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Students' learning paths about ratio and proportion in geometry: an analysis using Peirce's theory of signs

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In this paper we use Peirce's theory of signs to analyze the conceptions of four twelve-year-old students on ratio and proportion while solving geometric problems of the stretchers/shrinkers type. The students participated in a series of interdisciplinary activities based on the principle of the camera obscura. We will present the learning paths of these students as chains of signification described in terms of Peircean semiosis. We will emphasize the challenges that they encountered during the reconstruction of their initial ideas and we will classify their conceptualizations in terms of Peirce's classification of Signs.

Keywords: Peirce, Semiotics, Classes of Signs, Ratio and Proportion, Elementary School.

Introduction

In the 1990s semiotics has started to gain momentum as an analytical tool among researchers in the field of mathematics education (Presmeg, Radford, Roth & Kadunz, 2018; Sáenz-Ludlow & Kadunz, 2016). By and large semiotics is the study of how signs come to signify, a theory of “signification” (Presmeg, Radford, Roth, & Kadunz, 2016, p. 1). Mathematics, as a highly symbolic practice, calls for a study of the nature of sign systems in order to understand the processes of its teaching and learning (Sáenz-Ludlow, 2006). In the same decade, mathematics education research was marked by the development of theoretical constructs that acknowledged the social origins of knowledge and consciousness. In these paradigms, knowledge and consciousness are seen as products of communication embedded in experiences that take place in historically-, culturally- and geographically-inscribed contexts (Lerman, 2000). Knowledge and experience, then, are co-constructed, as the one partakes in shaping the other. As this process is replete with signs, semiotics may elucidate the production of mathematical meaning by learners, as well the shaping of contexts that support learners' conduct with mathematical content and practices.

In our paper, we will use Peircean semiotic theory to analyze instances of the development of four twelve-year-old students' ideas concerning ratio and proportion, as they participated in a series of interdisciplinary tasks based on the principle of camera obscura. Charles Sanders Peirce (1839–1914) was an American philosopher, logician, mathematician, scientist, and semiotician. Contrary to other semioticians' claim for an essentialist dyadic structure between a ‘signifier’ and a ‘signified’, Peirce suggested that a Sign signifies only when it is interpreted in one's mind. He argued for a triadic structure for the process of semiosis among the *Sign* or *Representamen* (signifier), the *Object* (signified), and the *Interpretant*. The Interpretant, the most innovative and distinctive feature in Peirce's account on semiosis, refers to the interpretation and nature of the Sign and Object relation. The Interpretant, as we will have the chance to see later on, has the potential to generate a new Sign, thus moving the signification process towards a higher level of interpretation and generalization (Sáenz-Ludlow & Kadunz, 2016).

By analyzing students' participation in these activities, we will explore how their learning paths constitute chains of signification described in terms of Peircean semiosis (Sáenz-Ludlow, 2003), as Signs transform to more mature thoughts that become a kind of new knowledge. We suggest that Peirce's ten classes of Signs may constitute an analytical 'tool' that can be used by researchers and teachers alike for deconstructing and classifying students' conceptualizations, not only in ratio and proportion tasks, but in other instructional situations as well. We present this analytical tool in the section that follows, along with an outline of Peirce's semiotics emphasizing the process of semiosis.

Theoretical considerations

Even though he provided various definitions, Peirce defined "a Sign as anything which is so determined by something else, called its Object, and so determines an effect upon a person, which effect I call its Interpretant, that the latter is thereby mediately determined by the former" (Peirce, 1998, pp. 478). The Interpretant, coming as the most groundbreaking and distinguishing characteristic of Peirce's account on semiosis, refers to the understanding that we have of the Representamen and Object relation. According to Peirce signification is not a dyadic relation, as a Sign signifies only when it is interpreted in one's mind. The Interpretant does not refer to the user but to the interpretation, the meaning that is generated for a user (Atkin, 2013).

The importance of Interpretant is clearly visible in the process of *semiosis*, in which Signs transform to other thoughts that are more mature and become a kind of knowledge. When the relation of an Object₍₁₎, a Representamen₍₁₎ and an Interpretant₍₁₎ is established, the Interpretant₍₁₎ is potentially converted to a Representamen₍₂₎ for a new Object₍₂₎ and determines a new and more advanced Interpretant₍₂₎, and so on. The conceptual process of *semiosis* is theoretically unlimited, as the chain of meaning-making continues to new Signs interpreting a prior Sign or set of Signs (Figure 1).

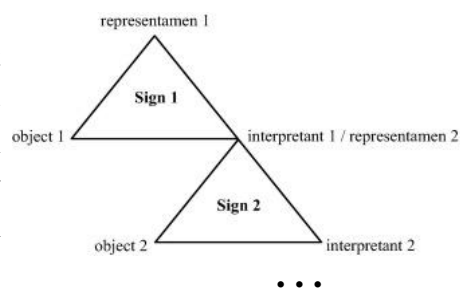


Figure 1: The infinite process of semiosis

However, in practice this process becomes limited by force of habit (Parker, 1998).

Peirce suggested that each of the three interrelated components of a Sign, namely Representamen, Object, and Interpretant, can be separated into a trichotomy. Representamens signify their Objects not through all their features, but in virtue of qualities, existential facts, or conventions and laws. In the first trichotomy, Signs then can be classified as *Qualisigns*, *Sinsigns*, and *Legisigns* respectively. The Representamen is the form in which the Sign appears, the spoken or written form of a word for instance, whereas the Sign is the whole meaningful ensemble. Peirce also believed that the nature of the Object constrains the nature of the Sign in terms of what is required for a successful signification. He thought about three categories of Objects, then, which were qualitative, existential or physical, and conventional or law-like in nature, constituting the second trichotomy. Thus, when a Sign reflects qualitative features of an Object, then the Sign is an *Icon*. When there is physical connection to its Object, then the sign is called an *Index*, and when there is a rule or law between it and its Object, then the sign is called a *Symbol*. The icon is an intuitively familiar trichotomy, as portraits, paintings, or diagrams fall under the *Icons* category. *Indices* may include fingers pointing to something or somewhere and proper names, while *Symbols* comprise broad speech acts like

assertion and judgment. Last, but not least, Peirce thought that Signs could be classified in terms of their relation with their Interpretant. In this third trichotomy, he suggested three categories by identifying qualities, existential facts, or conventional features as the basis for classifying the Sign in terms of its Interpretant. If the Sign determines an Interpretant by focusing our understanding of the Sign upon the qualitative features it employs in signifying its Object, then the Sign is classified as a *Rheme*. If a Sign determines an Interpretant by focusing our understanding of the sign upon the existential features that it employs in signifying an Object, then the Sign is a *Dicent*. Finally, if a Sign determines an Interpretant by focusing our understanding on some conventional or law-like features employed in signifying the Object, then the Sign is a *Delome*, or as Peirce frequently—but confusingly—called it, an *Argument* (Atkin, 2013; Colapietro, 1989; Freadman, 1996).

These three previous trichotomies produce combinations that are called ‘classes’. In order to categorize a Sign in these classes, we have to ask three questions: 1) What is the relation of the Sign with itself? (1st trichotomy), 2) What is the relation between the Sign and its Object? (2nd trichotomy), and 3) What is the relation between the Sign and its Object for its Interpretant? (3rd trichotomy). The kind of relation that answers the first question qualifies the second, which in turn qualifies the third. Once though we characterize a Representamen as, for example, a Qualisign, which is at the first level of the first trichotomy, the relation of the Sign to its Object and its Interpretant in the second and third trichotomy has to be at the first level as well, as the Sign cannot be related to an Object and an Interpretant from a higher level. According to these restrictions, then, the classes of Signs are ten (see Table 1). Even though the classes are hierarchically related, every link in the chain of signification does not necessarily constitute a more mature conceptual construct than those in the preceding links of the chain. Thus, in the highest level of Representamen, the Interpretant may be a Rheme (classes V, VI and VIII). Signs of this kind appear often in the mathematics classroom, as students may be familiar with representations of mathematical ideas but their interpretations about them may still be vague or confined to these representations. The chain of signification as outlined in the beginning of this section, along with Pierce’s taxonomy of Signs, may allow us to analyze the continuum of students’ conceptualization (i.e. learning path) of ‘complex’ concepts like ratio and proportion (Sáenz-Ludlow, 2003).

Table 1: Peirce’s classes of Signs

	S = R, O, I
I	Qualisign, Icon, Rheme
II	Sinsign, Icon, Rheme
III	Sinsign, Index, Rheme
IV	Sinsign, Index, Dicent
V	Legisign, Icon, Rheme
VI	Legisign, Index, Rheme
VII	Legisign, Index, Dicent
VIII	Legisign, Symbol, Rheme
IX	Legisign, Symbol, Dicent
X	Legisign, Symbol, Delome

An experimentation on ratio and proportion

We implemented the analytical tool that we outlined in the previous section in a data set produced from a research study that aimed to highlight students' ideas about ratio and proportion in geometry. Students’ ideas about these concepts were produced through their participation in a series of tasks that were part of an interdisciplinary experimentation. In particular, we wanted to address the two following questions: a) what are the conceptualizations of twelve-year-old students about ratio and proportion related to geometrical shapes; and b) how these conceptualizations could be modified through their participation in a series of interdisciplinary tasks. According to Lamon (1993), the

stretchers/shrinkers type of ratio and proportion problems, which have reference to geometry, pose difficulty for students in terms of recognizing their multiplicative structure. The students' strategies for solving ratio and proportion problems in a geometric context are the *nonconstructive strategies* of 'avoiding', 'visual or additive', and 'pattern-building'; the 'preproportional', 'qualitative proportional, and 'quantitative proportional' types of reasoning, which are considered as *constructive strategies*. The common ineffective strategy that students use in order to solve similar problems is the 'visual or additive strategy', which stresses the aforementioned students' difficulty with the multiplicative structure (Lamon, 1993).

Students worked in pairs for a total of seven 90-minute sessions over a period of 3 weeks. In this paper we will discuss the learning paths of four out of the eight students who participated in the study, those of George, Helen, Bill, and Peter. The tasks that comprised our research were based on an experimental construct that followed the principle of the rectilinear propagation of light in the *camera obscura*. We used this device in order to depict proportional geometrical shapes. In most tasks students had to investigate the similarity of rectangles formed by tiny led light sources placed on their vertices and their image on a screen placed behind the pinhole on a plane parallel to that of the rectangle. The use of this model gave us the opportunity to negotiate in a non-typical way the concepts of multiplicative relationships, internal and external ratios and also connect fractions with ratio and proportion. According to the Science and Mathematics programs of study, the students were familiar with the concepts at hand. Research data was comprised of the teacher-researcher's fieldnotes (first author), the fieldnotes of a 'non-participant' observer, the transcripts of the recordings in each pair, as well as the students written work.

In the tasks of the initial diagnostic assessment, which was silent and individual, we asked students to draw the parents of a little boy fish, whose body and tail were isosceles right-angled triangles of different size (students were familiar with this fish as it is the main character in a widely-read series of children's books in Greek). The only direction we gave to the students was that the parents needed to be similar to the boy but larger. In the second session of the experimentation we asked students to find the relation between two rectangles that were similar. Students were only aware that one of the rectangles was bigger in size than the other. In the third session, in one of the tasks, we gave students a number of rectangles of various sizes and we asked them to find which of these were similar. In the fourth session of the experimentation, we gave students a rectangle (7×3 cm) and we asked them to draw a similar one knowing only the length of one of its sides (11.2 cm). In a summative assessment during the final session, which was silent and individual as the initial diagnostic assessment, we asked students to investigate regular hexagons in terms of their similarity.

Students' chains of signification

In the task of the diagnostic assessment, the four students spontaneously measured the length of the sides of the triangles that comprised the little boy fish. They concluded that the triangles were isosceles, making no reference though to the measure of the angles in the two triangles. While drawing the parents, five of the students followed an additive strategy, without paying attention to the proportion between the size of the triangles of the body and tail of the boy fish and those of his parents. Helen and Bill tried to ensure that the triangles were isosceles to begin with, and that the measure of the interior angles matched roughly the ones in the boy's body and tail. So they drew the

two equal sides of the body and the tail of the parents by increasing their length by the same amount. For example, in drawing the father, one student increased the two sides from 5 to 8cm for the body and from 2.5 to 4.7cm for the tail. While drawing the third side of the triangles, though, the students realized that they could not enlarge this third side in same the manner that they had enlarged the other two. So, they increased the length of the third side as much as was needed in order to complete the triangles in their drawing. George designed one parent by doubling the sides of both triangles, thus paying attention to the interior angle in their drawings, but used an additive strategy for the other parent since they thought that they could multiply with a whole number (*Qualisign, Icon, Rheme*; in this case, the Sign constructed by the student is interpreted by the student as being of the same nature and quality with the given fish boy with O1: the multiplicative structure in the stretchers/shrinkers problems, R1: the students' drawings, and I1a: the use of multiplication and addition interchangeably and that ratio needs to be a whole number). Peter used multiplication for both parents, choosing the numbers 3 and 2 as operators for the father and the mother respectively (*Sinsign, Icon, Rheme*; the Sign is interpreted by the student as possibly standing for its Object with O1: the multiplicative structure in the stretchers/shrinkers problems, R1: the drawings and I1b: that multiplication as probably related to ratio and proportion).

In the second session, the four students spontaneously measured the lengths of their sides and Helen and Bill suggested that they should "find the difference between their lengths" (*Qualisign, Icon, Rheme*; the Sign is interpreted by the student as being of the same nature and quality with the given Representamen, with O2: the multiplicative structure of stretchers/shrinkers problems, R2: oral sentence, I2a: the relation between the two similar rectangles can be expressed by subtraction of their lengths). George, who had used a multiplicative strategy in drawing one parent and an additive in the other in the diagnostic assessment, disagreed by saying that "we need to divide to find out how many they are" (*Sinsign, Index, Rheme*; the Sign is interpreted by the student as possibly standing for another event-index, with O2: the multiplicative structure of stretchers/shrinkers problems, R2: oral sentence, I2b: division is related to ratio and proportion). At this point, Peter, who had used a multiplicative strategy in drawing both parents, disagreed, by saying "since we say twice or three times, we have multiplication" (*Legisign, Icon, Rheme*; is a type/law, a 'regularity of the indefinite future' that is interpreted by the student as possibly standing for its Object, with O2: the multiplicative structure of stretchers/shrinkers problems, R2: oral sentence, I2c: multiplication is related to ratio and proportion). Taking advantage of these statements, the researcher suggested that they should try both ideas, subtraction and division. In their first attempt, having completed the subtractions, they did not reach to a conclusion concerning the relation between the two rectangles. In their second attempt, instead of multiplication, which is directly related to the multiplicative strategy, the students decided to use division by dividing the larger by the smaller number. Having observed that the two quotients were equal when rounded to one decimal point, the four students concluded that we could ascertain whether two rectangles are similar by using division.

In the third session, the four students measured the sides of the rectangles and circled the similar ones on their sheet showing preproportional reasoning. Only George chose to express the relation between the similar rectangles by making a clear reference to the operator that constituted them as similar, showing qualitative proportional reasoning (*Legisign, Index, Rheme*; is a type interpreted by the student as possibly standing for its Object - another event, with O3: the multiplicative

structure of stretchers/shrinkers problems, R3: written expression, I3: division is related to ratio and proportion and the concept of unit plays a role to the result).

At the first part of the fourth session, the four students used the internal ratio ($\text{length } 1 \div \text{length } 2 = \text{breadth } 1 \div \text{breadth } 2$) in their calculations, and after they had drawn the desired rectangle they used the external ratio ($\text{length } 1 \div \text{breadth } 1 = \text{length } 2 \div \text{breadth } 2$) simply to check their answer without realizing that both ratios could be used interchangeably in order to express the desired relation between the rectangles (*Legisign, Icon, Rheme*; is a type/law, a ‘regularity of the indefinite future’ interpreted by the student as possibly standing for its Object, with O4: the internal and external ratio of similar shapes, R4: arithmetic operations, I4: the relation of similarity can be found with operations between lengths and widths, widths and lengths, and widths-widths and lengths-lengths and also ratios are pure numbers, without measurement). In the second part of this session, in an attempt to enhance the qualitative proportional reasoning through the use of mathematical terminology in the ‘stretchers/shrinkers’ tasks, we asked the students to express division in an alternative way. So expressions of the kind “seven divided by two” transformed to “seven over two,” especially from George and Peter, implicating thus a fractional terminology in the discussion (*Legisign, Symbol, Rheme*; is a type interpreted by the student as possibly standing for its Object-law, with O5: fractions are directly related to ratios, R5: oral sentences, I5: division can be alternatively expressed with a fraction and their terminology). All of the students though tended to always select the largest number as the numerator, as it was more convenient in the subsequent calculations. So we gave them a similar rectangle on the screen that was reduced in size and asked them to express the reduction with a fraction. All of them claimed that the answer for the reduction was “3.5 over 1.5, because in this way we begin from what it was (i.e. the length of the rectangle formed by the led light sources) and then we have the ‘reduced’ one” (i.e. the length of the rectangle on the screen).” The researcher then made a reference to maps their relation to the size of the actual land that they represent, and the four students immediately recognized that the ratio in the legend can be as small as 1 over 50000 (*Legisign, Symbol, Delome/Argument*; is a type interpreted by the student as semiotically standing for its Object-law, with O6: the role of numerators and denominators in ratios, R6: oral sentences, I6: flashback to maps and the various ratios in order to express the reduction may follow the same principles as in this situation). After this discussion, Bill, George, Helen, and Peter concluded that “the numerator expresses the representation on the map and the denominator the physical representation,” and therefore the fraction that expresses the relation between the rectangles can be larger or smaller than 1 when it represents a stretch or a shrink (*Legisign, Symbol, Rheme*; is a type interpreted by the student as possibly standing for its Object-law, with O7: the role of numerators and denominators in fractions, R7: oral sentences, I7: there is a fixed way to express the enlargement or zooming out and numerators and denominators determine the result).

In the tasks of the summative assessment, Bill, George, Helen, and Peter were able to transfer successfully the understanding that they had built in relation to similar rectangles in the sessions of the experimentation (*Legisign, Symbol, Dicient*; is a type interpreted by the student as physically standing for its Object-law, with O8: the multiplicative structure of stretchers/shrinkers problems and the role of numerators and denominators in ratios, R8: written sentences and written expressions, I8: the appropriate way to express ratios for enlargement or reduction, without measurements).

As we can see, the four students started the sessions (diagnostic assessment) with vague ideas about ratio and proportion in geometrical shapes. Only Peter used multiplicative strategy in all tasks while the other students used the additive strategy. In the instructional sessions that followed, it became evident that students produced arguments in order to support their statements. As a result of this effort some of the students constructed conceptualizations that fall in the upper levels in Peirce's list of Sign classes. In the summative session they managed to transfer their knowledge to a new context. The development of the four twelve-year-old students' conceptualizations concerning ratio and proportion while working on tasks that required the stretching and shrinking of rectangles and regular hexagons, were the Signs that the students constructed. Following Peirce's classification of Signs and the strategies students used while working on these tasks, the criteria used to classify the students' Signs can be summarized as below (Table 2):

Table 2: Signs constructed by the students

Sessions of Experimentation	Students' Signs according to ten classes of Peirce	
1 st : diagnostic assessment with individual tasks	Qualisign, Icon, Rheme (I) Sinsign, Icon, Rheme (II)	Bill, George, Helen Peter
2 nd : conservation of properties in stretcher/shrinker problems	Qualisign, Icon, Rheme (I) Sinsign, Index, Rheme, (III) Legisign, Icon, Rheme (V)	Bill, Helen George Peter
3 rd : stress of the multiplicative strategy	Legisign, Icon, Rheme (V) Legisign, Index, Rheme (VI)	Bill, George, Helen Peter
4 th : enhancement of multiplicative strategy by similar rectangles and ratios of lengths and widths	Legisign, Symbol, Rheme (VIII)	Bill, George, Helen, Peter in plenary session
division related to fraction, terminology, role of numerator and denominator, legend of maps	Legisign, Symbol, Delome (X) Legisign, Symbol, Rheme (VIII)	George, Peter Bill, George, Helen, Peter in plenary session
5 th : summative assessment with individual tasks	Legisign, Symbol, Dicent (IX)	Bill, George, Helen, Peter

Closing Remarks

The learning process is affected by many factors, such as students' ideas, the interaction between them, the context, the tasks, the representations involved, etc. This means that the learning process is neither rectilinear nor individual; it is 'fluid' and infinite, and includes a number of conceptualizations. Following Peirce's suggestions on how to make our ideas clear, the meaning of a concept or a sign consists in the habit to which it gives rise in the interpreter, be it either a habit of action or a habit of thought. This habit guides future actions, and is forever on trial in the light of upcoming experiences (Triandafillidis, 2002). We see students' chains of signification side by side to their activity sequences while they engage with tasks in an elaborated Hypothetical (Teaching) Learning Trajectory that a teacher/researcher has designed. By using Peirce's classes of Signs then, we expand the work of other researchers who have used Peirce's theory of Signs to analyze the learning process based on the triadic nature of a Sign's three components. Our work may also assist a teacher/researcher to concretize students' produced Signs, the ideas that are involved, to what extent students become familiar with the concepts, and also mark those characteristics of the tasks that turn students' conceptualizations to higher level Signs, i.e. habits of action or thought that

would endure future trials. In other words, it may help a teacher/researcher to apprehend turning points in students' activity sequences and the distance of a student's Sign from the mathematical goal(s) that she has set, as well as to take appropriate instructional actions in order to support the construction of Signs that are more mature and more relevant to the mathematical ideas at hand.

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