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# New results on Pareto spectra

JEAN-BERNARD BAILLON<sup>1</sup> and ALBERTO SEEGER<sup>2</sup>

**Abstract.** A Pareto eigenvalue of a matrix  $A$  of order  $n$  is a scalar  $\lambda \in \mathbb{R}$  for which the complementarity problem  $0 \preceq x \perp (Ax - \lambda x) \succeq 0$  admits a nonzero solution  $x \in \mathbb{R}^n$ . Pareto eigenvalues are also known as complementarity eigenvalues. They have found applications in graph theory, cone-constrained dynamical systems, and mathematical modelling in general. In this paper we continue our study on theoretical properties of Pareto spectra. Special attention is paid to classification of Pareto eigenvalues and cardinality issues.

*Mathematics Subject Classification:* 15A18, 15A39, 65H17.

*Key words:* Complementarity eigenvalue problem, Pareto eigenvalue, regular Pareto eigenvalue, strict complementarity, cardinality of Pareto spectra.

## 1 Introduction

The present paper is a continuation of our previous work [4] and therefore we use the same notation and terminology. Let  $\mathbb{M}_n$  be the linear space of real matrices of order  $n$ . The mathematical problem under consideration is that of evaluating the Pareto capacity of  $\mathbb{M}_n$ . Such expression was coined in Pinto da Costa and Seeger [11] and refers to the number

$$\mathbf{c}_n := \max_{A \in \mathbb{M}_n} \text{card}[\Pi(A)]. \quad (1)$$

Here,  $\Pi(A)$  stands for the Pareto spectrum or set of Pareto eigenvalues of  $A$ . By definition, a Pareto eigenvalue of  $A \in \mathbb{M}_n$  is a scalar  $\lambda \in \mathbb{R}$  for which the linear complementarity problem

$$x \succeq 0, \quad Ax - \lambda x \succeq 0, \quad \langle x, Ax - \lambda x \rangle = 0$$

admits a nonzero solution  $x \in \mathbb{R}^n$ . The notation  $\langle \cdot, \cdot \rangle$  refers to the usual inner product of  $\mathbb{R}^n$  and  $x \succeq 0$  means that  $x$  is nonnegative componentwisely. The interpretation of (1) is clear: such term corresponds to the maximal number of Pareto eigenvalues that can be found in a general matrix of order  $n$ . A matrix of order  $n$  is of full Pareto capacity if it belongs to

$$\mathfrak{S}_n := \{A \in \mathbb{M}_n : \text{card}[\Pi(A)] = \mathbf{c}_n\},$$

the solution set to the maximization problem (1). Matrices of full Pareto capacity are very difficult to characterize. The reader is conveyed to Baillon and Seeger [4] for historical comments on Pareto spectra and state-of-the-art knowledge on Pareto capacities. For applications of Pareto spectra in various fields of mathematics, see for instance [8, 9, 10, 14]. It is known that  $\mathbf{c}_1 = 1$ ,  $\mathbf{c}_2 = 3$ , and

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$\mathfrak{c}_3 = 9$ , but the exact value of  $\mathfrak{c}_n$  is a mystery starting from  $n = 4$ . There are several reasons why it is so hard to compute  $\mathfrak{c}_n$  in general. Firstly, the objective function

$$A \in \mathbb{M}_n \mapsto f(A) := \text{card}[\Pi(A)] \quad (2)$$

of the maximization problem (1) is highly unstable with respect to perturbations in its argument. More precisely, the cardinality of  $\Pi(A)$  may increase or decrease abruptly if the entries of  $A$  are subject to small perturbations. Topologically speaking,  $f$  is neither lower nor upper semicontinuous, cf. Proposition 4. Secondly, standard optimization techniques involving ascent directions and line search are hard to implement. This is because to identify an ascent direction for  $f$  at a current point  $A$  is far from being a trivial matter. And, thirdly, the evaluation of  $f$  at a given  $A$  is in itself a challenging numerical task. The organization of the paper is as follows.

- Section 2 classifies the Pareto eigenvalues of a matrix according to different criteria: simplicity, strictness, regularity, etc. This section serves to set up the mathematical background of our work and to fix some terminology.
- Section 3 focusses on regular Pareto eigenvalues and the associated concept of spectrally regular matrix. We show that the number of regular Pareto eigenvalues does not decrease if the matrix under consideration is subject to small perturbations. We also prove that spectrally regular matrices are points of continuity of the set-valued map  $\Pi : \mathbb{M}_n \rightarrow 2^{\mathbb{R}}$ .
- Section 4 states the double-plus-one rule. This rule asserts that the existence of a matrix of order  $n$  with  $r$  regular Pareto eigenvalues implies the existence of a matrix of order  $n + 1$  with at least  $2r + 1$  regular Pareto eigenvalues. Such a result confirms that  $\mathfrak{c}_n$  goes to infinity at least as fast as a constant times  $2^n$ . The double-plus-one rule serves also to sharpen the best lower bound for  $\mathfrak{c}_n$  that is known insofar.

## 2 Classification of Pareto eigenvalues

Theorem 1 serves to compute the Pareto spectrum of a matrix of moderate order. The symbol  $\mathcal{J}_n$  stands for the set of nonempty subsets of  $\mathbb{N}_n := \{1, \dots, n\}$  and  $A^J := [a_{i,j}]_{i,j \in J}$  stands for the principal submatrix of  $A$  with entries indexed by  $J \in \mathcal{J}_n$ . Our usual notation for the cardinality of a general set  $S$  is  $\text{card}(S)$ , but, for typographical convenience, the cardinality of an index set  $J$  is indicated with the shorter notation  $|J|$ . The notation  $u \succ 0$  means that  $u$  is a positive vector, its dimension being understood from the context.

**Theorem 1** (Seeger, 1999). *A scalar  $\lambda \in \mathbb{R}$  is a Pareto eigenvalue of  $A \in \mathbb{M}_n$  if and only if there exist an index set  $J \in \mathcal{J}_n$  and a vector  $u \in \mathbb{R}^{|J|}$  such that*

$$A^J u = \lambda u, \quad u \succ 0, \quad \sum_{j \in J} a_{i,j} u_j \geq 0 \quad \text{for all } i \notin J. \quad (3)$$

In order to compute the Pareto spectrum of  $A \in \mathbb{M}_n$  we must solve  $2^n - 1$  classical eigenvalue problems, one for each index set  $J$ . Of course, we must keep in mind the positivity condition  $u \succ 0$  and the slackness inequalities mentioned at the end of (3). Slackness inequalities are at the origin of many mathematical troubles and they must be handled with care. We say that

$$\Pi_J(A) := \{\lambda \in \mathbb{R} : \text{there exists } u \in \mathbb{R}^{|J|} \text{ satisfying (3)}\}$$

is the set of Pareto eigenvalues of  $A$  produced by  $J$ . The whole Pareto spectrum is obtained by passing to the union:  $\Pi(A) = \cup_{J \in \mathcal{J}_n} \Pi_J(A)$ . Pareto eigenvalues can be classified according to many different criteria: simplicity, strictness, regularity, etc.

**Definition 1.** A Pareto eigenvalue  $\lambda$  of a matrix  $A \in \mathbb{M}_n$  is called:

i) simple, if there exists an index set  $J$  such that (3) holds for some  $u \in \mathbb{R}^{|J|}$  and  $\lambda$  is algebraically simple as eigenvalue of  $A^J$ .

ii) strict, if there exists an index set  $J$  and a vector  $u \in \mathbb{R}^{|J|}$  such that

$$A^J u = \lambda u, \quad u \succ 0, \quad \sum_{j \in J} a_{i,j} u_j > 0 \quad \text{for all } i \notin J. \quad (4)$$

iii) regular, if there exists an index set  $J$  such that (4) holds for some  $u \in \mathbb{R}^{|J|}$  and  $\lambda$  is algebraically simple as eigenvalue of  $A^J$ .

iv) defective, if it is neither simple nor strict.

By combining Theorem 1 and the Perron-Frobenius theorem, we see that if the off-diagonal entries of  $A$  are positive, then the Pareto eigenvalues of  $A$  are all regular. Thus, the classification scheme of Definition 1 is of interest mainly for matrices with at least one nonpositive off-diagonal entry. The only difference between (3) and (4) is that in the later system the slackness inequalities are strict. Nonstrict Pareto eigenvalues are problematic from a numerical point of view. As explained in Adly and Seeger [1], nonstrict Pareto eigenvalues are hard to detect with an iterative algorithm like the semismooth Newton's method. Defective Pareto eigenvalues are even worse from an algorithmic perspective. A sort of ideal situation occurs when the Pareto eigenvalues of the matrix under consideration are all regular. Note that regularity is not a mere addition of simplicity and strictness. Indeed, regularity requires simplicity and strictness relative to the same index set.

**Example 1.** Consider a small order matrix with three Pareto eigenvalues:

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & -6 & 3 \end{bmatrix}.$$

The Pareto eigenvalue  $\lambda_1 = 0$  is produced by two index sets: with  $J = \{1, 2\}$  we see that  $\lambda_1$  is strict and with  $J = \{1\}$  we see that  $\lambda_1$  is simple. Note that  $\lambda_1$  is not regular, despite the fact that it is simple and strict. On the other hand, the index sets  $J = \{3\}$  and  $J = \{1, 3\}$  produce the regular Pareto eigenvalues  $\lambda_2 = 3$  and  $\lambda_3 = 4$ , respectively.

Regular Pareto eigenvalues arises frequently in practice and defective Pareto eigenvalues are rare. For instance, a matrix of order 3 may have 9 regular Pareto eigenvalues, but it cannot have more than 2 defective eigenvalues.

**Proposition 1.** A matrix of order 3 has at most 2 defective Pareto eigenvalues and this upper bound is attained.

*Proof.* A defective Pareto eigenvalue of a matrix of order 3 can be produced only by an index set of cardinality two. Hence, more than 3 defective Pareto eigenvalues is impossible. Suppose that

there exists a matrix  $A \in \mathbb{M}_3$  with 3 defective Pareto eigenvalues. In such a case, the system

$$\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ v_1 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{bmatrix} \begin{bmatrix} 1 \\ v_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ v_2 \end{bmatrix} \quad (6)$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} 1 \\ v_3 \end{bmatrix} = \lambda_3 \begin{bmatrix} 1 \\ v_3 \end{bmatrix} \quad (7)$$

$$a_{1,2} + a_{1,3}v_1 = 0 \quad (8)$$

$$a_{2,1} + a_{2,3}v_2 = 0 \quad (9)$$

$$a_{3,1} + a_{3,2}v_3 = 0 \quad (10)$$

is solvable with respect to the unknown variables  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$  and  $v_1, v_2, v_3 > 0$ . Since the  $\lambda_i$ 's have algebraic multiplicity two as classical eigenvalues, we have

$$2\lambda_1 = a_{2,2} + a_{3,3}, \quad 2\lambda_2 = a_{1,1} + a_{3,3}, \quad 2\lambda_3 = a_{1,1} + a_{2,2} \quad (11)$$

and

$$\begin{cases} (a_{2,2} - a_{3,3})^2 + 4a_{3,2}a_{2,3} = 0 \\ (a_{1,1} - a_{3,3})^2 + 4a_{3,1}a_{1,3} = 0 \\ (a_{1,1} - a_{2,2})^2 + 4a_{2,1}a_{1,2} = 0. \end{cases} \quad (12)$$

By substituting (11) into (5)-(7) we get in particular

$$\begin{cases} a_{3,3} - a_{2,2} = 2a_{2,3}v_1 \\ a_{3,3} - a_{1,1} = 2a_{1,3}v_2 \\ a_{2,2} - a_{1,1} = 2a_{1,2}v_3. \end{cases} \quad (13)$$

We distinguish two cases. The first case occurs when the off-diagonal entries of  $A$  are all nonzero. Hence,  $a_{1,2}a_{1,3} < 0$ ,  $a_{2,1}a_{2,3} < 0$ ,  $a_{3,1}a_{3,2} < 0$ ,  $a_{3,2}a_{2,3} < 0$ ,  $a_{3,1}a_{1,3} < 0$ ,  $a_{2,1}a_{1,2} < 0$ , and  $A$  has one of the following sign pattern:

$$\begin{bmatrix} * & - & + \\ + & * & - \\ - & + & * \end{bmatrix}, \quad \begin{bmatrix} * & + & - \\ - & * & + \\ + & - & * \end{bmatrix}.$$

In view of (13), the first sign pattern leads to  $a_{3,3} < a_{2,2} < a_{1,1} < a_{3,3}$ , which is a contradiction, and the second sign pattern leads to  $a_{2,2} < a_{3,3} < a_{1,1} < a_{2,2}$ , which is again a contradiction. We now consider the case in which  $A$  has a zero off-diagonal entry. By way of example we analyze the subcase  $a_{1,2} = 0$ , the other subcases can be treated in a similar way. Due to (8), we have also  $a_{1,3} = 0$ . The last two equations in (12) yield  $a_{1,1} = a_{3,3}$  and  $a_{1,1} = a_{2,2}$ , respectively. Hence,  $\lambda_1 = \lambda_2 = \lambda_3$ , contradicting the fact that the  $\lambda_i$ 's are distinct. We have proven in this way that a matrix of order 3 has at most 2 defective Pareto eigenvalues. To see that this upper bound is attained, consider for instance the matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & -1 & -4 \\ -2 & 1 & 7 \end{bmatrix}$$

whose Pareto eigenvalues are  $\lambda_1 = 1$  (defective, produced by  $J = \{1, 2\}$ ),  $\lambda_2 = 3$  (regular, produced by  $J = \{1, 2, 3\}$ ), and  $\lambda_3 = 5$  (defective, produced by  $J = \{1, 3\}$ ).  $\square$

This section ends with a characterization of strict Pareto eigenvalues in a similar spirit as in Adly and Seeger [1, Definition 5], i.e., without mentioning index sets or principal submatrices.

**Proposition 2.** *A real  $\lambda$  is a strict Pareto eigenvalue of  $A \in \mathbb{M}_n$  if and only if there exists a pair  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$  such that*

$$Ax - \lambda x - y = 0, \quad (14)$$

$$0 \preceq x \perp y \succeq 0, \quad (15)$$

$$x + y \succ 0. \quad (16)$$

*Proof.* Let  $J \in \mathcal{J}_n$  and  $u \in \mathbb{R}^{|J|}$  be in (4). Consider the nonzero vector  $x \in \mathbb{R}^n$  given by

$$x_i := \begin{cases} u_i & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}$$

and define  $y := Ax - \lambda x$ . Then  $(x, y)$  clearly satisfies (14)-(16). Conversely, let  $(x, y)$  be a pair as in (14)-(16). Then  $x \neq 0$ . If we take  $J := \{i \in \mathbb{N}_n : x_i > 0\}$  and define  $u \in \mathbb{R}^{|J|}$  as the vector formed with the positive components of  $x$ , then (4) holds true.  $\square$

Recall that two vectors  $x$  and  $y$  are complementary if they satisfy (15) and strictly complementary if they satisfy (15)-(16). Strictly complementary vectors are known to play a fundamental role in the theory of complementarity problems, so we do not need to further justify the introduction of condition (16).

### 3 On regular Pareto eigenvalues

What makes regular Pareto eigenvalues so special is a certain property called lower stability: the number of regular Pareto eigenvalues does not decrease if the matrix under consideration is subject to a small perturbation. This theme is developed next. Let  $\Pi^{\text{reg}}(A)$  denote the regular Pareto spectrum or set of regular Pareto eigenvalues of  $A$ . Of course, the inclusion  $\Pi^{\text{reg}}(A) \subseteq \Pi(A)$  holds for all  $A \in \mathbb{M}_n$ . The matrices in the set

$$\text{SR}(n) := \{A \in \mathbb{M}_n : \Pi^{\text{reg}}(A) = \Pi(A)\}$$

are called spectrally regular. The meaning of the acronym SR is clear. In plain English, a matrix is spectrally regular if all its Pareto eigenvalues are regular. It is not difficult to prove that  $\text{SR}(n)$  is a dense set in  $\mathbb{M}_n$ . By analogy with expression (1) we say that

$$\mathbf{c}_n^{\text{reg}} := \max_{A \in \mathbb{M}_n} \text{card}[\Pi^{\text{reg}}(A)] \quad (17)$$

is the regular Pareto capacity of the space  $\mathbb{M}_n$ . The inequality  $\mathbf{c}_n^{\text{reg}} \leq \mathbf{c}_n$  is obvious, but it is not clear whether these two integers are different after all. We know for a fact that  $\mathbf{c}_n^{\text{reg}} = \mathbf{c}_n$  when  $n \in \{1, 2, 3\}$ , but starting from  $n = 4$  we are in terra incognita: we barely know that

$$\begin{cases} 23 \leq \mathbf{c}_4^{\text{reg}} \leq \mathbf{c}_4 \leq 26, \\ 57 \leq \mathbf{c}_5^{\text{reg}} \leq \mathbf{c}_5 \leq 71, \end{cases} \quad (18)$$

and so on for larger values of  $n$ . The upper bound  $\mathbf{c}_4 \leq 26$  is announced in [4, p. 8] and it is obtained by combining [13, Lemma 5.1] and [4, Corollary 1]. The upper bound  $\mathbf{c}_5 \leq 71$  is announced in [11,

Table 5] and it is obtained by combining [13, Theorem 4.1] and a tedious case-by-case analysis of the sign pattern of an arbitrary matrix of order five. As justification of the lower bounds in (18) we mention that the particular matrices

$$\begin{bmatrix} 100 & 103 & -11 & -80 \\ 91 & 150 & -13 & -97 \\ 1 & 4 & 29 & -2 \\ 21 & 30 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 612 & 780 & 256 & -156 & -191 \\ 548 & 538 & 190 & -112 & -143 \\ 456 & 548 & 92 & -110 & -119 \\ 292 & 374 & 14 & -2 & -28 \\ 304 & 402 & 66 & -38 & -122 \end{bmatrix}$$

have 23 and 57 Pareto eigenvalues, respectively, all of them regular. Alternative matrices in the same vein are proposed in Chen and Qi [6], but these authors do not mention the regularity of the obtained Pareto eigenvalues.

The next proposition shows that the maximization problem (17) enjoys some favorable stability properties. We write first a perturbation lemma for eigenvalues and eigenvectors. We present a version that focusses on continuity and leaves differentiability issues aside.

**Lemma 1.** *Let  $\lambda \in \mathbb{R}$  be an algebraically simple eigenvalue of  $B_* \in \mathbb{M}_k$  with associated eigenvector  $u \in \mathbb{R}^k$ . Then there are continuous functions  $g : \mathcal{N} \rightarrow \mathbb{R}$  and  $\xi : \mathcal{N} \rightarrow \mathbb{R}^k$  on some open neighborhood of  $B_*$  such that*

$$g(B_*) = \lambda, \quad \xi(B_*) = u, \quad (19)$$

$$B\xi(B) = g(B)\xi(B) \quad \text{for all } B \in \mathcal{N}. \quad (20)$$

Furthermore,  $g(B)$  is an algebraically simple eigenvalue of  $B$ .

*Proof.* This result is part of the folklore. We give a short proof that is adapted from a note available on the web: A. Alexanderian, Dependence of Simple Eigenpairs to Differentiable Perturbations, November 2013. Let  $\phi : \mathbb{M}_k \times \mathbb{R} \rightarrow \mathbb{R}$  be given by  $\phi(B, t) := \det(tI_k - B)$  and let  $\partial_t \phi$  be the partial derivative of  $\phi$  with respect to  $t$ . Since  $\lambda$  is an algebraically simple eigenvalue of  $B_*$  we have  $\phi(B_*, \lambda) = 0$  and  $\partial_t \phi(B_*, \lambda) \neq 0$ . The implicit function theorem ensures the existence of an open neighborhood  $\mathcal{N}$  of  $B_*$  and a continuously differentiable functions  $g : \mathcal{N} \rightarrow \mathbb{R}$  such that  $g(B_*) = \lambda$  and  $\phi(B, g(B)) = 0$  for all  $B \in \mathcal{N}$ . This proves (20) and the first equality in (19). By a continuity argument we have  $\partial_t \phi(B, g(B)) \neq 0$ . Hence,  $g(B)$  is an algebraically simple eigenvalue of  $B$  and the eigenspace  $\text{Ker}[g(B)I_k - B]$  is a line. There is no loss of generality in assuming that  $u$  has unit length. Except for a sign, the eigenvector  $\xi(B)$  associated to  $g(B)$  is unique up to normalization. We select it so as to get the second equality in (19) and continuous dependence with respect to  $B$ . Of the two normalized eigenvectors associated to  $g(B)$ , we choose in fact the one whose angle with respect to  $u$  is smallest.  $\square$

Lemma 1 can also be proven by using Theorem 2.1 in Andrew et al. [2]. The next result concerns some topological properties of the maximization problem defining the regular Pareto capacity.

**Proposition 3.** *The maximization problem (17) has an objective function*

$$A \in \mathbb{M}_n \mapsto f^{\text{reg}}(A) := \text{card}[\Pi^{\text{reg}}(A)]$$

*that is lower semicontinuous and a solution set*

$$\mathfrak{S}_n^{\text{reg}} := \{A \in \mathbb{M}_n : \text{card}[\Pi^{\text{reg}}(A)] = \mathfrak{c}_n^{\text{reg}}\}$$

*that is open.*

*Proof.* A main ingredient of the proof is the set-valued map  $\Pi_J^{\text{reg}} : \mathbb{M}_n \rightarrow 2^{\mathbb{R}}$ , where  $\Pi_J^{\text{reg}}(A)$  is the set of regular Pareto eigenvalues of  $A$  produced by  $J$ . We consider  $J \in \mathcal{J}_n$  as an arbitrary but fixed index set. For notational convenience we write  $k := |J|$  and introduce the linear map  $L_J : \mathbb{M}_n \rightarrow \mathbb{M}_k$  given by  $L_J(A) := A^J$ . We divide the proof into three steps.

*Step 1:* lower stability of  $\Pi_J^{\text{reg}}$ . We claim the following stability result:

$$\left\{ \begin{array}{l} \text{Let } A \in \mathbb{M}_n \text{ and } \lambda \in \Pi_J^{\text{reg}}(A). \text{ Then there is a continuous} \\ \text{function } \psi : \mathcal{O} \rightarrow \mathbb{R} \text{ on some open neighborhood of } A \\ \text{such that } \psi(A) = \lambda \text{ and } \psi(C) \in \Pi_J^{\text{reg}}(C) \text{ for all } C \in \mathcal{O}. \end{array} \right. \quad (21)$$

Let  $A$  and  $\lambda$  be as in (21). Then the system (4) holds for some  $u \in \mathbb{R}^k$ . Since  $\lambda$  is algebraically simple as eigenvalue of  $A^J$ , Lemma 1 ensures the existence of an open neighborhood  $\mathcal{N}$  of  $A^J$  and continuous functions  $g : \mathcal{N} \rightarrow \mathbb{R}$  and  $\xi : \mathcal{N} \rightarrow \mathbb{R}^k$  satisfying  $g(A^J) = \lambda$ ,  $\xi(A^J) = u$ , and condition (20). Note that  $\mathcal{O} := \{C \in \mathbb{M}_n : L_J(C) \in \mathcal{N}\}$  is an open neighborhood of  $A$  and  $C^J \xi(C^J) = g(C^J) \xi(C^J)$  for all  $C \in \mathcal{O}$ . Over the set  $\mathcal{O}$  we consider the functions  $\psi := g \circ L_J$  and  $\eta := \xi \circ L_J$ . Since  $u = \xi(A^J) = \eta(A)$  satisfies (4), we have in particular

$$\eta(A) \succ 0, \quad \sum_{j \in J} a_{i,j} \eta_j(A) > 0 \text{ for all } i \notin J. \quad (22)$$

By applying a continuity argument and reducing the size of  $\mathcal{O}$  if necessary, we see that (22) is true not just for the reference matrix  $A$  but also for all  $C \in \mathcal{O}$ . On the other hand, we saw that  $\lambda = g(A^J) = \psi(A)$ . By a continuity argument again,  $\psi(C) = g(C^J)$  is an algebraically simple eigenvalue of  $C^J$ . The above discussion shows that  $\psi(C) \in \Pi_J^{\text{reg}}(C)$  for all  $C \in \mathcal{O}$  and completes the proof of (21).

*Step 2:* lower semicontinuity of  $f^{\text{reg}}$ . In fact,  $f^{\text{reg}}$  satisfies a property that is stronger than lower semicontinuity, namely, any matrix in  $\mathbb{M}_n$  is a local minimizer of  $f^{\text{reg}}$ . For proving this fact, we pick  $A \in \mathbb{M}_n$ , write  $r := f^{\text{reg}}(A)$ , and arrange the elements of  $\Pi^{\text{reg}}(A)$  in increasing order, say  $\lambda_1 < \dots < \lambda_r$ . For each  $i \in \mathbb{N}_r$ , we identify an index set  $J_i$  that produces  $\lambda_i$  and apply the lower stability result (21) with  $\lambda = \lambda_i$  and  $J = J_i$ . We get an open neighborhood  $\mathcal{O}_i$  of  $A$  and a continuous function  $\psi_i : \mathcal{O}_i \rightarrow \mathbb{R}$  such that  $\psi_i(A) = \lambda_i$  and  $\psi_i(C) \in \Pi_{J_i}^{\text{reg}}(C) \subseteq \Pi^{\text{reg}}(C)$  for all  $C \in \mathcal{O}_i$ . By considering  $\mathcal{O} := \bigcap_{i=1}^r \mathcal{O}_i$  and reducing this neighborhood of  $A$  even further if necessary, we get  $\psi_1(C) < \dots < \psi_r(C)$  and  $\{\psi_1(C), \dots, \psi_r(C)\} \subseteq \Pi^{\text{reg}}(C)$  for all  $C \in \mathcal{O}$ . Hence,  $f^{\text{reg}}(C) \geq r$  for all  $C \in \mathcal{O}$ .

*Step 3:* openness of  $\mathfrak{S}_n^{\text{reg}}$ . Let  $A \in \mathfrak{S}_n^{\text{reg}}$ . Since  $A$  is a local minimizer of  $f^{\text{reg}}$ , there is a neighborhood  $\mathcal{O}$  of  $A$  such that  $\mathfrak{c}_n^{\text{reg}} = f^{\text{reg}}(A) \leq f^{\text{reg}}(C) \leq \mathfrak{c}_n^{\text{reg}}$  for all  $C \in \mathcal{O}$ . Hence,  $\mathcal{O} \subseteq \mathfrak{S}_n^{\text{reg}}$  and  $A$  belongs to the interior of  $\mathfrak{S}_n^{\text{reg}}$ .  $\square$

**Corollary 1.** *The regular Pareto capacity of the space  $\mathbb{M}_n$  admits the characterization*

$$\mathfrak{c}_n^{\text{reg}} = \max_{A \in \text{SR}(n)} \text{card}[\Pi(A)]. \quad (23)$$

Formula (23) follows from the fact that  $\mathfrak{S}_n^{\text{reg}}$  is open and  $\text{SR}(n)$  is dense. In contrast with the case of  $\mathfrak{S}_n^{\text{reg}}$ , it is not clear whether  $\mathfrak{S}_n$  is open for  $n \geq 3$ . What is clear however is that  $\mathfrak{S}_n$  is not closed. Indeed, as mentioned in [11, Proposition 2], we have  $\Pi(tA) = t \Pi(A)$  for all  $A \in \mathbb{M}_n$  and all nonnegative  $t \in \mathbb{R}$ . As a consequence of this fact, we see that the zero matrix of order  $n$  belongs to topological closure of  $\mathfrak{S}_n$ , but not to  $\mathfrak{S}_n$  itself. The next result prevents us from being too optimistic concerning the topological properties of  $f$  and  $f^{\text{reg}}$ .

**Proposition 4.** *Let  $n \geq 2$ . Then  $f^{\text{reg}}$  is not upper semicontinuous and, on the other hand, the function  $f$  introduced in (2) is neither lower nor upper semicontinuous.*

*Proof.* Consider the matrix of order  $n$  defined by  $A(t, s) := tD + s(\mathbf{1}_n\mathbf{1}_n^\top - I_n)$ , where  $\mathbf{1}_n$  is the  $n$ -dimensional vector of ones,  $I_n$  is the identity matrix of order  $n$ , and  $D := \text{Diag}(d_1, \dots, d_n)$  with  $0 < d_1 < \dots < d_n < 1$ . Since the  $d_i$ 's are distinct, we have  $f(A(1, 0)) = n$ . By [13, Lemma 5.1] we can also write  $f(A(1, s)) = 1$  for all  $s < 0$ . This proves that  $f$  is not lower semicontinuous. The lack of upper semicontinuity in  $f^{\text{reg}}$  and  $f$  can be treated in tandem. Since  $A(0, 1) = \mathbf{1}_n\mathbf{1}_n^\top - I_n$  is the adjacency matrix of the complete graph on  $n$  vertices, Example 1 in Seeger [14] shows that

$$\Pi(A(0, 1)) = \Pi^{\text{reg}}(A(0, 1)) = \{0, 1, \dots, n-1\}.$$

In particular,  $f(A(0, 1)) = f^{\text{reg}}(A(0, 1)) = n$ . On the other hand, we claim that

$$f(A(t, 1)) \geq f^{\text{reg}}(A(t, 1)) \geq 2n - 1 \quad (24)$$

for all  $t \in ]0, 1[$  small enough. It suffices to check the second inequality in (24). The matrix

$$A(t, 1) = \begin{bmatrix} td_1 & 1 & \cdots & 1 \\ 1 & td_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & td_n \end{bmatrix}$$

is positive entrywisely. Hence,  $td_1, \dots, td_n$  are regular Pareto eigenvalues of this matrix. Now, for  $k \in \{2, \dots, n\}$ , let  $\rho_k$  be the spectral radius of the principal submatrix of order  $k$  on the upper-left corner of  $A(t, 1)$ . By the Perron-Frobenius theorem,  $\rho_k$  is a regular Pareto eigenvalue of  $A(t, 1)$ . We have  $td_1 < \dots < td_n < \rho_2 < \dots < \rho_n$  because, when  $t$  is small,  $\rho_k$  is slightly bigger than  $k-1$ . Hence,  $A(t, 1)$  has at least  $2n-1$  regular Pareto eigenvalues.  $\square$

*Remark 1.* David Sossa (Universidad de O'Higgins, Chile) carried out for us some numerical experiments with a random diagonal matrix  $D$  whose diagonal entries follow a uniform distribution on  $[0, 1]$ . For each value of  $n$  up to 12, he generated a sample of 100 random diagonal matrices of order  $n$  and in all instances he got  $f^{\text{reg}}(D + \mathbf{1}_n\mathbf{1}_n^\top - I_n) = 2^n - 1$ . The lower estimate in (24) is fairly conservative, but sufficient for proving the lack of upper semicontinuity in  $f^{\text{reg}}$  and  $f$ .

The next theorem gives a sufficient condition for a matrix to be a point of continuity of the Pareto spectral map  $\Pi$ . The lower and upper envelopes of a set-valued map  $\Phi : \mathbb{M}_n \rightarrow 2^{\mathbb{R}}$  are defined in the usual Painlevé-Kuratowski sense, i.e.,

$$\begin{aligned} \liminf_{C \rightarrow A} \Phi(C) &:= \left\{ \lambda \in \mathbb{R} : \lim_{C \rightarrow A} \text{dist}[\lambda, \Phi(C)] = 0 \right\}, \\ \limsup_{C \rightarrow A} \Phi(C) &:= \left\{ \lambda \in \mathbb{R} : \liminf_{C \rightarrow A} \text{dist}[\lambda, \Phi(C)] = 0 \right\}. \end{aligned}$$

For general material on Painlevé-Kuratowski limits, see the books of Aubin and Frankowska [3] or Rockafellar and Wets [12].

**Theorem 2.** *Suppose that  $A \in \mathbb{M}_n$  is spectrally regular. Then  $A$  is a point of continuity of  $\Pi : \mathbb{M}_n \rightarrow 2^{\mathbb{R}}$  in the sense that  $\liminf_{C \rightarrow A} \Pi(C) = \limsup_{C \rightarrow A} \Pi(C)$ . In particular, the points of continuity of  $\Pi$  form a dense set in  $\mathbb{M}_n$ .*

*Proof.* The stability result (21) was used in Proposition 3 to prove that  $f^{\text{reg}}$  is a lower semicontinuous function. The same stability result serves to show that  $\Pi^{\text{reg}} : \mathbb{M}_n \rightarrow 2^{\mathbb{R}}$  is a lower semicontinuous set-valued map in the sense that

$$\Pi^{\text{reg}}(A) \subseteq \liminf_{C \rightarrow A} \Pi^{\text{reg}}(C) \quad (25)$$

for all  $A \in \mathbb{M}_n$ . On the other hand, Seeger and Toriki [16, Proposition 1.2] proved that

$$\text{gr}(\Pi) := \{(A, \lambda) \in \mathbb{M}_n \times \mathbb{R} : \lambda \in \Pi(A)\}$$

is a closed set in the product space  $\mathbb{M}_n \times \mathbb{R}$ . Since  $\Pi$  has a closed graph, it follows that

$$\limsup_{C \rightarrow A} \Pi(C) \subseteq \Pi(A) \quad (26)$$

for all  $A \in \mathbb{M}_n$ . Now, the combination of (25) and (26) yields

$$\Pi^{\text{reg}}(A) \subseteq \liminf_{C \rightarrow A} \Pi^{\text{reg}}(C) \subseteq \liminf_{C \rightarrow A} \Pi(C) \subseteq \limsup_{C \rightarrow A} \Pi(C) \subseteq \Pi(A)$$

for all  $A \in \mathbb{M}_n$ . If we assume that  $A$  is spectrally regular, then the above chain of inclusions becomes of course a chain of equalities.  $\square$

### 3.1 Regularization of irregular Pareto eigenvalues

As said before, the number of regular Pareto eigenvalues does not decrease if the matrix under consideration is subject to an arbitrary perturbation of small size. From a perturbational point of view, irregular Pareto eigenvalues are not as stable as regular Pareto eigenvalues. The proof of Proposition 5 shows however that an irregular Pareto eigenvalue can be converted into a regular Pareto eigenvalue by means of a carefully chosen perturbation. For the reader's convenience, we recall a rank-one perturbation lemma pertaining to the realm of classical spectral analysis, cf. Ding and Zhou [7, Theorem 2.1].

**Lemma 2.** *Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the (possibly complex) eigenvalues of  $B \in \mathbb{M}_k$ , counting algebraic multiplicity. Consider the perturbed matrix  $\tilde{B} := B + up^\top$ , where  $u \in \mathbb{R}^k$  is an eigenvector of  $B$  associated to the eigenvalue  $\lambda_1$  and  $p \in \mathbb{R}^k$  is an arbitrary nonzero vector. Then the eigenvalues of  $\tilde{B}$  are  $\lambda_1 + \langle p, u \rangle, \lambda_2, \dots, \lambda_k$ , counting algebraic multiplicity.*

**Proposition 5.** *Suppose that  $A \in \mathbb{M}_n$  admits an irregular Pareto eigenvalue. Then any neighborhood of  $A$  contains a matrix with at least  $f^{\text{reg}}(A) + 1$  regular Pareto eigenvalues.*

*Proof.* Let  $\mathcal{O}$  be an open neighbourhood of  $A$ . Let  $r := f^{\text{reg}}(A)$  and  $\Pi^{\text{reg}}(A) = \{\mu_1, \dots, \mu_r\}$ . Let  $\lambda_1$  be an irregular Pareto eigenvalue of  $A$ . We must find a matrix in  $\mathcal{O}$  that has  $r + 1$  regular Pareto eigenvalues. To fix the ideas, suppose that  $\lambda_1$  is produced by the index set  $J$ . For the sake of convenience we write  $k := |J|$  and  $B := A^J$ . By applying a permutation similarity transformation on  $A$  if necessary, we can assume that

$$A = \begin{bmatrix} B & G \\ F & H \end{bmatrix},$$

i.e., the principal submatrix  $B$  can be placed on the upper-left corner of  $A$ . In such a case,

$$Bu = \lambda_1 u, \quad u \succ 0, \quad Fu \succeq 0$$

for some  $u \in \mathbb{R}^k$ . For a pedagogical reason, we distinguish three cases.

*Case 1:* non-strictness but simplicity, i.e., at least one component of  $Fu$  is equal to 0, but  $\lambda_1$  is algebraically simple as eigenvalue of  $B$ . We pick a small positive  $\varepsilon$  and form a new matrix  $\tilde{F}$  by adding  $\varepsilon$  to each nonnegative entry of  $F$ . Since  $\tilde{F}$  is near  $F$ , the perturbed matrix

$$\tilde{A} := \begin{bmatrix} B & G \\ \tilde{F} & H \end{bmatrix}$$

remains in  $\mathcal{O}$ . In addition to  $Bu = \lambda_1 u$  and  $u \succ 0$ , we have  $\tilde{F}u \succ 0$ . Hence,  $\lambda_1$  is a regular Pareto eigenvalue of  $\tilde{A}$ .

*Case 2:* strictness but not simplicity. In this case,  $Bu = \lambda_1 u$ ,  $u \succ 0$ ,  $Fu \succ 0$ , and  $\lambda_1$  has algebraic multiplicity  $d \in \{2, \dots, k\}$  as eigenvalue of  $B$ . Of course, if  $k = n$ , then  $B = A$  and the condition  $Fu \succ 0$  must be dropped. The eigenvalues of  $B$  are

$$\lambda_1 = \lambda_2 = \dots = \lambda_d, \lambda_{d+1}, \dots, \lambda_k,$$

where repetitions and/or complex numbers are possible within the group  $\{\lambda_{d+1}, \dots, \lambda_k\}$ . We pick a small positive  $\varepsilon$  and write  $\tilde{\lambda}_1 := \lambda_1 + \varepsilon$ . By applying Lemma 2 with  $p := \varepsilon \|u\|^{-2} u$ , we see that the eigenvalues of  $\tilde{B} := B + \varepsilon \|u\|^{-2} uu^\top$  are

$$\tilde{\lambda}_1, \lambda_2 = \dots = \lambda_d, \lambda_{d+1}, \dots, \lambda_k.$$

Since  $\tilde{B}$  is near  $B$ , the perturbed matrix

$$\tilde{A} := \begin{bmatrix} \tilde{B} & G \\ F & H \end{bmatrix}$$

remains in  $\mathcal{O}$ . We view  $\tilde{A}$  as a perturbed version of  $A$ , but it is important to underline that only the upper-left block of  $A$  is affected by a perturbation. Note that  $\tilde{B}u = \tilde{\lambda}_1 u$ ,  $u \succ 0$ ,  $Fu \succ 0$ , and that  $\tilde{\lambda}_1$  algebraically simple as eigenvalue of  $\tilde{B}$ . Hence,  $\tilde{\lambda}_1$  is a regular Pareto eigenvalue of  $\tilde{A}$ .

*Case 3:* defectiveness. We simultaneously perturb  $F$  as in Case 1 and  $B$  as in Case 2. In this way, we produce a perturbed matrix

$$\tilde{A} := \begin{bmatrix} \tilde{B} & G \\ \tilde{F} & H \end{bmatrix}$$

that remains in  $\mathcal{O}$  and satisfies  $\tilde{B}u = \tilde{\lambda}_1 u$ ,  $u \succ 0$ ,  $\tilde{F}u \succ 0$ , with  $\tilde{\lambda}_1$  algebraically simple. Hence,  $\tilde{\lambda}_1$  is a regular Pareto eigenvalue of  $\tilde{A}$ .

Now, let  $\tilde{A}$  be as in any of the three cases considered above. From the proof of Proposition 3, we see that  $\tilde{A}$  has at least  $r$  additional regular Pareto eigenvalues, say  $\tilde{\mu}_1, \dots, \tilde{\mu}_r$ , which are near  $\mu_1, \dots, \mu_r$ , respectively. Thus,  $\tilde{A}$  has at least  $r + 1$  regular Pareto eigenvalues.  $\square$

*Remark 2.* Proposition 5 generalizes to the case in which  $A \in \mathbb{M}_n$  has  $s$  irregular Pareto eigenvalues. These Pareto eigenvalues must be produced however by corresponding index sets that are mutually disjoint. Under such hypothesis, the conclusion is that any neighborhood of  $A$  contains a matrix with at least  $f^{\text{reg}}(A) + s$  regular Pareto eigenvalues.

This section end with an interesting application of Proposition 5. The next result not only yields Corollary 1 as a by-product, but it says something stronger.

**Proposition 6.** *Any solution to the maximization problem (17) is spectrally regular.*

*Proof.* Reasoning by absurd, suppose that  $A \in \mathbb{M}_n$  is a solution to (17) and has at least one irregular Pareto eigenvalue. As we saw in the proof of Proposition 5, it is possible to construct a nearby matrix  $\tilde{A} \in \mathbb{M}_n$  such that  $f^{\text{reg}}(\tilde{A}) \geq f^{\text{reg}}(A) + 1$ , contradicting the optimality of  $A$ .  $\square$

## 4 Double-plus-one rule

The purpose of this section is to prove a certain relationship between two consecutive regular Pareto capacities. As we shall see in a moment, the term  $\mathfrak{c}_n^{\text{reg}}$  can be used to derive a lower bound for the next term  $\mathfrak{c}_{n+1}^{\text{reg}}$ . To be more precise, Theorem 3 asserts that  $\mathfrak{c}_{n+1}^{\text{reg}}$  is bigger than twice the value of  $\mathfrak{c}_n^{\text{reg}}$ . Theorem 3 is a by-product of the following proposition that has an interest by its own.

**Proposition 7.** *Let  $A \in \mathbb{M}_n$  be a matrix with  $r$  regular Pareto eigenvalues. Then there exist a real  $w$  and positive vectors  $p, q \in \mathbb{R}^n$  such that the extended matrix*

$$M := \begin{bmatrix} A & q \\ p^\top & w \end{bmatrix} \quad (27)$$

has at least  $2r + 1$  Pareto eigenvalues.

*Proof.* Let  $\lambda_1, \dots, \lambda_r$  be the regular Pareto eigenvalues of  $A$ . The number of regular Pareto eigenvalues does not change if we multiply  $A$  by a positive scalar. Hence, we can assume that the  $\lambda_i$ 's are well-spaced, say  $|\lambda_i - \lambda_j| \geq 1$  for  $i \neq j$ . Since  $p \succ 0$  and  $q \succ 0$ , we have

$$\{w, \lambda_1, \dots, \lambda_r\} \subseteq \Pi^{\text{reg}}(M). \quad (28)$$

For avoiding repetitions on the right-hand side of (28), we suppose that  $w$  is not a regular Pareto eigenvalue of  $A$ . So,  $M$  has at least  $r + 1$  regular Pareto eigenvalues. We now explain how to select the triplet  $(w, p, q)$  so as to get at least  $r$  additional Pareto eigenvalues in  $M$ . These new Pareto eigenvalues, say  $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_r\}$ , will be constructed so as to avoid repetitions in the set  $\{w, \lambda_1, \dots, \lambda_r, \tilde{\lambda}_1, \dots, \tilde{\lambda}_r\}$ . We now enter into details. We focus on a particular element of  $\Pi^{\text{reg}}(A)$ , say  $\lambda_1$ . Suppose that  $\lambda_1$  is produced by the index set  $J$ . For notational convenience we write  $k := |J|$  and  $B := A^J$ . By applying a permutation similarity transformation on  $A$  if necessary, we can assume that

$$A = \begin{bmatrix} B & G \\ F & H \end{bmatrix}.$$

In such a case, there exists  $u \in \mathbb{R}^k$  such that

$$Bu = \lambda_1 u, \quad u \succ 0, \quad Fu \succ 0. \quad (29)$$

The vector  $u$  is unique up to normalization and the slackness condition  $Fu \succ 0$  is strict. Of course, if  $k = n$ , then  $B = A$  and the condition  $Fu \succ 0$  must be dropped. For constructing  $\tilde{\lambda}_1$  we use the information provided by (29) and examine with attention the solvability of the extended system

$$\begin{bmatrix} B & a \\ c^\top & w \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} v \\ 1 \end{bmatrix}, \quad (30)$$

$$v \succ 0, \quad (31)$$

$$Fv + b \succeq 0, \quad (32)$$

where  $a \succ 0$  is formed with the first  $k$  components of  $q$ , the vector  $b \succ 0$  is formed with the last  $n - k$  components of  $q$ , and  $c \succ 0$  is formed with the first  $k$  components of  $p$ . Note that the matrix on the right-hand side of (30) is a principal submatrix of  $M$ . The classical eigenvalue problem (30) decomposes into

$$Bv + a = \lambda v, \quad (33)$$

$$\langle c, v \rangle + w = \lambda. \quad (34)$$

If  $\lambda$  is not an eigenvalue of  $B$ , then equation (33) is equivalent to

$$v = R_\lambda^B a, \quad (35)$$

where  $t \mapsto R_t^B := (tI_k - B)^{-1}$  is the resolvent of  $B$ . Substituting (35) into (34) yields

$$\lambda = w + \langle c, R_\lambda^B a \rangle, \quad (36)$$

which is a rational equation in the variable  $\lambda$ . We distinguish two cases.

*Case  $k = 1$ .* In this case  $B$  is order one and has  $\lambda_1$  as unique entry,  $u = 1$ , and  $F$  is a positive column vector. Furthermore, (35) and (36) become

$$v = \frac{a_1}{\lambda - \lambda_1}, \quad (37)$$

$$\lambda = w + \frac{c_1 a_1}{\lambda - \lambda_1}, \quad (38)$$

respectively. The solutions to (38) are equal to the eigenvalues of the matrix

$$\begin{bmatrix} \lambda_1 & a_1 \\ c_1 & w \end{bmatrix}, \quad (39)$$

i.e., they are given by

$$\lambda_\pm := \frac{1}{2}(\lambda_1 + w) \pm \frac{1}{2}[(\lambda_1 - w)^2 + 4c_1 a_1]^{1/2}.$$

If  $w$  is negative and large enough in absolute value, then  $\lambda_+$  is slightly bigger than  $\lambda_1$ . So, if we are given in advance a positive  $\varepsilon$ , then we can adjust the parameter  $w$  so that (38) admits a solution  $\tilde{\lambda}_1$  satisfying  $0 < \tilde{\lambda}_1 - \lambda_1 < \varepsilon$ . By setting  $\lambda = \tilde{\lambda}_1$  in (37), we obtain  $v > 0$  and  $Fv + b \succ Fv \succ 0$ . Hence,  $\tilde{\lambda}_1$  is a strict Pareto eigenvalue of  $M$  because the slackness condition (32) holds strictly. In fact,  $\tilde{\lambda}_1$  is a regular Pareto eigenvalue of  $M$  because both eigenvalues of (39) are distinct.

*Case  $k \geq 2$ .* Since  $\lambda_1$  is algebraically simple as eigenvalue of  $B$ , there exists an invertible matrix  $S \in \mathbb{M}_k$  such that

$$SBS^{-1} = \Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & D \end{bmatrix},$$

where  $D \in \mathbb{M}_{k-1}$  does not admit  $\lambda_1$  as eigenvalue. In the parlance of matrix theory, we have used a similarity transformation to bring  $B$  to a block diagonal form. The block  $D$  captures all the eigenvalues of  $B$ , except  $\lambda_1$  which is intentionally left aside. In principle, the eigenvalues of  $D$  could be complex or have algebraic multiplicity greater than one. A direct computation shows that

$$R_t^B = S^{-1} \begin{bmatrix} \frac{1}{t - \lambda_1} & 0 \\ 0 & R_t^D \end{bmatrix} S,$$

where  $R_t^D$  is the resolvent of  $D$ . Hence, the rational equation (36) can be written as

$$\lambda - w - \langle \hat{\gamma}, R_\lambda^D \hat{\alpha} \rangle = \frac{\gamma_1 \alpha_1}{\lambda - \lambda_1}, \quad (40)$$

where  $(\alpha_1, \hat{\alpha})^\top := Sa$  and  $(\gamma_1, \hat{\gamma})^\top := S^{-\top}c$ . We claim that the parameters of equation (40) can be adjusted so as to get a solution arbitrarily close but distinct from  $\lambda_1$ . For proving this claim we

rely on three facts. Firstly, since  $\lambda_1$  is not an eigenvalue of  $D$ , the matrix  $\lambda_1 I_{k-1} - D$  is invertible and

$$\lim_{\lambda \rightarrow \lambda_1} R_\lambda^D \hat{\alpha} = R_{\lambda_1}^D \hat{\alpha}$$

is well defined. In particular,  $R_\lambda^D \hat{\alpha}$  remains bounded as  $\lambda$  goes to  $\lambda_1$ . Secondly, the coefficient  $\gamma_1$  is positive. Let  $\{e_1, \dots, e_k\}$  be the canonical basis of  $\mathbb{R}^k$ . Since  $u$  is the eigenvector associated  $\lambda_1$ , the matrix  $S$  can be chosen so as to satisfy the additional requirement  $Su = e_1$ . We see that

$$\gamma_1 = \langle S^{-\top} c, e_1 \rangle = \langle c, S^{-1} e_1 \rangle = \langle c, u \rangle$$

is positive because  $c$  and  $u$  are positive vectors. And, thirdly, we can assume that  $\alpha_1$  is nonzero, otherwise we slightly perturb the first  $k$  components of  $q$  and this produces a slight perturbation in  $\alpha_1$ . Again, we take  $w$  negative and large enough in absolute value. In such a case, (40) has a solution, say  $\tilde{\lambda}_1$ , that is arbitrarily close but distinct from  $\lambda_1$ . Whether  $\tilde{\lambda}_1$  is slightly bigger or slightly smaller than  $\lambda_1$  depends on the sign of  $\alpha_1$ . Indeed,  $\text{sign}(\tilde{\lambda}_1 - \lambda_1) = \text{sign}(\alpha_1)$ . Let  $\tilde{v}_1$  be the vector obtained by taking  $\lambda = \tilde{\lambda}_1$  in (35). We know already that the eigenpair  $(\tilde{\lambda}_1, \tilde{v}_1)$  is a solution to the eigenvalue problem (30). We must check that  $\tilde{v}_1$  is a positive vector. As said before, by increasing enough the absolute value of  $w$  we can render  $z := \tilde{\lambda}_1 - \lambda_1$  as close to 0 as desired. The substitution of  $\lambda = \tilde{\lambda}_1$  into (35) yields after simplification

$$\tilde{v}_1 = S^{-1} \left( \frac{\alpha_1}{z} e_1 + \begin{bmatrix} 0 \\ R_{z+\lambda_1}^D \end{bmatrix} \right) = \frac{\alpha_1}{z} u + S^{-1} \begin{bmatrix} 0 \\ R_{z+\lambda_1}^D \end{bmatrix}. \quad (41)$$

If  $z$  is sufficiently close to 0, then  $R_{z+\lambda_1}^D$  is near  $R_{\lambda_1}^D$ . Recall that  $\alpha_1$  and  $z$  have the same sign. We have shown in this way that  $\tilde{v}_1 \succ 0$  because  $u \succ 0$ , the factor  $\alpha_1/z$  is a huge positive real, and the last vector in (41) remains bounded as  $z$  moves around 0. The strict slackness inequality  $F\tilde{v}_1 + b \succ 0$  can be proven along the same lines: we write

$$F\tilde{v}_1 = \frac{\alpha_1}{z} Fu + FS^{-1} \begin{bmatrix} 0 \\ R_{z+\lambda_1}^D \end{bmatrix}$$

and recall that  $Fu \succ 0$ . Summarizing, whether  $k = 1$  or  $k \geq 2$ , in both cases  $\tilde{\lambda}_1$  is a strict Pareto eigenvalue of  $M$ . The construction of  $\tilde{\lambda}_2, \dots, \tilde{\lambda}_r$  is similar. In conclusion,  $M$  has at least  $r + 1$  regular Pareto eigenvalues plus an additional group of at least  $r$  strict Pareto eigenvalues.  $\square$

As we saw in the proof of Proposition 7, the terms  $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_r\}$  are strict Pareto eigenvalues of the matrix (27). These Pareto eigenvalues can be rendered regular by slightly perturbing the matrix under consideration so as to avoid algebraic multiplicity in the principal submatrices. For economy of language we say that a matrix  $M$  is amenable if each principal submatrix of  $M$  is free of algebraically multiple real eigenvalues. Amenable matrices of order  $k$  form a dense set in  $\mathbb{M}_k$ . We now are ready to prove the double-plus-one rule for regular Pareto capacities.

**Theorem 3.** *Regular Pareto capacities obey to the double-plus-one rule*

$$2\mathbf{c}_n^{\text{reg}} + 1 \leq \mathbf{c}_{n+1}^{\text{reg}} \quad (42)$$

for all  $n \geq 1$ .

*Proof.* Both sides in (42) are equal to 3 when  $n = 1$ . Let  $n \geq 2$  and pick any  $A_* \in \mathbb{M}_n$  such that  $f^{\text{reg}}(A_*) = r := \mathbf{c}_n^{\text{reg}}$ . We select  $w \in \mathbb{R}$ , positive vectors  $p, q \in \mathbb{R}^n$ , and a matrix  $A$  near  $A_*$ , so that

$$M := \begin{bmatrix} A & q \\ p^\top & w \end{bmatrix}$$

is amenable. Since  $\mathfrak{S}_n^{\text{reg}}$  is an open set, cf. Proposition 3, we may suppose that  $A$  has still  $r$  regular Pareto eigenvalues. We are then in the context of the proof of Proposition 7 with the additional assumption that  $M$  is amenable. Hence, there exists a matrix  $M$  of order  $n+1$  with at least  $2r+1$  regular Pareto eigenvalues. This proves (42).  $\square$

The recursive relation (42) initialized at  $\mathfrak{c}_1^{\text{reg}} = 1$  yields the lower estimate

$$2^n - 1 \leq \mathfrak{c}_n^{\text{reg}}. \quad (43)$$

Constructing a matrix of order  $n$  with  $2^n - 1$  regular Pareto eigenvalues is not difficult: we may consider for instance a matrix of the form  $D + \mathbf{1}_n \mathbf{1}_n^\top - I_n$ , where  $D$  is a random diagonal matrix whose diagonal entries follow a uniform distribution on the unit interval  $[0, 1]$ . Alternatively, we may consider Amara's example mentioned in the proof of [11, Proposition 3] or the example proposed in Brás et al. [5, Section 6]. By combining (42) and (43) we readily get

$$\mathfrak{c}_{n+1}^{\text{reg}} - \mathfrak{c}_n^{\text{reg}} \geq 2^n, \quad (44)$$

$$\mathfrak{c}_{n+2}^{\text{reg}} - 2\mathfrak{c}_{n+1}^{\text{reg}} + \mathfrak{c}_n^{\text{reg}} \geq 2^n, \quad (45)$$

for all  $n \geq 1$ . Inequalities (44) and (45) assert that the derivative and second derivative of the sequence  $\{\mathfrak{c}_n^{\text{reg}}\}_{n \geq 1}$  go both to infinity at least as fast as  $2^n$ . Another easy and interesting consequence of Theorem 3 reads as follows.

**Corollary 2.** *The quotient  $\mathfrak{c}_n^{\text{reg}}/2^n$  is increasing as function of  $n$ . In particular, the limit*

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{c}_n^{\text{reg}}}{2^n} = \sup_{n \geq 1} \frac{\mathfrak{c}_n^{\text{reg}}}{2^n} \quad (46)$$

*exists in  $[1, \infty]$ .*

The first statement of the corollary is an equivalent formulation of the double-plus-one rule (42). From (43) we see that the limit in (46) is greater than or equal to 1. A priori, we should not rule out the possibility of getting infinity on each side of (46). Checking whether the limit (46) is finite or infinite turns out to be a difficult question. The next proposition does not settle this issue but gives at least an alternative characterization of the limit (46).

**Proposition 8.** *Consider the sequence  $\{\delta_n\}_{n \geq 1}$  whose general term*

$$\delta_n := \mathfrak{c}_{n+1}^{\text{reg}} - (2\mathfrak{c}_n^{\text{reg}} + 1) \quad (47)$$

*corresponds to the gap between both sides of the double-plus-one rule (42). Then*

$$\mathfrak{c}_n^{\text{reg}} = 2^n - 1 + \sum_{k=1}^{n-1} 2^{n-1-k} \delta_k$$

*for all  $n \geq 2$ . In particular,*

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{c}_n^{\text{reg}}}{2^n} = 1 + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\delta_k}{2^k}. \quad (48)$$

The proof of Proposition 8 is easy and, therefore, omitted. Formula (48) shows that the limit (46) is finite if and only if the series

$$S := \sum_{k=1}^{\infty} \frac{\delta_k}{2^k} \quad (49)$$

is finite. The series (49) is impossible to compute in practice because the evaluation of the gap term (47) is in itself a difficult numerical task. However, if  $S_*$  and  $S^*$  are computable lower and upper bounds of (49), then  $1 + (1/2)S_*$  and  $1 + (1/2)S^*$  serve as lower and upper estimates of (46).

**Example 2.** As said before, we know that  $\mathfrak{c}_1^{\text{reg}} = 1$ ,  $\mathfrak{c}_2^{\text{reg}} = 3$ ,  $\mathfrak{c}_3^{\text{reg}} = 9$ , as well as  $23 \leq \mathfrak{c}_4^{\text{reg}} \leq 26$  and  $\mathfrak{c}_5^{\text{reg}} \geq 57$ . With this preliminary information we get  $\delta_1 = 0$ ,  $\delta_2 = 2$ , and

$$\frac{\delta_3}{2^3} + \frac{\delta_4}{2^4} \geq \frac{9}{8}.$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{c}_n^{\text{reg}}}{2^n} \geq 1 + \frac{1}{2} \left[ \frac{\delta_1}{2} + \frac{\delta_2}{2^2} + \frac{\delta_3}{2^3} + \frac{\delta_4}{2^4} \right] \geq \frac{29}{16}. \quad (50)$$

We could sharpen this lower bound by bringing additional gap terms into the picture, say  $\delta_5$  and  $\delta_6$ . However, it is not clear to us which is the general growth pattern of the  $\delta_k$ 's.

The next theorem is a generalization of the double-plus-one rule. The formulation of Theorem 4 is certainly more involved and obscure than that of Theorem 3, but the proof follows similar steps. By a slight abuse of language, we refer to inequality (51) as the max-convolution rule. The term on the left-hand side of (51) would be a genuine max-convolution if we drop the factor  $2^d$ . Note that (51) can be viewed as a strong form of super-additivity for the sequence  $\{\mathfrak{c}_n^{\text{reg}}\}_{n \geq 1}$ .

**Theorem 4.** *For all  $n \geq 2$ , we have*

$$\max_{\substack{\ell+d=n \\ \ell, d \geq 1}} \{2^d \mathfrak{c}_\ell^{\text{reg}} + \mathfrak{c}_d^{\text{reg}}\} \leq \mathfrak{c}_n^{\text{reg}}. \quad (51)$$

*Proof.* We give only a sketch of the proof, because writing down all the details would take several pages. Let  $\ell, d$  be positive integers. We must prove that

$$2^d \mathfrak{c}_\ell^{\text{reg}} + \mathfrak{c}_d^{\text{reg}} \leq \mathfrak{c}_{\ell+d}^{\text{reg}}. \quad (52)$$

The case  $d = 1$  corresponds to (42) and the case  $\ell = 1$  corresponds to (44). So, we may suppose that  $\ell, d \geq 2$ . Let  $r := \mathfrak{c}_\ell^{\text{reg}}$  and  $s := \mathfrak{c}_d^{\text{reg}}$ . We pick  $A \in \mathfrak{S}_\ell^{\text{reg}}$  and  $W \in \mathfrak{S}_d^{\text{reg}}$  and form the matrix

$$M := \begin{bmatrix} A & Q \\ P & W \end{bmatrix},$$

where  $P, Q$  are rectangular matrices whose entries are all positive. Note that  $M$  is a generalization of (27). Recall that  $\mathfrak{S}_\ell^{\text{reg}}$  and  $\mathfrak{S}_d^{\text{reg}}$  are open sets, cf. Proposition 3. So, by proceeding to a small perturbation if necessary, we may assume that  $M$  is amenable. Since

$$\Pi^{\text{reg}}(W + tI_d) = \Pi^{\text{reg}}(W) + \{t\}$$

for all  $t \in \mathbb{R}$ , any shift of  $W$  remains in  $\mathfrak{S}_d^{\text{reg}}$ . So, we may suppose that  $\Pi^{\text{reg}}(A) \cap \Pi^{\text{reg}}(W) = \emptyset$  and that the diagonal entries of  $W$  are negative and large in absolute value. By following similar steps as in the proof of Proposition 7, we can show that  $\Pi^{\text{reg}}(M)$  contains the following elements:

$$\left\{ \begin{array}{l} \text{the } s \text{ regular Pareto eigenvalues of } W, \\ \text{the } r \text{ regular Pareto eigenvalues of } A, \text{ say } \{\lambda_1, \dots, \lambda_r\}, \\ \text{for each } L \in \mathcal{J}_d, \text{ a new group } \{\lambda_1^L, \dots, \lambda_r^L\} \text{ of distinct reals.} \end{array} \right. \quad (53)$$

In order to get the element  $\lambda_1^L$  for instance, we start by writing  $Bu = \lambda_1 u$ ,  $u \succ 0$ ,  $Fu \succ 0$ , as in the proof of Proposition 7, and then we solve an extended system of the form

$$\begin{bmatrix} B & Q_1 \\ P_1 & W^L \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} = \lambda \begin{bmatrix} v \\ z \end{bmatrix}, \quad (54)$$

$$v, z \succ 0, \quad (55)$$

$$Fv + Q_2 z \succ 0, \quad (56)$$

where  $W^L$  is the principal submatrix of  $W$  induced by the index set  $L$ . The matrices  $P_1, Q_1, Q_2$  are defined in an obvious way. The idea is that the matrix on the left-hand side of (54) is a principal submatrix of  $M$ . In particular,  $P_1$  is a submatrix of  $P$  and  $Q_1, Q_2$  are submatrices of  $Q$ . By slightly perturbing  $M$  if necessary, we may suppose that the elements listed in (53) are all distinct. Hence,  $M$  is of order  $\ell + d$  and has at least  $s + r + (2^d - 1)r = 2^d \mathfrak{c}_\ell^{\text{reg}} + \mathfrak{c}_d^{\text{reg}}$  regular Pareto eigenvalues.  $\square$

**Corollary 3.** *For all positive integers  $\ell, k, d$ , we have*

$$\mathfrak{c}_{k\ell}^{\text{reg}} \geq \frac{2^{k\ell} - 1}{2^\ell - 1} \mathfrak{c}_\ell^{\text{reg}}, \quad (57)$$

$$\mathfrak{c}_{k\ell+d}^{\text{reg}} \geq \mathfrak{c}_d^{\text{reg}} + \frac{2^{k\ell} - 1}{2^\ell - 1} 2^d \mathfrak{c}_\ell^{\text{reg}}. \quad (58)$$

*Proof.* The proof is based on the max-convolution inequality (52). For proving (57) we fix  $\ell$  and use mathematical induction on  $k$ . The details are omitted. Inequality (58) is obtained afterward by combining (52) and (57).  $\square$

#### 4.1 Lower bounds for Pareto capacities

It is possible to construct a matrix of order  $n$  with more than  $2^n - 1$  Pareto eigenvalues, but this is rather tricky. The best lower bound for  $\mathfrak{c}_n$  that is known insofar is

$$\varrho_n := (3/2)2^n - 3.$$

The inequality  $\varrho_n \leq \mathfrak{c}_n$  was obtained in Seeger and Vicente-Pérez [17, Theorem 2.2] by examining the Pareto spectrum of a matrix

$$\begin{bmatrix} s^2 & s^3 & s^4 & \dots \\ -s^3 & s^4 & s^5 & \dots \\ -s^4 & s^5 & s^6 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (59)$$

that depends on a positive parameter  $s$ . Under the assumption that  $s$  is transcendental and bigger than  $1 + \sqrt{2}$ , these authors proved (by using five full pages!) that (59) has exactly  $\varrho_n$  Pareto eigenvalues. Since the Pareto eigenvalues of (59) are all regular,  $\varrho_n$  serves also as lower bound for  $\mathfrak{c}_n^{\text{reg}}$ . The next proposition goes a step further and shows that a matrix of order  $n$  could have more than  $\varrho_n$  regular Pareto eigenvalues.

**Proposition 9.** *Suppose that there exists a matrix of order  $m$  with at least  $r$  regular Pareto eigenvalues. Then*

$$\left( \frac{r+1}{2^m} \right) 2^n - 1 \leq \mathfrak{c}_n^{\text{reg}} \quad (60)$$

for all  $n \geq m$ .

*Proof.* The proof is by induction on  $n$ . Inequality (60) is obvious for  $n = m$ . We suppose that (60) is true for a given  $n$  and examine the case  $n + 1$ . By using the double-plus-one rule and the induction hypothesis we get

$$\mathfrak{c}_{n+1}^{\text{reg}} \geq 2 \mathfrak{c}_n^{\text{reg}} + 1 \geq 2 \left[ \left( \frac{r+1}{2^m} \right) 2^n - 1 \right] + 1 = \left( \frac{r+1}{2^m} \right) 2^{n+1} - 1.$$

Hence, inequality (60) holds true also for  $n + 1$ .  $\square$

The statement of Proposition 9 amounts to saying that

$$2^k (\mathfrak{c}_n^{\text{reg}} + 1) - 1 \leq \mathfrak{c}_{n+k}^{\text{reg}} \quad (61)$$

for all positive integers  $n$  and  $k$ . The double-plus-one rule (42) is recovered from (61) by taking  $k = 1$ . Recall that in Section 3 we gave a particular example of matrix of order 5 with 57 Pareto regular eigenvalues. The next corollary is obtained by applying Proposition 9 with  $(m, r) = (5, 57)$ .

**Corollary 4.** *For all  $n \geq 5$  we have*

$$(29/16) 2^n - 1 \leq \mathfrak{c}_n^{\text{reg}}. \quad (62)$$

Note that (62) is consistent with (50). The factor 29/16 is bigger than the old factor 3/2 suggested by Seeger and Vicente-Pérez [17]. The improvement is not so considerable, but it is an improvement after all, cf. Table 1. By using (62) we see for instance that a matrix of order 20 could have more than 1.9 million regular Pareto eigenvalues! Proposition 9 leaves open the possibility of getting something even better than (62). A natural strategy would be to search for an admissible pair  $(m, r)$  such that  $(r+1)/2^m$  is as large as possible. That  $(m, r)$  is admissible means that there exists a matrix of order  $m$  with at least  $r$  regular Pareto eigenvalues.

**Corollary 5.** *Inequality (62) can be successively sharpened as follows:*

$$(233/128) 2^n - 1 \leq \mathfrak{c}_n^{\text{reg}} \quad \text{for all } n \geq 8, \quad (63)$$

$$(117/64) 2^n - 1 \leq \mathfrak{c}_n^{\text{reg}} \quad \text{for all } n \geq 9, \quad (64)$$

$$(941/512) 2^n - 1 \leq \mathfrak{c}_n^{\text{reg}} \quad \text{for all } n \geq 10. \quad (65)$$

*Proof.* We have not found yet a matrix of order 6 with more than 115 regular Pareto eigenvalues. In principle, the existence of such a matrix should not be discarded. Similarly, we have not found yet a matrix of order 7 with more than 231 regular Pareto eigenvalues. However, we found matrices of order 8, 9, and 10, with 465, 935, and 1881, regular Pareto eigenvalues, respectively. For instance,

$$\begin{bmatrix} 712 & 780 & 256 & -156 & -191 & 1 & 1 & 1 \\ 548 & 638 & 190 & -112 & -143 & 1 & 1 & 1 \\ 456 & 548 & 192 & -110 & -119 & 1 & 1 & 1 \\ 292 & 374 & 14 & 98 & -28 & 1 & 1 & 1 \\ 304 & 402 & 66 & -38 & -22 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 16 & -2 & 8 \\ 1 & 1 & 1 & 1 & 1 & 6 & 8 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 & -1 & 12 \end{bmatrix}$$

is a matrix of order 8 with 465 regular Pareto eigenvalues. By applying Proposition 9 to the admissible pairs  $(8, 465)$ ,  $(9, 935)$ , and  $(10, 1881)$ , we get (63), (64), and (65), respectively.  $\square$

n	Ref. [17] (3/2)2 <sup>n</sup> - 3	Corollary 4 (29/16)2 <sup>n</sup> - 1	Corollary 5 (233/128)2 <sup>n</sup> - 1	Corollary 5 (117/64)2 <sup>n</sup> - 1	Corollary 5 (941/512)2 <sup>n</sup> - 1
5	45	<b>57</b>	—	—	—
6	93	<b>115</b>	—	—	—
7	189	<b>231</b>	—	—	—
8	381	463	<b>465</b>	—	—
9	765	927	931	<b>935</b>	—
10	1533	1855	1863	1871	<b>1881</b>
11	3069	3711	3727	3743	<b>3763</b>
12	6141	7423	7455	7487	<b>7527</b>
20	1572861	1900543	1908735	1916927	<b>1927167</b>

Table 1: Lower bounds for  $\mathfrak{c}_n^{\text{reg}}$ . Best results are in bold.

Further sharpenings are still possible but this requires finding admissible pairs that perform better than (10, 1881). This can be done with the help of costly numerical experimentation or by exploiting the max-convolution rule. The cost of numerical experimentation is prohibitive if  $n$  is not of moderate size, so we explain next how to take advantage of the max-convolution rule.

**Example 3.** By way of example, suppose that we need a lower bound for  $\mathfrak{c}_{22}^{\text{reg}}$ . By using (65) we get  $\mathfrak{c}_{22}^{\text{reg}} \geq 7708671$ . It is possible to get a sharper lower bound by using iteratively the max-convolution inequality (52). For instance, starting from  $\mathfrak{c}_5^{\text{reg}} \geq 57$ , we get successively

$$\begin{aligned} \mathfrak{c}_{10}^{\text{reg}} &\geq 2^5 \mathfrak{c}_5^{\text{reg}} + \mathfrak{c}_5^{\text{reg}} \geq 1881 \\ \mathfrak{c}_{20}^{\text{reg}} &\geq 2^{10} \mathfrak{c}_{10}^{\text{reg}} + \mathfrak{c}_{10}^{\text{reg}} \geq 1928025 \\ \mathfrak{c}_{22}^{\text{reg}} &\geq 2^2 \mathfrak{c}_{20}^{\text{reg}} + \mathfrak{c}_2^{\text{reg}} \geq 7712103. \end{aligned}$$

Since  $22 = 4 \times 5 + 2$ , the lower bound  $\mathfrak{c}_{22}^{\text{reg}} \geq 7712103$  can be obtained also by applying inequality (58) with  $(k, \ell, d) = (4, 5, 2)$ .

Table 2 and Table 3 display the lower bounds for  $\mathfrak{c}_{\ell+d}^{\text{reg}}$  obtained with the help of the max-convolution inequality (52) and the best known lower bounds for  $\mathfrak{c}_\ell^{\text{reg}}$  and  $\mathfrak{c}_d^{\text{reg}}$ . We rely on the following available information:  $\mathfrak{c}_1^{\text{reg}} = 1$ ,  $\mathfrak{c}_2^{\text{reg}} = 3$ ,  $\mathfrak{c}_3^{\text{reg}} = 9$ ,  $\mathfrak{c}_4^{\text{reg}} \geq 23$ ,  $\mathfrak{c}_5^{\text{reg}} \geq 57$ .

$\ell$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
4	47	95	193	391	793	1587	3175	6353	12711
5	<b>115</b>	<b>231</b>	<b>465</b>	<b>935</b>	<b>1881</b>	<b>3763</b>	<b>7527</b>	<b>15057</b>	<b>30119</b>
6	<b>231</b>	463	929	1863	3737	7475	14951	29905	59815
7	463	927	1857	3719	7449	14899	29799	59601	119207
8	931	1863	3729	7463	14937	29875	59751	119505	239015
9	1871	3743	7489	14983	29977	59955	119911	239825	479655
10	<b>3763</b>	<b>7527</b>	<b>15057</b>	<b>30119</b>	<b>60249</b>	<b>120499</b>	<b>240999</b>	<b>482001</b>	<b>964007</b>
11	<b>7527</b>	15055	30113	60231	120473	240947	481895	963793	1927591
12	15055	30111	60225	120455	240921	481843	963687	1927377	3854759

Table 2: Lower bounds for  $\mathfrak{c}_{\ell+d}^{\text{reg}}$  obtained by using (52). Best results are in bold.

In Table 2, the best lower bounds are indicated in bold. As we can see, the best result are obtained with  $\ell = 5$  and  $\ell = 10$ . This explains why in Table 3 we are taking  $\ell$  as a multiple of 5. Note that the factor

$$\frac{61696857 + 1}{2^{25}} = \frac{30848429}{16777216} = 1.838709652 \quad (66)$$

$\ell$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
15	120499	240999	482001	964007	1928025
20	3856051	7712103	15424209	30848423	61696857

Table 3: Continuation of Table 2.

is bigger than the factor

$$\frac{1881 + 1}{2^{10}} = \frac{941}{512} = 1.837890625$$

used in Corollary 5, but the improvement is marginal. We mention in passing that (66) is almost equal to  $57/31 = 1.838709677$ , a term that seems to be somewhat special. Indeed, by writing (57) with  $\ell = 5$ , we readily get

$$\lim_{n \rightarrow \infty} \frac{c_n^{\text{reg}}}{2^n} = \lim_{k \rightarrow \infty} \frac{c_{5^k}^{\text{reg}}}{2^{5^k}} \geq \lim_{k \rightarrow \infty} \left( \frac{1}{2^{5^k}} \frac{2^{5^k} - 1}{2^5 - 1} c_5^{\text{reg}} \right) = \frac{c_5^{\text{reg}}}{2^5 - 1} \geq \frac{57}{31}.$$

In fact, from Corollary 2 and the above discussion we see that

$$\lim_{n \rightarrow \infty} \frac{c_n^{\text{reg}}}{2^n} = \sup_{\ell \geq 1} \frac{c_\ell^{\text{reg}}}{2^\ell - 1}. \quad (67)$$

In conclusion, we have derived three alternative characterizations for the limit of the quotient  $c_n^{\text{reg}}/2^n$  as  $n$  goes to infinity, namely, (46), (48), and (67). We have not been able insofar to take advantage of these characterizations for obtaining the precise numerical value of this limit. This being said, the path has been paved to young researchers who would like to contribute to this area. There are many open questions that deserve attention, among which we would like to mention the following items:

- As said already, the equality  $c_n^{\text{reg}} = c_n$  holds at least for  $n \in \{1, 2, 3\}$ . It remains unclear whether  $c_n^{\text{reg}}$  and  $c_n$  could be different for some  $n \geq 4$ . Thanks to Corollary 1, the equality  $c_n^{\text{reg}} = c_n$  is true for a given  $n$  if and only if  $\mathfrak{S}_n$  contains a spectrally regular matrix. This observation may shed some light on the present question.
- If  $c_n^{\text{reg}}$  and  $c_n$  were not equal in general, then it would be useful to know whether these terms have at least the same asymptotic behavior. A somewhat related question is that of knowing whether the Pareto capacity  $c_n$  obeys to the double-plus-one rule.
- The starting point of this work was the study of the term  $c_n$ . Equally interesting is the analysis of the Pareto capacity of some special subsets of  $\mathbb{M}_n$ . For instance, if  $\mathcal{G}_n$  is the set of connected undirected graphs of order  $n$  and  $\mathbb{A}_n := \{A_G : G \in \mathcal{G}_n\}$  is the set of associated adjacency matrices, then

$$\mathfrak{a}_n := \max_{A \in \mathbb{A}_n} \text{card}[\Pi(A)] = \max_{G \in \mathcal{G}_n} \text{card}[\Pi(A_G)]$$

is a number that plays a certain role in graph theory. It has been shown in Fernandes et al. [8] that  $\mathfrak{a}_n$  grows faster than any polynomial in  $n$ , that is to say,  $\lim_{n \rightarrow \infty} \mathfrak{a}_n/n^p = \infty$  for all positive integer  $p$ . There are good reasons to believe that  $\mathfrak{a}_n$  grows at exponential rate in  $n$ , but this is something that remains conjectural, cf. Seeger and Sossa [15, Conjecture 1].

In this paper and in our previous work [4] we have accomplished a number of advances concerning the analysis of Pareto capacities, but the debate is far from being closed.

## References

- [1] S. Adly and A. Seeger. A nonsmooth algorithm for cone-constrained eigenvalue problems. *Comp. Optim. Appl.*, 49 (2011), 299–318.
- [2] A.L. Andrew, K.W.E. Chu, and P. Lancaster. Derivatives of eigenvalues and eigenvectors of matrix functions. *SIAM J. Matrix Anal. Appl.*, 14 (1993), 903–926.
- [3] J.P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, Boston, 1990.
- [4] J.-B. Baillon and A. Seeger. On the maximal number of Pareto eigenvalues in a matrix of given order. *Linear Multilinear Algebra*, June 2019, DOI: 10.1080/03081087.2019.1624681.
- [5] C. Brás, A. Iusem, and J. Júdice. On the quadratic eigenvalue complementarity problem. *J. Global Optim.*, 66 (2016), 153–171.
- [6] X.-D. Chen and Y.-C. Qi. The Pareto eigenvalue problem of some matrices (Chinese). *J. Shanghai Second Polytechnic Univ.*, 27 (2010), 54–64.
- [7] J. Ding and A. Zhou. Eigenvalues of rank-one updated matrices with some applications. *Applied Math. Letters*, 20 (2007), 1223–1226.
- [8] R. Fernandes, J. Júdice, and V. Trevisan. Complementarity eigenvalue of graphs. *Linear Algebra Appl.*, 527 (2017), 216–231.
- [9] J.B. Hiriart-Urruty and A. Seeger. A variational approach to copositive matrices. *SIAM Review* 52 (2010), 593–629.
- [10] G. Holubová and P. Nečesal. A note on the relation between the Fučík spectrum and Pareto eigenvalues. *J. Math. Anal. Appl.*, 427 (2015), 618–628.
- [11] A. Pinto da Costa and A. Seeger. Cone-constrained eigenvalue problems: theory and algorithms. *Comp. Optim. Appl.*, 45 (2010), 25–57.
- [12] R.T. Rockafellar and R.J.-B. Wets. *Variational Analysis*. Springer-Verlag, Berlin, 1998.
- [13] A. Seeger. Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. *Linear Algebra Appl.* 292 (1999), 1–14.
- [14] A. Seeger. Complementarity spectral analysis of connected graphs. *Linear Algebra Appl.*, 543 (2018), 205–225.
- [15] A. Seeger and D. Sossa. On cardinality of complementarity spectra of connected graphs. *Linear Algebra Appl.*, November 2019, Doi: 10.1016/j.laa.2019.11.012.
- [16] A. Seeger and M. Torik. On eigenvalues induced by a cone-constraint. *Linear Algebra Appl.*, 372 (2003), 181–206.
- [17] A. Seeger and J. Vicente-Pérez. On cardinality of Pareto spectra. *Electron. J. Linear Algebra*, 22 (2011), 758–766.