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Understanding powers–individual concepts and common misconceptions

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The topic of powers and exponents is generally considered one major obstacle for learners in schools and universities. Our main goal is to answer the question: How can we bridge the gap between individual conceptions in students' minds and the relevant mental models (Grundvorstellungen) to conceptualize powers? With reference to this learning object, two tasks from our test instrument and initial results are presented, which are part of our diagnostic tool to enable the uncovering of students' individual images and misconceptions in the area of powers, by means of comparing the prescriptive with the descriptive level of relevant mental models.

Keywords: Grundvorstellung, conceptual change, understanding, mental models, powers.

Introduction

Even though the subject of power expressions and its modern exponential notation has been dealt with for a long time (Euler, 1770), it seems that students of all ages still face difficulties when confronted with them, as well as with exponentiation in general. That has been documented over the years, mainly in school settings (e.g. Confrey & Smith, 1994; Pitta-Pantazi et al., 2007; Avcu, 2010). The key issue seems to be the lack of understanding of basic rules and algorithms used to manipulate powers. To guard against such occurrences, there is a need for a *conceptual change* in students' minds (vom Hofe & Blum, 2016, p. 237), in order to gain a deeper understanding on the conceptual level. Hahn and Prediger (2008) have proposed that opportunities for conceptual change should even precede the introduction of rules on the procedural level, or as Sfard (1995) puts it: "Operational thinking must be replaced by structural"¹. This proposition considers the distinction of knowledge into *conceptual* and *procedural* knowledge (Hiebert & Lefevre, 1986). Taking the above into consideration, we pose a twofold question: Which mental models (*Grundvorstellungen*, vom Hofe & Blum, 2016) should students form for a confident and flexible manipulation of power expressions, and which individual models do they have? To answer this, we first concern ourselves with the identification of adequate mental models through a subject matter analysis and an expert survey. On the basis thereof, we carry out a design research study, where we go from a first pilot study to developing test items, over to a number of three cycles, to aid us in achieving a final version of our diagnosis instrument. Our goal is to find out what students' individual mental models are and whether they are consistent with the desirable normative models. In the present article, after setting the theoretical background, we will provide some insights into our methodology and evaluate the potential explanatory power on a small item sample of the test instrument.

¹ An extended discussion of this can be found in Long (2005, pp. 62f).

Theoretical Background

In this section we will present the theoretical approaches used in the study to operationalize understanding. First, an outline of the construct of mental models or *Grundvorstellungen* (vom Hofe & Blum, 2016) together with related terminology used in international literature will be given. Furthermore, the various transformations between representations will be displayed, since they play an essential role for successfully dealing with mathematical concepts. This sets the basis for the generation of the items used in the test instrument.

The concept of Grundvorstellungen (GVs)

Particularly when modeling with the help of mathematics, one needs to activate the suitable GV(s) to translate between the “real word” and the “world of mathematics”, which in turn—when successfully performed—means, that the topic has been understood (Bossé, Adu-Gyamfi, & Cheetham, 2011, p. 117). The concept of *Grundvorstellungen*, or *mental models*, has a long tradition in German subject-matter didactics. It is used to describe the connection between mathematics and reality on a conceptual level (vom Hofe & Blum, 2016). Three aspects of GVs can be distinguished. The *normative aspect* constitutes the basic idea of the concept; it is to be understood prescriptively as the desired status to be reached when understanding a concept. The *descriptive aspect* characterizes the individual perceptions and notions of learners; it is, therefore, a representation of the students’ conceptualization. Finally, the *constructive aspect* compares the normative and the descriptive aspect to provide some insight concerning students’ difficulties of understanding, as well as offer solutions for overcoming misconceptions (Hefendehl-Hebeker et al., 2019).

Considering that the concept has also been referred to as a theory, our view is the one offered by Radford, who mentions: “... Mogens Niss (1999) contends that a theory in math education has two goals. First, it entails a descriptive purpose, aimed at increasing understanding of the phenomena studied. Second, it has a normative purpose, aimed at developing instructional design.” (Radford, 2012, p. 4). Moreover, Hefendehl-Hebeker et al. (2019) add: “... the approach of the GV concept combines normative and descriptive methods with a constructive aim. In this sense, analyses of students’ work based on the GV concept typically do not remain at the descriptive level but lead to indications of a constructive ‘repairing’ of the analyzed problems.” (p. 35).

To differentiate and clarify the concept from internationally used terms we should mention the theoretical proximity of GVs to *concept image* and *concept definition* (Tall & Vinner, 1981). Although both approaches refer to understanding of mathematics using Piaget’s cognitive psychology (vom Hofe 1998) and constructivism (Hahn & Prediger 2008) as background theories, GVs are mostly understood in their normative level, as a didactic category for teachers to make explicit which mental models are appropriate for understanding the specific mathematical concept or operation. To differ clearly between the normative mental models according to the mathematical concept and the descriptive mental models in students’ minds, some authors replace the descriptive aspect with the term *individual mental models* (e.g. Prediger 2008). Vom Hofe (1998) uses the term *basic model* as a prescriptive notion and *individual image* as a descriptive notion, explaining his integrative view as a “holistic view of mathematical thinking and doing,” offering “methodical and

theoretical possibilities for combining normative didactic conceptions with descriptive and interpretive working methods” (p. 326).

Furthermore, Soto-Andrade and Reyes-Santander (2011) compare Grundvorstellungen with Lakoff & Nunez’s (2000) *conceptual metaphors*. The similarity of both is constituted by a similar mechanism of transmission: Real experience is the source domain allowing for concept formation, the goal is to provide abstract concepts with meaning. However, they differ in the aspect of their implementation in practice and in that through GVs, preexisting mathematical concepts are represented, rather than constructed. In this paper we will use the terms *normative* and *descriptive aspect* of GVs to mean *mental models* for the former and *individual models* for the latter, as used in Prediger (2008), according to Fischbein’s definition of a mental model as a “meaningful interpretation of a phenomenon or a concept” (Fischbein, 1989, p. 12). Following the conceptual change theory, we will refer to the descriptive aspect in a competence-oriented meaning, rather than adhere to a deficit-oriented analysis. The formation of GVs is the key element of concept formation, thus, also entails conceptual change (Kleine, Jordan, & Harvey, 2005).

Grundvorstellungen of Powers

From the normative point of view, there are four aspects that can be extracted and considered as mental models for powers with natural numbers as both the base and the exponent. Following a literature review², a subject matter analysis and an expert survey our preliminary results were presented in a first draft (Itsios & Barzel, 2018). After further expert talks, the identified mental models were redrafted and rephrased. These are:

- (GV1) *Repeated Multiplication (RM)*: this mental model can appear in two different ways—a more dynamic one (RM-d), describing a temporal-successive process where new objects emerge, e.g. the repeated doubling of a number of bacteria per hour, and a more static one (RM-s), focusing on the end result of a repeated action, as a whole or an object, describing the end result of a repeated action, e.g. a binary tree.
- (GV2) *Combinatory Conception (CC)*: this refers to the number of possible permutations with repetition (number of n-tuples), e.g. there are 10^5 possible combinations for a lock with five cylinders, each using a digit from 0 to 9.
- (GV3) *Stretching by the same factor (SF)*: this represents a magnification without the emergence of new objects, e.g. 5^2 can mean zooming into an image twice with a five-fold magnification.

Transformations among Different Representations

When learning a mathematical concept, students should engage in three actions as proposed by Lesh, Post and Behr (1987): *recognition* of the concept in multiple representations, flexible *manipulation* within these representations and *translation* between different representations. A

² Although we do not adopt the theoretical framework of combining embodied mathematics with conceptual metaphors, we acknowledge the work done in Mowat (2010), as this offers a database of mathematics teachers’ concept images on exponentiation.

stable use of multiple representational systems can be a guarantee for an easier development of concepts in mathematics and demonstrating mathematical understanding can mean being competent in the manipulation of these systems. These systems can vary from equations, graphs and tables to diagrams, charts or even verbal and written forms (Goldin, 2002, p. 197; Pape & Tchoshanov, 2001, p. 120). There is even a stipulation in the specific content of powers, that conceptual connections between representations is a prerequisite for deeper understanding of concepts (Nataraj & Thomas, 2012, p. 557). These transformations among representations set the basis for the design of our test instrument, which will be introduced in the next section.

Design and Methodology

A first draft of the test instrument was designed using 20 items in total, including three open-ended questions and 17 calculation tasks. For the construction of the items, the aforementioned GVs, as well as most appropriate transformations among representations were taken into consideration, to make the test as comprehensive as possible. This was then tested with a sample of 117 subjects (87 civil and mechanical engineering and 30 university mathematics students) and after a first assessment all responses were categorized according to the frequency of their occurrence. Then, together with the right answer, the three most frequent wrong answers were used as possible choices for each item. The final version comprises the initial three open-ended questions, one calculation task and 16 single-choice items. Following a baseline diagnosis, a total of 176 Grade 10 students from two different schools³ were tested in the main study, with the purpose of determining individual conceptions as well as to detect which GVs can be activated. For the purposes of this article we will present two items regarding the transformations from symbolic to graphical, as well as from symbolic to verbal.

Item Sample

Figure 1 shows two of the three open-ended questions (items A2 and A3), in which the diagnostic potential of individual mental models becomes evident. With them, we can explore the ideas that are prevalent in learners' minds about their conceptual understanding of power expressions and their connection to everyday life situations.

<p>A2. Draw a picture representing the expression 2^3. Explain your picture briefly.</p> <p>A3. Try to come up with a situation from your everyday life, that can be described with the expression 3^2.</p>

Figure 1: Test items A2 and A3

Item A2 requires the translation from symbolic to graphical and we expect that mainly Grundvorstellung RM-s will be activated. By requiring a brief explanation, we add a second transformation that students must perform, namely graphical to verbal. Since practically none of the test subjects' answers includes an explanation, we will consider item A2 to address only the transformation symbolic to graphical. Item A3 requires the translation from symbolic to verbal and

³ One comprehensive school (Gesamtschule) and one secondary school (Realschule).

the Grundvorstellungen RM-d, CC and SF are expected to be activated. In case RM-s is activated, the translation should still be considered to be under symbolic to verbal, not symbolic to graphical, since the task requires a verbalization of the situation, regardless of the fact that the given situation could describe a spatial arrangement.

Some results

Our assumption concerning the results is the following: if an appropriate GV is activated, then there will be a plausible answer given. We make no claims to the opposite, namely, if no reasonable answer is given, as to whether the corresponding GV is present in the student's mind or not. We analyzed the data by categorizing each response to a corresponding pattern, within a small expert group to ensure interrater agreement. In Table 1, the results to item A2 are presented together with the respective response frequencies and in each case an example is given, followed by the relevant GV that most likely could be supported due to the proximity to an individual model. Only a total of 51 Grade 10 students answered this question, including isolated cases with answers that are not hereby listed, because we could not diagnose the structure of those responses. In these drawings we often encountered indications of repeated multiplication (RM), with correct answers as well as misconceptions. 16 answers corresponded to a more static view (RM-s, 1st row), where 2^3 is misunderstood as multiplication, presented in a spatial-simultaneous arrangement. We also saw substantial dynamic versions, for example a sequence (2nd row), which we see either as figured numbers or just as cardinality of the relevant numbers. We also attributed RM-d, when the variables were given as symbols, where we can say that the calculation is perceived correctly, but without any hints as to whether the meaning can be properly elaborated (3rd row).

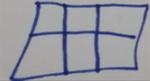
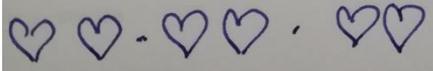
Response pattern	N	Example	GV
Type of calculation misconception (multiplication)	16		RM-s
Cardinal numbers as symbolic sequence	7		RM-d
Repeated multiplication with symbolic variables	7		RM-d

Table 1: Results and response frequencies to test item A2

What can be extracted from these results is that learners have not yet adequately shown that they can activate and use the necessary GVs to deal with the given task. However, their individual models, albeit not wrong, remain confined within the world of mathematics and symbols, rather than extending to real-life applications. Table 2 shows the answers given to test item A3, with a total of 46 Grade 10 students answering this question. Some isolated cases have not been listed under the response patterns, showing—again—no structure or diagnostic value. The last column shows, as previously in Table 1, the GV that in our opinion has to be activated, in order for the initial presented idea to make sense in the context of the given power expression.

Here we encountered cases where 3^2 is misunderstood as multiplication, presented in a temporal-successive way (1st row). The most common given response that can be considered feasible is shown in the 2nd row, where the power expression is viewed as a 3-by-3 spatial-simultaneous arrangement. Very few students (3) tried to present a situation, such that the Combinatory Conception could be identified, but failed to present it accurately (3rd row).

Response pattern	N	Example	GV
Type of calculation misconception (multiplication)	17	I am ill. I have to take pills. Two in the morning, two at noon and two in the evening.	<i>RM-d</i>
Area	7	I want to find out the area of a square-shaped room to lay a new carpet. Each side of the room has a length of 3m. I calculate $3^2 = 9$, then I know that I need a 9m^2 carpet.	<i>RM-s</i>
Combinatorics	3	I own 3 dresses, 3 handbags and 3 pairs of shoes that I can combine.	<i>CC</i>

Table 2: Results and response frequencies to test item A3

Looking at the first row of Table 2, we can identify a misconception that could be used as an opportunity for supporting learning. Further support, e.g. interviews, could be structured in a way that students are asked to compare the result of 3^2 with their own result (six), leading to a cognitive dissonance. This, in turn, could be the starting point of a conceptual change within their individual images to fit into the normatively set mental models.

Discussion

In all presented cases, we can observe that multiplicative thinking is predominant, with more than 50% of given answers showing that the two variables in question are combined in a multiplicative fashion, mostly assuming commutativity⁴. That can be seen in the first example of Table 2, where the numbers 2 and 3 of the expression are combined in a way that results in a multiplication context. After categorizing the response patterns for items A2 and A3 we identify the main problem to be a fixation on previous arithmetic operations of a lower level (multiplication). This problem encompasses depicting a number raised to the third power (see Table 1, first example given) as an array or a two-dimensional area, rather than a three-dimensional object, for example a cube.

Looking at the results of items A2 and A3 we can assume, that the main problem overall is a misconception of the mathematical object of exponents and powers. Confusing multiplication with exponentiation seems to be a principal reason for the demonstrated findings. A possible explanation for these findings is the lack of context variety offered to students when introducing the concept,

⁴ Responses to item A3 included both cases, where 2 and 3 were the multiplier and the multiplicand, respectively, and vice versa.

leading to a false notion of what exponentiation is and when it can be applied, even if plain calculation tasks can be performed faultlessly.

Regarding this TWG, our contribution is twofold. First, by expanding the range of Grundvorstellungen to include the normative GVs for powers, as mentioned above. Second, with our example we have shown how the concept of GVs can help to bridge the gap between individual models in students' minds and the relevant mental models which are necessary for the specific conceptualization. The method of connecting and comparing preexisting ideas and misconceptions with the normative level of GVs shows promising prospects to adequately describe our results, potentially leading to an accurate diagnosis and treatment of these misconceptions.

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