). On the other hand, we show (see Theorem 2.2 and Remark 2.2) that the Liouville theorem satisfied by an invariant measure μ for the flow associated with b can be regarded as a divergence-curl result such that for any \mathbb{Z}^d -periodic C^1 -function ψ ,

$$\int_{Y_d} b(y) \cdot \nabla \psi(y) d\mu(y) = 0. \quad (1.9)$$

Our new approach consists in combining the tools (1.8) and (1.9) to obtain the limit of $X(t, x)/t$ as $t \rightarrow \infty$ for some dynamical systems (1.1). So, we get the desired asymptotics for the dynamical system (1.7) (see Proposition 3.2) assuming that the \mathbb{Z}^d -periodic function a is nonnegative and may vanish contrary to [11, 13], and that the vector ξ satisfies one of the two following conditions which extend in any dimension the above two-dimensional conditions obtained by Peirone (see Remark 3.3 and Proposition 3.4) and Tassa:

- There exists $T > 0$ such that $T\xi \in \mathbb{Z}^d$, which leads us to a limit of $X(t, x)/t$ depending on x .
- For any $k \in \mathbb{Z}^d \setminus \{0\}$, $\xi \cdot k \neq 0$, which leads us to the limit

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = a^* \xi \quad \text{with} \quad a^* := \begin{cases} \left(\int_{Y_d} a^{-1}(y) dy \right)^{-1} & \text{if } a > 0 \text{ in } \mathbb{R}^d \\ 0 & \text{if } a \text{ vanishes in } \mathbb{R}^d. \end{cases}$$

When a vanishes at different points in the torus, the unique ergodicity of the flow (1.5) is not satisfied, while the asymptotics of the flow holds true for b in (1.6). The former asymptotics result easily extends (see Corollary 3.1) to the case where the vector field b is rectifiable to a fixed direction ξ through a diffeomorphism Φ on the torus, *i.e.* $b = a \circ \Phi \nabla \Phi^{-1} \xi$ for some \mathbb{Z}^d -periodic nonnegative function a . Finally, again using (1.8) and (1.9) we obtain the zero vector limit in (1.6) when b is a current field (see Proposition 3.5), *i.e.* $b = A \nabla v$ with A a regular \mathbb{Z}^d -periodic nonnegative symmetric matrix-valued conductivity and v a regular \mathbb{Z}^d -periodic potential.

Notation

- Y_d for $d \geq 1$, denotes the d -dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, which is identified to the cube $[0, 1)^d$ in \mathbb{R}^d .
- $C_c^k(\mathbb{R}^d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^d)$ with compact support.
- $C_{\sharp}^k(Y_d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$ which are \mathbb{Z}^d -periodic, *i.e.*

$$\forall k \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d, \quad f(x+k) = f(x).$$
- $L_{\sharp}^p(Y_d)$ for $p \geq 1$, denotes the space of the real-valued functions in $L_{\text{loc}}^p(\mathbb{R}^d)$ which are \mathbb{Z}^d -periodic.
- $\mathcal{M}(Y_d)$ denotes the space of the Radon measures on Y_d , and $\mathcal{M}_p(Y_d)$ denotes the space of the probability measures on Y_d .
- $\mathcal{D}'(\mathbb{R}^d)$ denotes the space of the distributions on \mathbb{R}^d .

Definitions and recalls

Let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector-valued function in $C_{\sharp}^1(Y_d)^d$. Consider the dynamical system

$$\begin{cases} \frac{\partial X}{\partial t}(t, x) = b(X(t, x)), & t \in \mathbb{R} \\ X(0, x) = x \in \mathbb{R}^d. \end{cases} \quad (1.10)$$

The solution $X(\cdot, x)$ of (1.10) which is known to be unique (see, *e.g.*, [8, Section 17.4]) is associated with the flow $(T_t)_{t \in \mathbb{R}}$, defined by

$$T_t(\varphi)(x) := \varphi(X(t, x)) \quad \text{for } \varphi \in C_{\sharp}^0(Y_d) \text{ and } x \in \mathbb{R}^d, \quad (1.11)$$

which satisfies the semi-group property

$$\forall s, t \in \mathbb{R}, \quad T_{s+t} = T_s \circ T_t, \quad (1.12)$$

and is well defined in the torus since

$$\forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d, \forall k \in \mathbb{Z}^d, \quad X(t, x+k) = X(t, x) + k. \quad (1.13)$$

Property (1.13) follows immediately from the uniqueness of the solution X to (1.10) combined with the \mathbb{Z}^d -periodicity of b .

A solution $X(\cdot, x)$ to (1.10) is said to be *periodic in the torus* if there exists $\tau > 0$ and $k \in \mathbb{Z}^d$ such that

$$\forall t \in \mathbb{R}, \quad X(t+\tau, x) = X(t, x) + k. \quad (1.14)$$

If $k = 0$ the solution is said to be periodic in \mathbb{R}^d .

A measure μ in $\mathcal{M}_p(Y_d)$ is said to be *invariant for the flow* T_t (see, *e.g.*, [5, Chap. 2]) if

$$\forall t \in \mathbb{R}, \forall \psi \in C_{\sharp}^0(Y_d), \quad \int_{Y_d} T_t(\psi)(y) d\mu(y) = \int_{Y_d} \psi(X(t, y)) d\mu(y) = \int_{Y_d} \psi(y) d\mu(y), \quad (1.15)$$

or equivalently, the image measure of μ by the flow T_t agrees with μ .

2 Some general results

2.1 Existence of invariant probability measures for the flow

When a flow preserves the set of the continuous functions on a compact metric space, the existence of an invariant probability measure for the flow is a classical statement which can be derived thanks to a weak compactness argument applied to sequences of probability measures defined from the Birkhoff time averages in (1.5) (see, *e.g.*, [5, Theorem 1, Section 1.8] in the discrete time case). The following result adapts this statement restricting it to the limit points of the Birkhoff time averages for a given function.

Proposition 2.1 *Let $b \in C_{\sharp}^1(Y_d)^d$. There exists an invariant probability measure on Y_d for the flow T_t (1.11) associated with b . Moreover, let $g \in C_{\sharp}^0(Y_d)$ and $x \in \mathbb{R}^d$ be fixed, and let $(t_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that $\lim_n t_n = \infty$. Then, for any limit point a of the sequence $(u_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ defined by*

$$u_n := \frac{1}{t_n} \int_0^{t_n} g(X(s, x)) ds, \quad n \in \mathbb{N}, \quad (2.1)$$

there exists a probability measure $\mu \in \mathcal{M}_p(Y_d)$ (depending a priori on x and g) which is invariant for the flow T_t and satisfies

$$a = \int_{Y_d} g(y) d\mu(y). \quad (2.2)$$

Proof. As above mentioned the following lemma is classical in ergodic theory. For the reader's convenience its proof is postponed to the Appendix.

Lemma 2.2 *Let $x \in \mathbb{R}^d$, let $(r_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that $\lim_n r_n = \infty$, and let $\nu_n \in \mathcal{M}_p(Y_d)$, $n \in \mathbb{N}$, be the probability measure defined by*

$$\int_{Y_d} f(y) d\nu_n(y) = \frac{1}{r_n} \int_0^{r_n} f(X(s, x)) ds \quad \text{for } f \in C_{\sharp}^0(Y_d). \quad (2.3)$$

Then, there exists a subsequence $(\nu_{n_k})_{k \in \mathbb{N}}$ of $(\nu_n)_{n \in \mathbb{N}}$ which converges weakly $$ to some probability measure $\mu \in \mathcal{M}_p(Y_d)$ which is invariant for the flow T_t .*

Let a be a limit point of the sequence $(u_n)_{n \in \mathbb{N}}$ (2.1), namely

$$a = \lim_{n \rightarrow \infty} \frac{1}{t_{\theta(n)}} \int_0^{t_{\theta(n)}} g(X(t, x)) dt,$$

for some subsequence $(t_{\theta(n)})_{n \in \mathbb{N}}$ of $(t_n)_{n \in \mathbb{N}}$. Set $r_n := t_{\theta(n)}$, and consider the associated sequence $(\nu_n)_{n \in \mathbb{N}}$ of probability measures on Y_d given by (2.3). By Lemma 2.2 we can extract a subsequence $(\nu_{n_k})_{k \in \mathbb{N}}$ which converges weakly $*$ to some invariant measure $\mu \in \mathcal{M}_p(Y_d)$ for the flow T_t . We thus have

$$\forall f \in C_{\sharp}^0(Y_d), \quad \lim_{k \rightarrow \infty} \int_{Y_d} f(y) d\nu_{n_k}(y) = \int_{Y_d} f(y) d\mu(y),$$

which implies in particular that

$$a = \lim_{k \rightarrow \infty} \frac{1}{r_{n_k}} \int_0^{r_{n_k}} g(X(s, x)) ds = \lim_{k \rightarrow \infty} \int_{Y_d} g(y) d\nu_{n_k}(y) = \int_{Y_d} g(y) d\mu(y).$$

□

Remark 2.1 *The second assertion of Proposition 2.1 may provide a simple way to find the limit of Birkhoff's time averages for a specific function, when the general form of the limit in Birkhoff's theorem is unknown. For instance, if the support of any invariant probability measure for the flow is contained in a subset F of Y_d and if $g \in C_{\#}^0(Y_d)$ vanishes on F , then*

$$\forall x \in Y_d, \quad \lim_{t \rightarrow \infty} \left[\frac{1}{t} \int_0^t g(X(s, x)) ds \right] = 0,$$

since any limit point of these averages is zero by (2.2).

2.2 A criterium for the asymptotics of the flow

It is known (see, e.g., [5, Theorem 2, Section 1.8] for the discrete case) that the uniqueness of an invariant measure $\mu \in \mathcal{M}_p(Y_d)$ for the flow T_t is equivalent to the pointwise property:

$$\forall x \in \mathbb{R}^d, \forall f \in C_{\#}^0(Y_d), \quad \lim_{t \rightarrow \infty} \left(\frac{1}{t} \int_0^t f(X(s, x)) ds \right) = \int_{Y_d} f(y) d\mu(y). \quad (2.4)$$

The following result which is new as the best we know, allows us to restrict condition (2.4) to $f = b$ and to derive the asymptotics of the flow at each point in \mathbb{R}^d , by assuming the uniqueness of the averages of b with respect to the invariant probability measures on Y_d for the flow rather than the uniqueness of an invariant measure for the flow.

Theorem 2.1 *Let $b \in C_{\#}^1(Y_d)^d$. Define the sets*

$$\mathcal{I}_b := \left\{ \mu \in \mathcal{M}_p(Y_d) : \mu \text{ is invariant for the flow } T_t \right\} \quad \text{and} \quad C_b := \left\{ \int_{Y_d} b d\mu : \mu \in \mathcal{I}_b \right\}. \quad (2.5)$$

Then, \mathcal{I}_b is a nonempty set, and C_b is a nonempty compact convex set of \mathbb{R}^d . Moreover, the following equivalence holds for any $\zeta \in \mathbb{R}^d$,

$$C_b = \{\zeta\} \quad \Leftrightarrow \quad \forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \zeta. \quad (2.6)$$

Proof. By virtue of Proposition 2.1 the sets \mathcal{I}_b and C_b are nonempty. The set C_b is a convex set in \mathbb{R}^d since \mathcal{I}_b is clearly convex. Using the compactness of $\mathcal{M}_p(Y_d)$ for the weak $*$ topology, we get that C_b is a closed set in \mathbb{R}^d . We also have $C_b \subset [0, \|b\|_{\infty}]$, so that C_b is a compact set of \mathbb{R}^d .

Now, assume that $C_b = \{\zeta\}$. Let $x \in \mathbb{R}^d$, let $(t_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that $\lim_n t_n = \infty$, and define the sequence $(u_n)_{n \in \mathbb{N}}$ by (2.1) with the function $g := b \cdot \xi$ for $\xi \in \mathbb{R}^d$. Let a be a limit point of the sequence $(u_n)_{n \in \mathbb{N}}$. By Proposition 2.1 there exists an invariant measure $\mu \in \mathcal{M}_p(Y_d)$ for the flow T_t satisfying

$$a = \int_{Y_d} b(y) \cdot \xi d\mu(y),$$

which by hypothesis implies that $a = \zeta \cdot \xi$. Hence, $\zeta \cdot \xi$ is the unique limit point of the bounded sequence $(u_n)_{n \in \mathbb{N}}$ which thus converges to $\zeta \cdot \xi$. Therefore, due to the arbitrariness of the sequence $(t_n)_{n \in \mathbb{N}}$ we obtain that

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \lim_{t \rightarrow \infty} \left(\frac{1}{t} \int_0^t b(X(s, x)) ds \right) = \zeta.$$

Conversely, assume that the right-hand side of (2.6) holds, which implies that

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \left(\frac{1}{t} \int_0^t b(X(s, x)) ds \right) = \zeta.$$

Then, integrating over Y_d the former equality with respect to any probability measure $\mu \in \mathcal{I}_b$, then applying successively Lebesgue's dominated convergence theorem and Fubini's theorem, we get that

$$\begin{aligned} \zeta &= \lim_{t \rightarrow \infty} \int_{Y_d} \left(\frac{1}{t} \int_0^t b(X(s, x)) ds \right) d\mu(x) \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\int_{Y_d} b(X(s, x)) d\mu(x) \right) ds = \int_{Y_d} b(x) d\mu(x). \end{aligned}$$

which shows that $C_b = \{\zeta\}$. This concludes the proof of (2.6). \square

2.3 Liouville's theorem and a divergence-curl result

Liouville's theorem provides a criterium for a probability measure on a smooth compact manifold in \mathbb{R}^d (see, e.g., [5, Theorem 1, Section 2.2]) to be invariant for the flow. The next result revisits this theorem in $\mathcal{M}_p(Y_d)$ in association with a divergence-curl result on the torus.

Theorem 2.2 *Let $b \in C_{\sharp}^1(Y_d)^d$ and let $\mu \in \mathcal{M}_p(Y_d)$. We define the Radon measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^d)$ on \mathbb{R}^d by*

$$\int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x) := \int_{Y_d} \varphi_{\sharp}(y) d\mu(y) \quad \text{where} \quad \varphi_{\sharp}(\cdot) := \sum_{k \in \mathbb{Z}^d} \varphi(\cdot + k) \quad \text{for } \varphi \in C_c^0(\mathbb{R}^d). \quad (2.7)$$

Then, μ is invariant for the flow T_t if, and only if, one of the two following conditions is satisfied:

$$\operatorname{div}(\tilde{\mu} b) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \quad (2.8)$$

$$\forall \psi \in C_{\sharp}^1(Y_d), \quad \int_{Y_d} b(y) \cdot \nabla \psi(y) d\mu(y) = 0. \quad (2.9)$$

Proof. Assume that μ is invariant for the flow, i.e. (1.15). Let $\varphi \in C_c^1(\mathbb{R}^d)$. Since by (1.13) we have for any $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$,

$$[\varphi(X(t, \cdot))]_{\sharp}(y) = \sum_{k \in \mathbb{Z}^d} \varphi(X(t, y + k)) = \sum_{k \in \mathbb{Z}^d} \varphi(X(t, y) + k) = \varphi_{\sharp}(X(t, y)), \quad (2.10)$$

it follows from (2.7) and the invariance of μ that

$$\begin{aligned} \forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) &= \int_{Y_d} [\varphi(X(t, \cdot))]_{\sharp}(y) d\mu(y) = \int_{Y_d} \varphi_{\sharp}(X(t, y)) d\mu(y) = \\ &= \int_{Y_d} \varphi_{\sharp}(y) d\mu(y) = \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x). \end{aligned}$$

Taking the derivative of the former expression with respect to t , we get that

$$\forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} b(X(t, x)) \cdot \nabla \varphi(X(t, x)) d\tilde{\mu}(x) = 0,$$

which at $t = 0$ yields

$$\forall \varphi \in C_c^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} b(x) \cdot \nabla \varphi(x) d\tilde{\mu}(x) = 0, \quad (2.11)$$

namely the variational formulation of the distributional equation (2.8).

Conversely, assume that equation (2.8) holds true. Let $\varphi \in C_c^1(\mathbb{R}^d)$ and define the function $\phi \in C^1(\mathbb{R} \times \mathbb{R}^d)$ by $\phi(t, x) := \varphi(X(t, x))$. By the semi-group property (1.12) we have for any $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\partial}{\partial s}(\phi(s+t, X(-s, x))) &= \frac{\partial}{\partial s}(\phi(t, x)) = 0 \\ &= \frac{\partial \phi}{\partial s}(s+t, X(-s, x)) - b(X(-s, x)) \cdot \nabla_x \phi(s+t, X(-s, x)), \end{aligned}$$

which at $s = 0$ gives the classical transport equation

$$\forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d, \quad \frac{\partial \phi}{\partial t}(t, x) = b(x) \cdot \nabla_x \phi(t, x). \quad (2.12)$$

Hence, since $\varphi(X(t, \cdot))$ is in $C^1(\mathbb{R}^d)$ and has a compact support independent of t when t lies in a compact set of \mathbb{R} , we deduce from (2.12) and (2.8) that

$$\forall t \in \mathbb{R}, \quad \frac{d}{dt} \left(\int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) \right) = \int_{\mathbb{R}^d} b(x) \cdot \nabla_x (\varphi(X(t, x))) d\tilde{\mu}(x) = 0,$$

or equivalently,

$$\forall t \in \mathbb{R}, \quad \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) = \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x).$$

On the other hand, we have the following result.

Lemma 2.3 ([2], Lemma 3.5) *For any smooth \mathbb{Z}^d -periodic function $\psi \in C_{\#}^{\infty}(Y_d)$, there exists a smooth function with compact support $\varphi \in C_c^{\infty}(\mathbb{R}^d)$ such that $\psi = \varphi_{\#}$.*

Hence, using relation (2.10) and definition (2.7) we get that for any $\psi \in C_{\#}^{\infty}(Y)$,

$$\begin{aligned} \forall t \in \mathbb{R}, \quad \int_{Y_d} \psi(X(t, y)) d\mu(y) &= \int_{Y_d} \varphi_{\#}(X(t, y)) d\mu(y) = \int_{\mathbb{R}^d} \varphi(X(t, x)) d\tilde{\mu}(x) = \\ &= \int_{\mathbb{R}^d} \varphi(x) d\tilde{\mu}(x) = \int_{Y_d} \psi(y) d\mu(y), \end{aligned}$$

which shows that μ is invariant for the flow. We have just proved the equivalence between the invariance of μ for the flow and the distributional equation (2.8) satisfied by $\tilde{\mu}$.

Finally, the equivalence between (2.8), or equivalently (2.11), and (2.9) is a straightforward consequence of the relation

$$\forall \varphi \in C_c^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} b(x) \cdot \nabla \varphi(x) d\tilde{\mu}(x) = \int_{Y_d} b(y) \cdot \nabla \varphi_{\#}(y) d\mu(y)$$

which itself follows from $[b \cdot \nabla \varphi]_{\#} = b \cdot \nabla \varphi_{\#}$ and (2.7), combined with Lemma 2.3. \square

Remark 2.2 Equation (2.9) can be considered as the divergence free of the vector-valued measure μb in the torus Y_d , while equation (2.8) is exactly the divergence free of the vector-valued measure $\tilde{\mu} b$ in the space \mathbb{R}^d . Equation (2.9) is also equivalent to

$$\forall \nabla \psi \in C_{\#}^0(Y_d)^d, \quad \int_{Y_d} b(y) \cdot \nabla \psi(y) d\mu(y) = \left(\int_{Y_d} b(y) d\mu(y) \right) \cdot \left(\int_{Y_d} \nabla \psi(y) dy \right), \quad (2.13)$$

since

$$\nabla \psi \in C_{\#}^0(Y_d)^d \Leftrightarrow \left(x \mapsto \psi(x) - x \cdot \int_{Y_d} \nabla \psi(y) dy \right) \in C_{\#}^1(Y_d).$$

Condition (2.13) has to be regarded as a divergence-curl result combining the invariant measure μ for the divergence free vector field μb and the Lebesgue measure for the gradient field $\nabla \psi$.

3 Application to the asymptotics of the flow

First of all, we apply the tools of Section 2 to the one-dimensional case.

3.1 The one-dimensional case

We have the following result.

Proposition 3.1 *Let $b \in C_{\#}^1(Y_1)$. We have the following alternative:*

(i) *If $b \neq 0$ sur Y_1 , then $\underline{b}/b(y) dy$ is the unique invariant measure for the flow T_t associated with b , and*

$$\forall x \in \mathbb{R}, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \underline{b} := \left(\int_{Y_1} \frac{dy}{b(y)} \right)^{-1}. \quad (3.1)$$

(ii) *If b does vanish in Y_1 , then*

$$\forall x \in \mathbb{R}, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0. \quad (3.2)$$

Proof.

Case (i). Let $\mu \in \mathcal{M}_p(Y_1)$ be an invariant measure for the flow T_t . By the condition (2.8) of Theorem 2.2 there exists a constant $C \in \mathbb{R}$ such that $\tilde{\mu} b = C$ in \mathbb{R} , or equivalently,

$$\forall \varphi \in C_c^0(\mathbb{R}), \quad \int_{\mathbb{R}} \varphi(x) b(x) d\tilde{\mu}(x) = C \int_{\mathbb{R}} \varphi(x) dx.$$

Hence, by the definition (2.7) of $\tilde{\mu}$ we have

$$\forall \varphi \in C_c^0(\mathbb{R}), \quad \int_{Y_1} \varphi_{\#}(y) b(y) d\mu(y) = \int_{\mathbb{R}} \varphi(x) b(x) d\tilde{\mu}(x) = C \int_{\mathbb{R}} \varphi(x) dx = C \int_{Y_1} \varphi_{\#}(y) dy.$$

Therefore, from Lemma 2.3 we deduce that

$$\forall \psi \in C_{\#}^0(Y_1), \quad \int_{Y_1} \psi(y) b(y) d\mu(y) = C \int_{Y_1} \psi(y) dy. \quad (3.3)$$

Taking $\psi = 1/b$ in (3.3) we get that $C = \underline{b}$, which implies that $\mu(dy) = \underline{b}/b(y) dy$ is thus the unique invariant measure for the flow.

As a consequence the set \mathcal{S}_b defined by (2.5) is a singleton, and $C_b = \{\underline{b}\}$. Therefore, due to the equivalence (2.6) of Theorem 2.1 we obtain the desired asymptotics (3.1). We could also have concluded directly thanks to the unique ergodicity theorem. However, the unique ergodicity theorem does not apply in the following case, while Theorem 2.1 does.

Case (ii). Let $\mu \in \mathcal{M}_p(Y_1)$ be an invariant measure for the flow T_t . The equality $\tilde{\mu}b = C$ still holds true in \mathbb{R} for some constant $C \in \mathbb{R}$, as well as equality (3.3). Take the function $\psi := (|b| + \varepsilon)^{-1}$ for $\varepsilon > 0$, in equality (3.3), and make ε tend to 0. Then, using that b is regular and vanishes in Y_1 (which implies that $1/b \notin L^1_{\sharp}(Y_d)$), and applying successively Beppo-Levi's theorem and (3.3), it follows that

$$|C| \times \infty = |C| \times \int_{Y_1} \frac{dy}{|b(y)|} = \lim_{\varepsilon \rightarrow 0} \left| \int_{Y_1} \frac{C dy}{|b(y)| + \varepsilon} \right| = \liminf_{\varepsilon \rightarrow 0} \left| \int_{Y_1} \frac{b(y)}{|b(y)| + \varepsilon} d\mu(y) \right| \leq 1, \quad (3.4)$$

which implies that $C = 0$. Finally, taking $\psi = 1$ in the equality (3.3) with $C = 0$, leads us to $C_b = \{0\}$, which by virtue of the equivalence (2.6) of Theorem 2.1 yields the asymptotics of the flow (3.2).

As a by-product, using Lebesgue's dominated convergence theorem in the third integral of (3.4), we also get the equality

$$\int_{\{b(y) \neq 0\}} \frac{b(y)}{|b|} d\mu(y) = 0. \quad (3.5)$$

□

3.2 The rectifiable case

3.2.1 The case with a fixed direction

This section deals with the case where the vector field b has a fixed direction, namely

$$b(y) = a(y) \xi, \quad y \in Y_d, \quad (3.6)$$

for some nonnegative function $a \in C^1_{\sharp}(Y_d)$ which may vanish, and for a given vector $\xi \in \mathbb{R}^d$ with $|\xi| = 1$.

We have the following result.

Proposition 3.2 *Let $b \in C^1_{\sharp}(Y_d)^d$ be given by (3.6) with $a \geq 0$ and $\xi \in \mathbb{R}^d$ with $|\xi| = 1$.*

(i) *If $\xi \cdot k \neq 0$ for any $k \in \mathbb{Z}^d \setminus \{0\}$, then we have*

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = a^* \xi \quad \text{with} \quad a^* := \begin{cases} \underline{a} & \text{if } a > 0 \text{ in } Y_d \\ 0 & \text{if } a \text{ vanishes in } Y_d, \end{cases} \quad (3.7)$$

where \underline{a} is the harmonic mean of a .

(ii) *If there exists $T > 0$ such that $T\xi \in \mathbb{Z}^d$, then we have*

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = a^*(x) \xi \quad \text{with} \quad a^*(x) := \begin{cases} \left(\frac{1}{T} \int_0^T \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))} \right)^{-1} \xi & \text{if } \forall u \in \mathbb{R}, a(u\xi + \Pi_{\xi^\perp}(x)) \neq 0 \\ 0 & \text{if } \exists u \in \mathbb{R}, a(u\xi + \Pi_{\xi^\perp}(x)) = 0, \end{cases} \quad (3.8)$$

where Π_{ξ^\perp} denotes the projection on the hyperplane ξ^\perp orthogonal to ξ .

Remark 3.1 *The two-dimensional framework is completely covered by the disjoint cases (i) and (ii) of Proposition 3.2, since*

$$\exists k \in \mathbb{Z}^d \setminus \{0\}, \quad \xi \cdot k = 0 \Leftrightarrow \exists T > 0, \quad T\xi \in \mathbb{Z}^2.$$

This equivalence does not hold in higher dimension.

Proof of Proposition 3.2.

Case (i). Assume that for any $k \in \mathbb{Z}^d \setminus \{0\}$, $\xi \cdot k \neq 0$.

First assume that a does not vanish in Y_d . Let $x \in \mathbb{R}^d$. Since the vector field b is parallel to the fixed direction ξ , we have

$$\forall t \in \mathbb{R}, \quad X(t, x) = (X(t, x) \cdot \xi) \xi + \Pi_{\xi^\perp}(X(t, x)) \quad \text{and} \quad \Pi_{\xi^\perp}(X(t, x)) = \Pi_{\xi^\perp}(x), \quad (3.9)$$

which implies that

$$\begin{cases} \frac{\partial(X \cdot \xi)}{\partial t}(t, x) = a((X \cdot \xi)(t, x) \xi + \Pi_{\xi^\perp}(x)), & t \in \mathbb{R} \\ (X \cdot \xi)(0, x) = x \cdot \xi. \end{cases} \quad (3.10)$$

Then, the solution $X \cdot \xi$ to equation (3.10) is given by

$$\forall t \in \mathbb{R}, \quad X(t, x) \cdot \xi = F_x^{-1}(t + F_x(x \cdot \xi)) \quad \text{where} \quad F_x(t) := \int_0^t \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))}. \quad (3.11)$$

By approximating in $C_{\sharp}^0(Y_d)$ the continuous \mathbb{Z}^d -periodic function $1/a$ by Fejér's type trigonometric polynomials, and noting that by hypothesis

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad \lim_{t \rightarrow \infty} \left(\frac{1}{t} \int_0^t e^{-2i\pi s \xi \cdot k} ds \right) = 0,$$

it follows that

$$\lim_{t \rightarrow \infty} \frac{F_x(t)}{t} = \int_{Y_d} \frac{dy}{a(y)} = \underline{a},$$

which taking into account that $F_x(X(t, x) \cdot \xi) \underset{t \rightarrow \infty}{\sim} t$ with $X(t, x) \cdot \xi \rightarrow \infty$ as $t \rightarrow \infty$, implies that

$$\lim_{t \rightarrow \infty} \frac{X(t, x) \cdot \xi}{t} = \underline{a}.$$

Therefore, we get that

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \lim_{t \rightarrow \infty} \frac{(X(t, x) \cdot \xi) \xi + \Pi_{\xi^\perp}(x)}{t} = \underline{a} \xi.$$

Otherwise, if a vanishes in Y_d let us prove that

$$\forall \mu \in \mathcal{I}_{a\xi}, \quad \int_{Y_d} a(y) d\mu(y) = 0,$$

or equivalently, $C_{a\xi} = \{0\}$. Then, Theorem 2.1 will allow us to conclude. Assume by contradiction that there exists a measure $\mu \in \mathcal{I}_{a\xi}$ such that

$$\int_{Y_d} a(y) d\mu(y) > 0.$$

Set $a_\varepsilon := a + \varepsilon$ for $\varepsilon > 0$, and define the probability measure (recall that $a \geq 0$)

$$d\mu_\varepsilon(x) := \left(\int_{Y_d} \frac{a(y)}{a_\varepsilon(y)} d\mu(y) \right)^{-1} \frac{a(x)}{a_\varepsilon(x)} d\mu(x), \quad (3.12)$$

which is well defined since

$$\int_{Y_d} \frac{a(y)}{a_\varepsilon(y)} d\mu(y) \geq \frac{1}{\|a\|_\infty + \varepsilon} \int_{Y_d} a(y) d\mu(y) > 0.$$

Since μ is invariant for the flow associated with $a\xi$, by equality (2.9) we have

$$\forall \varphi \in C_{\sharp}^1(Y_d), \quad \int_{Y_d} a_\varepsilon(x) \xi \cdot \nabla \varphi(x) d\mu_\varepsilon(x) = \left(\int_{Y_d} \frac{a(y)}{a_\varepsilon(y)} d\mu(y) \right)^{-1} \int_{Y_d} a(x) \xi \cdot \nabla \varphi(x) d\mu(x) = 0,$$

hence $\mu_\varepsilon \in \mathcal{I}_{a_\varepsilon \xi}$. But from the former case $a > 0$ combined with Theorem 2.1 we deduce that

$$C_{a_\varepsilon \xi} = \left\{ \int_{Y_d} a_\varepsilon(y) \mu_\varepsilon(dy) \xi \right\} = \{ \underline{a_\varepsilon} \xi \} \quad \text{or equivalently} \quad \int_{Y_d} a_\varepsilon(y) \mu_\varepsilon(dy) = \underline{a_\varepsilon}.$$

This combined with the expression of $a_\varepsilon(x) d\mu_\varepsilon(x)$ given by (3.12) leads us to the equality

$$\left(\int_{Y_d} \frac{a(y)}{a_\varepsilon(y)} d\mu(y) \right) \underline{a_\varepsilon} = \left(\int_{Y_d} \frac{a(y)}{a_\varepsilon(y)} d\mu(y) \right) \left(\int_{Y_d} a_\varepsilon(x) \mu_\varepsilon(dx) \right) = \int_{Y_d} a(x) \mu(dx).$$

Then, applying Beppo-Levi's theorem and using that a is regular and vanishes in Y_d (which implies that $1/a \notin L_{\sharp}^1(Y_d)$), it follows that

$$0 < \int_{Y_d} a(x) \mu(dx) = \left(\int_{Y_d} \frac{a(y)}{a_\varepsilon(y)} d\mu(y) \right) \underline{a_\varepsilon} \leq \underline{a_\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \left(\int_{Y_d} \frac{dy}{a(y)} \right)^{-1} = 0,$$

which yields a contradiction. Therefore, we get that $C_{a\xi} = \{0\}$, and by the convergence (2.6) of Theorem 2.1 we obtain that

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0.$$

Case (ii). Now, assume that there exists $T > 0$ such that $T\xi \in \mathbb{Z}^d$. Let $x \in \mathbb{R}^d$. We have to distinguish two cases.

If $a(u\xi + \Pi_{\xi^\perp}(x)) > 0$ for any $u \in \mathbb{R}$, then by the T -periodicity of $u \mapsto a(u\xi + \Pi_{\xi^\perp}(x))$ we have

$$\lim_{t \rightarrow \infty} \frac{F_x(t)}{t} = \frac{1}{T} \int_0^T \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))},$$

hence

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \left(\frac{1}{T} \int_0^T \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))} \right)^{-1} \xi.$$

On the contrary, if there exists $u_0 \in \mathbb{R}$ such that $a(u_0\xi + \Pi_{\xi^\perp}(x)) = 0$, then the part (ii) of Proposition 3.1 applies to the one-dimensional solution $X \cdot \xi$ to (3.10), where $u \mapsto a(u\xi + \Pi_{\xi^\perp}(x))$ is T -periodic and vanishes at the point u_0 . We thus deduce that

$$\lim_{t \rightarrow \infty} \frac{X(t, x) \cdot \xi}{t} = 0, \quad \text{then} \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0,$$

which concludes the proof. \square

3.2.2 Rectification to a fixed direction

In this section we extend the result of the former section thanks to a diffeomorphism on the torus.

Definition 3.3 *A mapping $\Phi \in C^1(\mathbb{R}^d)^d$ is said to be a C^1 -diffeomorphism on the torus if Φ satisfies the following conditions:*

- $\det(\Phi(x)) \neq 0$ for any $x \in \mathbb{R}^d$,
- there exist a matrix $A \in \mathbb{Z}^{d \times d}$ with $|\det(A)| = 1$, and a \mathbb{Z}^d -periodic mapping $\Phi_{\sharp} \in C^1_{\sharp}(Y_d)^d$ such that

$$\forall x \in \mathbb{R}^d, \quad \Phi(x) = Ax + \Phi_{\sharp}(x). \quad (3.13)$$

Note that the invertibility of A and the periodicity of Φ_{\sharp} in (3.13) imply that Φ is a proper function (*i.e.*, the inverse image by the function of any compact set in \mathbb{R}^d is a compact set). Hence, by virtue of Hadamard-Caccioppoli's theorem [4] (also called Hadamard-Lévy's theorem) the mapping Φ is actually a C^1 -diffeomorphism on \mathbb{R}^d . Also noting that due to $A^{-1} \in \mathbb{Z}^{d \times d}$, we have

$$\forall k \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d, \quad \begin{cases} \Phi(x+k) - \Phi(x) = Ak & \in \mathbb{Z}^d \\ \Phi^{-1}(x+k) - \Phi^{-1}(x) = A^{-1}k & \in \mathbb{Z}^d, \end{cases}$$

Φ well defines an isomorphism on the torus.

Using a diffeomorphism on the torus the result of Proposition 3.2 can be extended to the following general result.

Corollary 3.1 *Consider a vector field b such that there exist a C^1 -diffeomorphism Φ on the torus given by (3.13), a nonnegative function $a \in C^1_{\sharp}(Y_d)^d$ and a vector $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, such that*

$$\forall x \in \mathbb{R}^d, \quad b(x) = a(\Phi(x)) \nabla \Phi(x)^{-1} \xi. \quad (3.14)$$

Then, if ξ satisfies one of the two conditions of Proposition 3.2 we get that

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = a^*(\Phi(x)) A^{-1} \xi, \quad (3.15)$$

where $a^(y)$ is given by formula (3.7) or formula (3.8).*

Remark 3.2 *A similar result as Corollary 3.1 was proved by Tassa [13, Theorem 2.1] in dimension two assuming that the first coordinate b_1 of b does not vanish in \mathbb{R}^2 , and that there exists an invariant measure for the flow associated with b , having a density in $C^1_{\sharp}(Y_2)$ with respect to Lebesgue's measure. More precisely, under these two conditions Tassa showed in the spirit of Kolmogorov's theorem the existence of a C^1 -diffeomorphism on the torus satisfying (3.14) with the matrix $A = I_2$. In the present case, we only assume the weaker rectification formula (3.14) in any dimension $d \geq 1$, with any diffeomorphism Φ on the torus and a possibly vanishing nonnegative function a .*

Conversely, consider a vector field $b \in C^1_{\sharp}(Y_d)^d$ satisfying (3.14) with $a > 0$. By making the change of variables $\varphi(x) = \psi(y)$ with $y = \Phi(x)$ for any function $\varphi \in C^{\infty}(\mathbb{R}^d)$, it is easy to check that

$$\int_{\mathbb{R}^d} \frac{\det(\nabla \Phi(x))}{a(\Phi(x))} b(x) \cdot \nabla \varphi(x) dx = \int_{\mathbb{R}^d} \xi \cdot \nabla \psi(y) dy = 0,$$

which implies that

$$\operatorname{div}(\sigma b) = 0 \text{ in } \mathbb{R}^d \quad \text{with} \quad \sigma := \left(\int_{Y_d} \frac{\det(\nabla\Phi(y))}{a(\Phi(y))} dy \right)^{-1} \frac{\det(\nabla\Phi)}{a \circ \Phi} > 0. \quad (3.16)$$

Since the function σ is \mathbb{Z}^d -periodic, by virtue of Theorem 2.2 (3.16) means that the density probability measure $\sigma(x) dx$ is invariant for the flow associated with b .

Therefore, the rectification formula (3.14) of a nonvanishing vector field b can be regarded as a (more restrictive) substitute to (3.16) in dimension $d > 2$ for which Kolmogorov's theorem does not apply.

Proof of Corollary 3.1. Define

$$Y(t, y) := \Phi(X(t, x)) \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^d \quad \text{with} \quad y := \Phi(x). \quad (3.17)$$

By the chain rule and (3.14) we have

$$\begin{cases} \frac{\partial Y}{\partial t}(t, y) = \nabla\Phi(X(t, x)) \frac{\partial X}{\partial t}(t, x) = \nabla\Phi(X(t, x)) b(X(t, x)) = a(\Phi(X(t, x))) \xi = a(Y(t, y)) \xi \\ Y(0, y) = \Phi(X(0, x)) = \Phi(x) = y, \end{cases}$$

which shows that Y is the solution to the dynamical system associated with the vector field $a \xi$. Hence, if ξ satisfies one of the two conditions of Proposition 3.2, we get the asymptotics of the flow

$$\forall y \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{Y(t, y)}{t} = a^*(y) \xi,$$

where $a^*(y)$ is given by formula (3.7) or formula (3.8). Therefore, by the boundedness of Φ_{\sharp} in (3.13) we have for any $x \in \mathbb{R}^d$ and $y := \Phi(x)$,

$$\frac{X(t, x)}{t} = \frac{\Phi^{-1}(Y(t, y))}{t} = A^{-1} \left[\frac{A(\Phi^{-1}(Y(t, y)))}{t} \right] = A^{-1} \left[\frac{Y(t, y)}{t} \right] + \frac{O(1)}{t} \xrightarrow{t \rightarrow \infty} a^*(y) A^{-1} \xi,$$

which yields the desired asymptotics (3.15). \square

Remark 3.3 Peirone [11, Theorem 3.1] proved that in dimension two and for a nonvanishing vector field $b \in C_{\sharp}^1(Y_2)^2$, there is no periodic solution in \mathbb{R}^2 to system (1.10) (see the definition after (1.14)), and that the limit of $X(t, x)/t$ as $t \rightarrow \infty$ does exist for any $x \in \mathbb{R}^2$ with the following alternative:

- If system (1.10) has no periodic solution in the torus according to (1.14), then the limit of $X(t, x)/t$ as $t \rightarrow \infty$ is independent of $x \in \mathbb{R}^2$ as in (3.7).
- If system (1.10) has a periodic solution in the torus, then the limit of $X(t, x)/t$ as $t \rightarrow \infty$ may depend on $x \in \mathbb{R}^2$ as in (3.8).

Actually, in any dimension the alternative on the vector ξ in Corollary 3.1 (see also Proposition 3.2) is naturally connected to the former alternative due to the following result.

Proposition 3.4 In the framework of Corollary 3.1 assume in addition that

$$\exists x \in \mathbb{R}^d, \quad \forall t \in \mathbb{R}, \quad a(t\xi + \Pi_{\xi^\perp}(x)) > 0. \quad (3.18)$$

Then, the following equivalence holds:

$$\exists T > 0, \quad T\xi \in \mathbb{Z}^d \Leftrightarrow \text{system (1.10) has a periodic solution in } Y_d \text{ but not in } \mathbb{R}^d. \quad (3.19)$$

Moreover, the implication (\Leftarrow) of (3.19) is always true. When condition (3.18) is not satisfied, the implication (\Rightarrow) of (3.19) does not hold in general.

Proof. First note that the definition (3.13) of the C^1 -diffeomorphism on the torus Φ implies that a solution $X(\cdot, x)$ is periodic in the torus if, and only if, the function $Y(\cdot, \Phi(x))$ is periodic in the torus. Therefore, it is enough to prove the result in the case $b = a\xi$.

If system (1.10) with $b = a\xi$ has a periodic solution $X(\cdot, x)$ in the torus but not in \mathbb{R}^d , then there exists $\tau > 0$ and $k \in \mathbb{Z}^d \setminus \{0\}$ such that by (3.9) we have for any $t \in \mathbb{R}$,

$$\Pi_{\xi^\perp}(x) = \Pi_{\xi^\perp}(X(t + \tau, x)) = \Pi_{\xi^\perp}(X(t, x) + k) = \Pi_{\xi^\perp}(X(t, x)) + \Pi_{\xi^\perp}(k) = \Pi_{\xi^\perp}(x) + \Pi_{\xi^\perp}(k).$$

Hence, we obtain that $\Pi_{\xi^\perp}(k) = 0$, or equivalently, there exists $T > 0$ such that $T\xi = \pm k \in \mathbb{Z}^d$.

Conversely, assume that (3.18) is satisfied for some $x \in \mathbb{R}^d$, and there exists $T > 0$ such that $T\xi = k \in \mathbb{Z}^d$. The solution $X(\cdot, x)$ of (1.10) is given by (3.9) and (3.11). Define

$$\tau := \int_0^T \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))}.$$

Then, by the T -periodicity of the function $t \mapsto a(t\xi + \Pi_{\xi^\perp}(x))$ we have

$$\begin{aligned} F_x(x \cdot \xi + T) &= \int_0^{x \cdot \xi + T} \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))} \\ &= \int_0^{x \cdot \xi} \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))} + \int_{x \cdot \xi}^{x \cdot \xi + T} \frac{ds}{a(s\xi + \Pi_{\xi^\perp}(x))} \\ &= F_x(x \cdot \xi) + \tau = F_x(X(\tau, x) \cdot \xi) \quad \text{by (3.11),} \end{aligned}$$

which implies that $X(\tau, x) \cdot \xi = x \cdot \xi + T$. Moreover, the functions $t \mapsto X(t + \tau, x) \cdot \xi$ and $t \mapsto X(t, x) \cdot \xi + T$ are solutions to the system $z'(t) = a(z(t)\xi + \Pi_{\xi^\perp}(x))$, and agree at $t = 0$. Hence, by a uniqueness argument these two solutions are equal, which yields

$$\forall t \in \mathbb{R}, \quad X(t + \tau, x) \cdot \xi = X(t, x) \cdot \xi + T = (X(t, x) + k) \cdot \xi.$$

Moreover, again by (3.9) and recalling that $\Pi_{\xi^\perp}(k) = 0$ we have

$$\forall t \in \mathbb{R}, \quad \Pi_{\xi^\perp}(X(t + \tau, x)) = \Pi_{\xi^\perp}(x) = \Pi_{\xi^\perp}(X(t, x)) = \Pi_{\xi^\perp}(X(t, x) + k).$$

The two previous identities imply that

$$\forall t \in \mathbb{R}, \quad X(t + \tau, x) = X(t, x) + k.$$

namely, $X(\cdot, x)$ is periodic in the torus.

Consider the vector field $b(x) := a(x)e_1$ with $a(x) := \frac{1}{\pi} \cos^2(\pi x_1) \geq 0$ for $x \in \mathbb{R}^d$. Here, the vector $\xi = e_1$ clearly satisfies the left-hand side of (3.19). However, since

$$\forall x \in \mathbb{R}^d, \quad a\left(\frac{1}{2}e_1 + \Pi_{e_1^\perp}(x)\right) = \frac{1}{\pi} \cos^2\left(\frac{\pi}{2}\right) = 0,$$

condition (3.18) is not satisfied. Moreover, for any $x \in \mathbb{R}^d$ the solution $X(\cdot, x)$ to (1.10) is given by

$$\left\{ \begin{array}{l} X_1(t, x) = \begin{cases} x_1 & \text{if } x_1 = \frac{1}{2} + n, \quad n \in \mathbb{Z} \\ \frac{1}{\pi} \arctan(t + \tan(\pi x_1)) + n & \text{if } x_1 \in (-\frac{1}{2} + n, \frac{1}{2} + n), \quad n \in \mathbb{Z} \end{cases} \\ X_i(t, x) = x_i \text{ for } i \geq 2 \end{array} \right. \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

It follows that any periodic solution in the torus is necessarily stationary and *a fortiori* periodic in \mathbb{R}^d in the sense of (1.14). Therefore, the implication (\Rightarrow) of (3.19) does not hold. \square

3.3 The case of a current field

The following result deals with the case of a current field $b = A\nabla v$ defined with some matrix-valued conductivity A and some electric field ∇v with zero average.

Proposition 3.5 *Assume that*

$$b = A\nabla v \text{ in } Y_d \quad \text{with} \quad \begin{cases} A \in C_{\sharp}^1(Y_d)^{d \times d}, & A = A^T \geq 0 \text{ in } Y_d, \\ v \in C_{\sharp}^2(Y_d)^d. \end{cases} \quad (3.20)$$

Then, the flow T_t associated with b satisfies the asymptotics

$$\forall x \in \mathbb{R}^d, \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = 0.$$

Proof. Let $\mu \in \mathcal{M}_p(Y_d)$ be an invariant measure for the flow T_t . By the div-curl relation (2.13) combined with the zero average of ∇v , we have

$$\int_{Y_d} \underbrace{A(y)\nabla v(y) \cdot \nabla v(y)}_{\geq 0} \mu(dy) = \int_{Y_d} b(y) \cdot \nabla v(y) \mu(dy) = 0,$$

which implies that $A\nabla v \cdot \nabla v = 0$ μ -a.e. in Y_d . Since the matrix-valued A is symmetric and nonnegative, by the Cauchy-Schwarz inequality we deduce that $A\nabla v = 0$ μ -a.e. in Y_d , and thus

$$\int_{Y_d} b(y) \mu(dy) = \int_{Y_d} A(y)\nabla v(y) \mu(dy) = 0.$$

Therefore, $C_b = \{0\}$ and the equivalence (2.6) of Theorem 2.1 allows us to conclude. \square

A Proof of Lemma 2.2

Since Y_d is a compact metrizable space, there exists a subsequence $(\nu_{n_k})_{k \in \mathbb{N}}$ of $(\nu_n)_{n \in \mathbb{N}}$ which converges weakly $*$ to some probability measure $\mu \in \mathcal{M}_p(Y_d)$, namely for any $f \in C_{\sharp}^0(Y_d)$,

$$\int_{Y_d} f(y) d\nu_{n_k}(y) = \frac{1}{r_{n_k}} \int_0^{r_{n_k}} f(X(s, x)) ds \xrightarrow{k \rightarrow \infty} \int_{Y_d} f(y) d\mu(y). \quad (\text{A.1})$$

Let us prove that μ is invariant for the flow T_t . For the sake of simplicity denote $\tau_k := r_{n_k}$ and $\mu_k := \nu_{n_k}$. Let $t \in \mathbb{R}$ and $f \in C_{\sharp}^0(Y_d)$. By the semi-group property of the flow (1.12) we have

$$\int_{Y_d} (T_t f)(y) d\mu_k(y) = \frac{1}{\tau_k} \int_0^{\tau_k} f(X(s+t, x)) ds.$$

By the change of variable $r = s + t$, it follows that

$$\begin{aligned} \int_{Y_d} (T_t f)(y) d\mu_k(y) &= \frac{1}{\tau_k} \int_t^{t+\tau_k} f(X(r, x)) dr \\ &= \frac{1}{\tau_k} \int_0^{\tau_k} f(X(r, x)) dr + \frac{1}{\tau_k} \int_{\tau_k}^{t+\tau_k} f(X(r, x)) dr - \frac{1}{\tau_k} \int_0^t f(X(r, x)) dr \end{aligned}$$

Since f is bounded and $t \in \mathbb{R}$ is fixed, we deduce from (A.1) that

$$\lim_{k \rightarrow \infty} \int_{Y_d} (T_t f)(y) d\mu_k(y) = \int_{Y_d} f(y) d\mu(y).$$

However, by the definition of μ we also have

$$\lim_{k \rightarrow \infty} \int_{Y_d} (T_t f)(y) d\mu_k(y) = \int_{Y_d} (T_t f)(y) d\mu(y).$$

Hence, we get that

$$\forall t \in \mathbb{R}, \forall f \in C_{\#}^0(Y_d), \quad \int_{Y_d} (T_t f)(y) d\mu(y) = \int_{Y_d} f(y) d\mu(y),$$

which implies that μ is invariant for the flow T_t . □

References

- [1] Y. BRENIER: “Remarks on some linear hyperbolic equations with oscillatory coefficients”, *Proceedings of the Third International Conference on Hyperbolic Problems* (Uppsala 1990) Vol. I, II, Studentlitteratur, Lund (1991), 119-130.
- [2] M. BRIANE: “Isotropic realizability of a strain field for the two-dimensional incompressible elasticity system”, *Inverse Problems*, **32** (6) (2016), 22 pp.
- [3] M. BRIANE: “Isotropic realizability of fields and reconstruction of invariant measures under positivity properties. Asymptotics of the flow by a nonergodic approach”, *SIAM J. Appl. Dyn. Sys.*, **18** (4) (2019), 1846-1866.
- [4] R. CACCIOPPOLI: “Sugli elementi uniti delle trasformazioni funzionali: un teorema di esistenza e unicit ed alcune sue applicazioni”, *Rend. Sem. Mat. Padova*, **3** (1932), 1-15.
- [5] I.P. CORNFELD, S.V. FOMIN & YA.G. SINAIĀ: *Ergodic Theory*, translated from the Russian by A.B. Sosinskii, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245**, Springer-Verlag, New York, 1982, 486 pp.
- [6] F. GOLSE: “Moyennisation des champs de vecteurs et EDP” (French), [The averaging of vector fields and PDEs], *Journées Équations aux Dérivées Partielles*, Saint Jean de Monts 1990, Exp. no. **XVI**, École Polytech. Palaiseau, 1990, 17 pp.
- [7] F. GOLSE: “Perturbations de systèmes dynamiques et moyennisation en vitesse des EDP” (French), [On perturbations of dynamical systems and the velocity averaging method for PDEs], *C. R. Acad. Sci. Paris Sér. I Math.*, **314** (2) (1992), 115-120.
- [8] M.W. HIRSCH, S. SMALE & R.L. DEVANEY: *Differential equations, Dynamical Systems, and an Introduction to Chaos*, Second edition, *Pure and Applied Mathematics* **60**, Elsevier Academic Press, Amsterdam, 2004, 417 pp.
- [9] T.Y. HOU & X. XIN: “Homogenization of linear transport equations with oscillatory vector fields”, *SIAM J. Appl. Math.*, **52** (1) (1992), 34-45.
- [10] A.N. KOLMOGOROV: “On dynamical systems with an integral invariant on the torus” (Russian), *Doklady Akad. Nauk SSSR (N.S.)*, **93** (1953), 763-766.
- [11] R. PEIRONE: *Convergence of solutions of linear transport equations*, *Ergodic Theory Dynam. Systems*, **23** (3) (2003), 919-933.

- [12] *Dynamical Systems II, Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics* (Translated from the Russian), Edited by E.Ya Sinai, Encyclopaedia of Mathematical Sciences **2**, Springer-Verlag Berlin 1989, 281 pp.
- [13] T. TASSA: “Homogenization of two-dimensional linear flows with integral invariance”, *SIAM J. Appl. Math.*, **57** (5) (1997), 1390-1405.