Eñe product in the transalgebraic class

Ricardo Pérez-Marco

EÑE PRODUCT IN THE TRANSALGEBRAIC CLASS

RICARDO PÉREZ-MARCO

Abstract

We define transalgebraic functions on a compact Riemann surface as meromorphic functions except at a finite number of punctures where they have finite order exponential singularities. This transalgebraic class is a topological multiplicative group. We extend the action of the eñe product to the transcendental class on the Riemann sphere. This transalgebraic class, modulo constant functions, is a commutative ring for the multiplication, as the additive structure, and the eñe product, as the multiplicative structure. In particular, the divisor action of the eñe product by multiplicative convolution extends to these transalgebraic divisors. The polylogarithm hierarchy appears related to transalgebraic eñe poles of higher order.

Contents

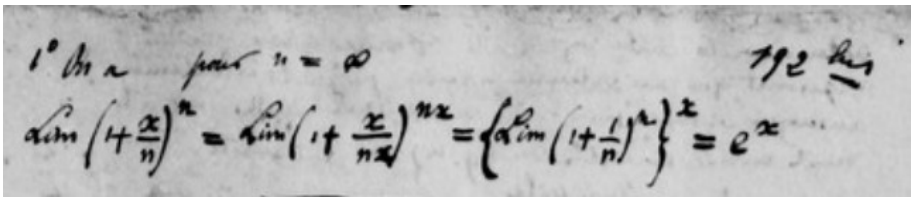

Figure 1. From Évariste Galois' manuscripts, Ms 2108, Institut de France, [17].

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1. Introduction and Euler miscellanea

One of Euler's major discoveries is the transalgebraic nature of the exponential function as the unique "polynomial", normalized to take the value 1 at 0 as well as its derivative, with only one zero of infinite order at ∞ (see [16]). Euler writes

$$e^{-z} = \left(1 - \frac{z}{\infty}\right)^{+\infty} .$$

Hence, the exponential appears as a "transalgebraic polynomial". Observe that in this heuristic formula both infinite symbols are of a different nature: One is an infinite point (∞ in the Riemann sphere) and the other an infinite number ($+\infty$ is the infinite order of the zero). Obviously, the proper justification of this heuristic comes from Euler formula,

$$e^z = \lim_{n\to+\infty} \left(1 + \frac{z}{n}\right)^n$$

A one line proof, assuming known the existence of the elementary limit for $e = 2.71828182846\ldots$

$$e = \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n$$

can be found in Galois manuscripts [17] (these may be course notes from his professor P.-L. É. Richard at Lycée Louis-le-Grand, so the authorship is unclear),

$$\lim_{n\to+\infty} \left(1 + \frac{z}{n}\right)^n = \lim_{n\to+\infty} \left(1 + \frac{z}{nz}\right)^{nz} = \lim_{n\to+\infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^z = e^z$$

Note, that we do not want to use the logarithm function in the proof since the proper order to develop the theory is to define the exponential first. We also have a purely geometric proof that is the proper geometrization of Euler's transalgebraic heuristics, by using Carathéodory's convergence Theorem of the uniformizations for the Carathéodory convergence of the log-Riemann surfaces of $\sqrt[n]{z}$ to the log-Riemann surface of $\log z$ when $n \to +\infty$ (see [4] for details, and [5] for background on log-Riemann surfaces). The construction of the log-Riemann surface of $\log z$ does not require the previous definition of the logarithm (on the contrary, we can define the logarithm function from it).

The main property of the exponential can be derived "à la Euler" as follows (using $\infty^{-2} << \infty^{-1}$):

$$\begin{aligned}
e^{z_1}.e^{z_2} &= \left(1 + \frac{z_1}{\infty}\right)^{+\infty} . \left(1 + \frac{z_2}{\infty}\right)^{+\infty} \\
&= \prod_{+\infty} \left(1 + \frac{z_1}{\infty}\right) . \left(1 + \frac{z_2}{\infty}\right) \\
&= \prod_{+\infty} \left(1 + \frac{z_1 + z_2}{\infty} + \frac{z_1.z_2}{\infty^2}\right) \\
&= \prod_{+\infty} \left(1 + \frac{z_1 + z_2}{\infty}\right) \\
&= \left(1 + \frac{z_1 + z_2}{\infty}\right)^{+\infty} \\
&= e^{z_1+z_2}
\end{aligned}$$

The exponential function is the link between the additive and the multiplicative structure on $\mathbb{C}$. It serves also as the link between the multiplicative and the eñe product structure.

The eñe product.

We briefly recall the definition of the eñe product (see [22]). Given two polynomials $P, Q \in \mathbb{C}[z]$, normalized such that

$$P(0) = Q(0) = 1$$

say

$$P(z) = 1 + a_1 z + a_2 z^2 + \ldots = \prod_{\alpha} \left(1 - \frac{z}{\alpha}\right)$$

$$Q(z) = 1 + b_1 z + b_2 z^2 + \ldots = \prod_{\beta} \left(1 - \frac{z}{\beta}\right)$$

where (α) and (β) are the respective zeros counted with multiplicity, then we define the eñe product ([22]) by

$$P \star Q(z) = \prod_{\alpha,\beta} \left(1 - \frac{z}{\alpha\beta}\right)$$

Therefore, the divisor of $P \star Q$ is the multiplicative convolution of the divisors of P and Q. If we write,

$$P \star Q(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

then for $n \geq 1$,

$$c_n = C_n(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) = -n a_n b_n + \ldots$$

where $C_n \in \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ is a universal polynomial with integer coefficients and the dots in the left side of the formula represents a polynomial on $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}$ (see [22]). This allows to define the eñe product $f \star g$ of two formal power series

$$f(z) = 1 + a_1 z + a_2 z^2 + \ldots$$

$$g(z) = 1 + b_1 z + b_2 z^2 + \ldots$$

with coefficients $a_n, b_n \in A$, in a general commutative ring A by

$$f \star g(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

with $c_n = C_n(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$ (so without any reference to the zeros that we don't have in this general setting).

We refer to [22] for the rich algebraic and analytic properties of the eñe product. From the previous Eulerian heuristic of the exponential of a polynomial with an infinite order zero at ∞, it is natural to expect that the eñe product with an exponential must make sense and be an exponential. We do have a much more precise result: The exponential linearizes the eñe product. More precisely, we have a linear the exponential form of the eñe product (Theorem 4.1 from [22]) in the following sense: If we write the power series in exponential form

$$f(z) = e^{F(z)}$$

$$g(z) = e^{G(z)}$$

with formal power series (that have a finite non-zero radius of convergence when f and g are polynomials),

$$F(z) = F_1 z + F_2 z^2 + \ldots$$

$$G(z) = G_1 z + G_2 z^2 + \ldots$$

then we have

$$f \star g(z) = e^{H(z)}$$

with

$$H(z) = -F_1G_1z - 2F_2G_2z^2 - 3F_3G_3z^3 + \ldots = -\sum_{k=0}^{+\infty} kF_kG_k\, z^k$$

We denote $\star_e$ the linearized exponential form of the eñe product

$$F \star_e G(z) = -\sum_{k=0}^{+\infty} kF_kG_k\, z^k$$

Note that we could have defined the eñe product for polynomials in this formal way, but we would miss the original interpretation with the convolution of zeros (in particular because the disk of convergence of the exponential form never contains zeros!).

With similar heuristic ideas as before we can derive à la Euler the exponential form of the eñe product (for a rigorous proof see Theorem 4.1 from [22]). Write as before

$$f(z) = e^{F(z)} = \left(1 + \frac{F(z)}{\infty}\right)^{+\infty}$$
$$g(z) = e^{G(z)} = \left(1 + \frac{G(z)}{\infty}\right)^{+\infty}$$

Then, note the double ∞ on the products and $\infty^{-3} << \infty^{-2}$, and compute

$$\begin{aligned}
f \star g(z) &= \left(1 + \frac{F(z)}{\infty}\right)^{+\infty} \star \left(1 + \frac{G(z)}{\infty}\right)^{+\infty} \\
&= \prod_{+\infty,+\infty} \left(1 + \sum_{k\geq 1} \frac{F_k}{\infty} z^k\right) \star \left(1 + \sum_{k\geq 1} \frac{G_k}{\infty} z^k\right) \\
&= \prod_{+\infty,+\infty} \left(1 + \sum_{k} \left(-k \frac{F_k}{\infty}.\frac{G_k}{\infty} + \mathcal{O}\left(\frac{1}{\infty^3}\right)\right) z^k\right) \\
&= \prod_{+\infty,+\infty} \left(1 + \frac{1}{\infty.\infty} \sum_{k\geq 1} -kF_kG_k\, z^k\right) \\
&= \left(1 + \frac{(F \star_e G)(z)}{\infty.\infty}\right)^{(+\infty).(+\infty)} \\
&= e^{(F \star_e G)(z)}
\end{aligned}$$

Motivated by these transalgebraic heuristic considerations, the purpose of this article is to extend the eñe product to the class of transalgebraic functions, a class of functions with exponential singularities that we define precisely in next section. This is a good demonstration of the dual analytic-algebraic character of the eñe product.

2. The transalgebraic class on a compact Riemann surface.

We first define exponential singularities. The exponential singularities of finite order play the role of zeros of infinite order following Euler's heuristics. We are mostly interested in this section in the case of the Riemann sphere, or 1-dimensional projective space over $\mathbb{C}$, $X = \overline{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ (genus $g = 0$), but there is little extra effort to define the transalgebraic class of functions $\mathcal{T}(X)$ for a general compact Riemann surface X. We denote $\mathcal{M}(X)$ the space of meromorphic functions on X, and $\mathcal{M}(X)^*$ the non-zero meromorphic functions.

Definition 2.1. *A point $z_0 \in \mathbb{C}$ is an exponential singularity of f if for some neighborhood U of z_0, f is a holomorphic function $f : U - \{z_0\} \to \mathbb{C}$, f has no zeros nor poles on U and f does not extend meromorphically to U.*

The exponential singularity $z_0 \in \mathbb{C}$ of f is of finite order $1 \leq d = d(f, z_0) < +\infty$ if d is the minimal integer such that

$$\limsup_{z \to z_0} |z - z_0|^d \log |f(z)| < +\infty$$

If no such finite order d exists, the exponential singularity is of infinite order and $d = d(f, z_0) = +\infty$.

Let X be a Riemann surface. A point $z_0 \in X$ is an exponential singularity for f if it is an exponential singularity in a local chart. The order $d \geq 1$ has the same definition and is independent of the local chart.

Observe that we cannot have $d = 0$ because f would be bounded in a pointed neighborhood of z_0 and by Riemann's removability Theorem f will have an holomorphic extension at z_0. Note also that the definition means that f has no monodromy around z_0 (i.e. f is holomorphic in a pointed neighborhood of z_0), and that z_0 is not a regular point nor a pole for f. Also z_0 is an exponential singularity for f if and only if it is also one for f^{-1}. We have a more precise result.

Proposition 2.2. *The point $z_0 \in X$ is an exponential singularity for f if and only if there is a local chart z where $z_0 = 0$ and f can be written in a neighborhood U of 0 as*

$$f(z) = z^n e^{h(z)}$$

where $n \in \mathbb{Z}$ and $h : U - \{0\} \to \mathbb{C}$ is holomorphic. The integer $n \in \mathbb{Z}$ if the residue of the logarithmic differential $d \log f = f'/f\, dz$ at $z = 0$. The order d is finite if and only if h is meromorphic with a pole of order d at z_0. The order of the pole of the logarithmic differential of f at z_0 is $d + 1 \geq 2$.

Proof. The problem is local and we can assume $z_0 = 0$ and take U a simply connected neighborhood of 0 in a local chart with local variable z where f has no zeros nor poles on U. Then, the logarithmic derivative f'/f is holomorphic on $U - \{0\}$ and has a Laurent expansion

$$\frac{f'(z)}{f(z)} = \sum_{n \in \mathbb{Z}} a_n z^n$$

The residue a_{-1} is an integer $n \in \mathbb{Z}$ since f has no monodromy around 0. We take for h the holomorphic function on $U - \{z_0\}$ defined by

$$h(z) = c_0 + \sum_{n \in \mathbb{Z}^*} \frac{a_{n-1}}{n} z^n$$

where $c_0 \in \mathbb{C}$ is an arbitrary constant to be chosen later. Then we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z} + h'(z)$$

and, choosing the constant c_0 properly we have

$$f(z) = z^n e^{h(z)}$$

as desired. Conversely, such an expression has clearly an exponential singularity at $z_0 = 0$. From this expression, the order d is finite if and only if h has finite polar part of order d which is equivalent to have a pole of order $d+1$ for the logarithmic derivative of f. □

Proposition 2.3. *If f has an exponential singularity at z_0 then in any pointed neighborhood of z_0 the function f takes any value $c \in \mathbb{C}^*$ infinitely often.*

Proof. This is a direct application of Picard's Theorem since f does not take the values $0, \infty \in \overline{\mathbb{C}}$ in a small pointed neighborhood of z_0. □

Definition 2.4. *For a compact Riemann surface X and a finite number of punctures $S \subset X$, we define the space $\mathcal{F}(X,S)$ as the set of non-zero meromorphic functions f on $X - S$ such that S is a set of zeros, poles, or exponential essential singularities of f or of its meromorphic extension. We define also*

$$\mathcal{F}(X) = \bigcup_{S \subset X; S \text{ finite}} \mathcal{F}(X,S)$$

Examples.

• For $X = \overline{\mathbb{C}}$ and $S = \emptyset$ we have that $\mathcal{F}(\overline{\mathbb{C}}, \emptyset) = \mathbb{C}(z)^*$ is the space of non-zero rational functions.

• For $X = \overline{\mathbb{C}}$ and $S = \{\infty\}$, $\mathcal{F}(\overline{\mathbb{C}}, \{\infty\})$ is the set of functions $f(z) = R(z)e^{h(z)}$ where $R \in \mathbb{C}(z)^*$ and $h : \mathbb{C} \to \mathbb{C}$ is an entire function, in the chart $z \in \mathbb{C} = \overline{\mathbb{C}} - \{\infty\}$. The exponential singularity at infinite is of finite order $d \geq 1$ if and only if $h \in \mathbb{C}[z]$ is a polynomial of degree $d \geq 1$. For $d = 0$, h must be constant and f is a non-zero rational function.

Proposition 2.5. *The spaces $\mathcal{F}(X,S)$ and $\mathcal{F}(X)$ endowed with the multiplication of functions are multiplicative abelian groups. If $f \in \mathcal{F}(X)$ then f has a finite number of zeros, poles and exponential singularities.*

Proof. These spaces are multiplicative groups from the remark made previously that $z_0 \in X$ is an exponential singularity for f if and only if it is one for f^{-1}. The finiteness of exponential singularities follows from the finiteness of S. By compactness, an infinite sequence of zeros must have an accumulation point on X. It cannot accumulate a point of $X - S$ or f would be identically zero. It can neither accumulate a point of S since they all have small zero free pointed neighborhoods. Hence we must have a finite number of zeros. The same argument for poles (or applied to f^{-1}) gives a finite number of them. □

Definition 2.6. *Let X be a compact Riemann surface, $S \subset X$ a finite subset, and $n \geq 0$. We define $\mathcal{F}_n(X) \subset \mathcal{F}(X)$ (resp. $\mathcal{F}_n(X,S) \subset \mathcal{F}(X,S)$) as the subset of functions having at most n zeros and poles (both <u>not</u> counted with multiplicity), and exponential singularities (resp. with the exponential singularities located at S). We define $\mathcal{T}_n(X) \subset \mathcal{F}_n(X) \subset \mathcal{F}(X)$, resp. $\mathcal{T}_n(X,S) \subset \mathcal{F}_n(X,S) \subset \mathcal{F}(X,S)$, as the subset of functions with finite order exponential singularities.*

We have the filtrations

$$\mathcal{F}(X) = \bigcup_{n\geq 0} \mathcal{F}_n(X)$$
$$\mathcal{F}(X,S) = \bigcup_{n\geq 0} \mathcal{F}_n(X,S)$$

We define the class of transalgebraic functions $\mathcal{T}(X)$ *and* $\mathcal{T}(X,S)$ *by*

$$\mathcal{T}(X) = \bigcup_{n\geq 0} \mathcal{T}_n(X)$$
$$\mathcal{T}(X,S) = \bigcup_{n\geq 0} \mathcal{T}_n(X,S)$$

We define also $\mathcal{M}_n(X) \subset \mathcal{F}_n(X)$ *to be the subset of meromorphic functions. We have, for* $n \geq 0$,

$$\mathcal{M}_n(X) \subset \mathcal{T}_n(X) \subset \mathcal{F}_n(X)$$
$$\mathcal{M}_n(X) \subset \mathcal{T}_n(X,S) \subset \mathcal{F}_n(X,S)$$

and

$$\mathcal{M}(X) \subset \mathcal{T}(X) \subset \mathcal{F}(X)$$
$$\mathcal{M}(X) \subset \mathcal{T}(X,S) \subset \mathcal{F}(X,S)$$

Remarks.

• The space $\mathcal{F}_0(X) = \mathbb{C}^*$ is the set of non-zero constant functions.

• For $X = \overline{\mathbb{C}}$ and $n = 1$, $\mathcal{F}_1(\overline{\mathbb{C}})$ is the set of functions which are Moebius conjugated to some $e^{h(z)}$ where h is an entire function. To see this, observe first that the function cannot be a constant nor a rational non constant function since for such a function the number of zeros and poles would be at least 2. For $f \in \mathcal{F}_1(\overline{\mathbb{C}})$ We can send the unique singularity to ∞ by a Moebius map, and f would be of the form $e^{h(z)}$ with h holomorphic in $\mathbb{C}$. It follows that $\mathcal{T}_1(\overline{\mathbb{C}})$ are those functions Moebius conjugated to $e^{h(z)}$ where $h(z) \in \mathbb{C}[z]$ is a polynomial.

Proposition 2.7. *The spaces* $\mathcal{T}(X)$ *and* $\mathcal{T}(X,S)$ *endowed with the multiplication are groups.*

Proof. From previous remarks they are invariant by taking inverses and their are clearly multiplicative invariant. □

We define transalgebraic divisors.

Definition 2.8. *For* $f \in \mathcal{F}(X)$, $S(f)$ *denotes the set of exponential singularities of* f. *We define the transalgebraic divisor of* f *as the formal sum*

$$\mathrm{Div}(f) = \sum_{\rho\in X} n_\rho.(\rho) + \sum_{\rho\in S(f)} d_\rho.(\rho)_\infty$$

where $n_\rho \in \mathbb{Z}$ *is the positive, resp. negative, order of the zero, resp. pole, at* ρ *or* n_ρ *is the residue of the logarithmic derivative of* $d\log f$ *at* ρ, *or* $n_\rho = 0$ *if* ρ *is neither a zero nor pole nor an exponential singularity, and* $1 \leq d_\rho \leq +\infty$ *is the order of the exponential singularity at* ρ *when* ρ *is an exponential singularity, i.e.* $d_\rho + 1$ *is the order of the polar part of* $d\log f$ *at* ρ. *The integer* $d_\rho = d_\rho(f)$ *is also called the transalgebraic degree of* f *at* ρ.

The algebraic part of the divisor is

$$\mathrm{Div}_0(f) = \sum_{\rho \in X} n_\rho.(\rho)$$

and the transcendental part of the divisor is

$$\mathrm{Div}_\infty(f) = \sum_{\rho \in S(f)} d_\rho.(\rho)_\infty$$

so that

$$\mathrm{Div}(f) = \mathrm{Div}_0(f) + \mathrm{Div}_\infty(f) \ .$$

The support of the transcendental part of the divisor is $\mathrm{supp}(\mathrm{Div}_\infty(f)) = S(f)$. *The support of the algebraic part* $\mathrm{supp}(\mathrm{Div}_0(f))$ *is the classical support of* $\mathrm{Div}_0(f)$.

The support of the transalgebraic divisor of f *is the finite subset of* X

$$\mathrm{supp}(\mathrm{Div}(f)) = \mathrm{supp}(\mathrm{Div}_0(f)) \cup \mathrm{supp}(\mathrm{Div}_\infty(f)) \ .$$

With these notations, we have $f \in \mathcal{F}(X, S(f)) \cap \mathcal{F}_{|\mathrm{supp}(f)|}$.

We recall that the set of compact subsets of a compact metric space is endowed with a natural Hausdorff distance and a Hausdorff topology. Any distance on X defining its compact surface topology defines the same Haussdorf topology on compact subsets. We fix one such distance d_H and we define a topology on $\mathcal{F}(X)$ that is independent of the choice of d_H.

Definition 2.9. *We define the topology on* $\mathcal{F}(X)$ *of uniform convergence out of the support of the divisor by defining a sequence* (f_k) *convergent to* $f \in \mathcal{F}$ *if*

$$\mathrm{supp}(\mathrm{Div}(f_k)) \to \mathrm{supp}(\mathrm{Div}(f))$$

in Hausdorff topology, and $f_k \to f$ *uniformly on compact sets out of* $\mathrm{supp}(\mathrm{Div}(f))$.

We can construct a bases of neighborhoods $(U_\epsilon(f))_{\epsilon>0}$ of an element $f \in \mathcal{F}(X)$ for this topology by taking for $\epsilon > 0$, the ϵ-Hausdorff neighborhood of $\mathrm{supp}(\mathrm{Div}(f))$, $V_\epsilon(\mathrm{supp}(\mathrm{Div}(f)))$ and defining $U_\epsilon(f)$ to be the subset of $g \in \mathcal{F}(X)$ such that $\mathrm{supp}(\mathrm{Div}(g)) \in V_\epsilon(\mathrm{supp}(\mathrm{Div}(f)))$, i.e.

$$d_H(\mathrm{supp}(\mathrm{Div}(g)), \mathrm{supp}(\mathrm{Div}(f))) < \epsilon$$

and

$$||g - f||_{C^0(X - W_\epsilon(\mathrm{supp}(\mathrm{Div}(f))))} < \epsilon$$

where $W_\epsilon(\mathrm{supp}(\mathrm{Div}(f)))$ denotes the ϵ-neighborhood of $\mathrm{supp}(\mathrm{Div}(f))$ in X.

Proposition 2.10. *The groups* $\mathcal{M}(X)^*$, $\mathcal{M}(X,S)^*$, $\mathcal{T}(X)$, $\mathcal{T}(X,S)$, $\mathcal{F}(X)$ *and* $\mathcal{F}(X,S)$ *are topological groups.*

Proof. For the groups of fuctions with singularities in S, the multiplication and inverse are continuous since the set of zeros, poles and singularities are restricted to S. The larger groups are unions of those according to the filtration from Definition 2.4). □

Proposition 2.11. *The subgroup* $\mathcal{F}(X,S)$ *and the subspace* $\mathcal{F}_n(X)$ *are closed in* $\mathcal{F}(X)$.

Proof. We prove first that $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$. Consider a sequence $(f_k) \subset \mathcal{F}_n(X)$ such that $f_k \to f \in \mathcal{F}(X)$. The cardinal of finite sets is upper semi continuous for the Hausdorff topology, hence $|\mathrm{supp}(\mathrm{Div}(f))| \leq n$. Moreover, by Hurwitz Theorem the limit function f has no singularities and cannot have zeros nor poles in $X - \mathrm{supp}(\mathrm{Div}(f))$ (note that f cannot be the constant function

0 or ∞ since $0, \infty \notin \mathcal{F}$). Hence $f \in \mathcal{F}_n(X)$. Now the subspace $\mathcal{F}_n(X, S)$ is closed in $\mathcal{F}(X)$ with the same proof. □

The space $\mathcal{M}_n(X)^*$ is not closed as we can see using the Euler example where $X = \overline{\mathbb{C}}$ with

$$f_k(z) = \left(1 + \frac{z}{k}\right)^k \in \mathcal{M}_2(\overline{\mathbb{C}})^*$$

but $f_k(z) \to e^z \notin \mathbb{C}(z) = \mathcal{M}(\overline{\mathbb{C}})$.

We can define a transalgebraic degree:

Definition 2.12. *Let $f \in \mathcal{T}(X)$. The total transalgebraic degree of f is*

$$d_\infty(f) = \sum_{\rho \in S(f)} (d_\rho(f) + 1)$$

We also define $d_0(f) = |\operatorname{supp}(\operatorname{Div}_0(f)) - S(f)|$. We define the space $\mathcal{T}^{d_0, d_\infty}(X) \subset \mathcal{T}(X)$ as the subspace of those $f \in \mathcal{T}(X)$ with

$$d_\infty(f) = d_\infty$$
$$d_0(f) = d_0 \ .$$

For a finite set $S \subset X$ we define

$$\mathcal{T}^{d_0, d_\infty}(X, S) = \mathcal{T}^{d_0, d_\infty}(X) \cap \mathcal{T}(X, S)$$

Observe that for $d_\infty = 0$,

$$\mathcal{T}^{d_0, 0}(X) = \mathcal{M}_{d_0}(X)$$

and for $d_\infty \geq 1$,

$$\mathcal{T}^{d_0, d_\infty}(X) \subset \mathcal{T}_{d_0 + d_\infty - 1}(X)$$

since for any $f \in \mathcal{T}^{d_0, d_\infty}(X)$ we have $|\operatorname{supp}(\operatorname{Div}(f))| \leq d_0(f) + d_\infty(f) - 1$. Also we have $|S(f)| \leq d_\infty(f)$.

The main Theorem in this section is that the closure of $\mathcal{M}_n(X)$ is in the transalgebraic class. More precisely,

Theorem 2.13. *The closure of $\mathcal{M}_n(X)$ in $\mathcal{F}(X)$ is*

$$\overline{\mathcal{M}_n(X)} \subset \bigcup_{\substack{d_0, d_\infty \geq 0 \\ d_0 + d_\infty \leq n}} \mathcal{T}^{d_0, d_\infty}(X) \subset \mathcal{M}_n(X) \cup \mathcal{T}_{n-1}(X) \subset \mathcal{T}_n(X) \subset \mathcal{T}(X)$$

For $n \leq |S|$, the closure of $\mathcal{M}_n(X, S)$ is

$$\overline{\mathcal{M}_n(X, S)} \subset \bigcup_{\substack{d_0, d_\infty \geq 0 \\ d_0 + d_\infty \leq n}} \mathcal{T}^{d_0, d_\infty}(X, S) \subset \mathcal{M}_n(X, S) \cup \mathcal{T}_{n-1}(X, S) \subset \mathcal{T}_n(X, S) \subset \mathcal{T}(X, S)$$

The following two Lemmas are clear from the local analysis.

Lemma 2.14. *Let $f \in \mathcal{M}(X)$ be a meromorphic function, $f : X \to \overline{\mathbb{C}}$. The poles and zeros of f correspond bijectively to simple poles of the logarithmic derivative form $f'/f\, dz = d \log f$. The residue at these simple poles is the positive, resp. negative, multiplicity of the zero, resp. pole, of f.*

Lemma 2.15. *Let $f \in \mathcal{T}(X)$. The logarithmic derivative $d \log f$ is a meromorphic differential $d \log f \in \Omega^1(X)$ with integer residues. Conversely, if $f \in \mathcal{F}(X)$ is such that $d \log f \in \mathcal{M}\Omega^1(X)$ has integer residues at poles, then $f \in \mathcal{T}(X)$.*

The following Lemma shows that the only way to have a creation of an exponential singularity ρ of transalgebraic degree $d_\rho(f)$ at the limit for a converging sequence $f_k \to f$ of rational functions is to have $d_\rho(f)+1$ distinct sequences of poles and zeros of the f_k converging to ρ (there must be both, poles and zeros).

Lemma 2.16. *Let $(f_k) \subset \mathcal{M}_n(X)$ converging to $f \in \mathcal{F}(X)$ and let $\rho \in S(f) \subset X$. Then there exists at least $d_\rho(f)+1$ distinct sequences of poles and zeros of the (f_k) converging to ρ.*

Proof. Consider a local chart at ρ and the logarithmic derivative f_k'/f_k in this chart that has simple poles corresponding to zeros and poles of f_k. We have $f_k'/f_k \to f'/f$ and at the limit we have at ρ a pole of order $d_\rho(f)+1$ for f'/f. The result follows from Rouché's Theorem. Note that poles and zeros can also annihilate each other is the have the same multiplicity. □

Proof of Theorem 2.13. We consider a sequence of meromorphic functions $(f_k) \subset \mathcal{M}_n(X)$ converging to $f \in \mathcal{F}(X)$. The associated sequence of meromorphic logarithmic derivatives $d\log f_k$ have the support of their divisor Hausdorff converging to $\mathrm{supp}(\mathrm{Div}(f))$, and uniformly on compact sets outside $X - \mathrm{supp}(f)$ we have $d\log f_k \to d\log f$. If m poles of the $d\log f_k$ converge to a pole of $d\log f$ then the order of the pole is less or equal to $m-1$ and the residue is an integer as the sum of integer residues of $d\log f_k$ of the converging points. Hence using the second Lemma 2.15 and counting poles we have that $f \in \mathcal{T}^{d_0,d_\infty}(X)$. We have the same proof for the closure of $\mathcal{M}_n(X,S)$. □

Observe that from the proof we have that if the residues of the poles of the $d\log f_k$ (or the order of the zeros or poles of f_k) that collapse into a pole of $d\log f$ are bounded, then the pole of $d\log f$ must be simple. Hence, the only way to have higher order poles for $d\log f$ is when poles and zeros of f_k collapse into a point, the orders are unbounded, but their sum is asymptotically constant.

We have a converse, and any $f \in \mathcal{T}(x)$ can be approximated by a sequence $(f_k) \subset \mathcal{M}(X)$, and each exponential singularity can be realized as a limit of poles and zeros, but these more precise results will be studied elsewhere, since we are interested in this article on the simpler case $X = \overline{\mathbb{C}}$. We specialize the above results to $X = \overline{\mathbb{C}}$.

Observe that $\mathcal{M}_n = \mathcal{M}_n(\overline{\mathbb{C}})$ (we drop the dependence on X since $X = \overline{\mathbb{C}}$ from now on) is the group of non-zero rational functions $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with support of cardinal bounded by n, i.e. such that

$$|\mathrm{supp}(R)| = |R^{-1}(0) \cup R^{-1}(\infty)| \leq n \ .$$

We abuse the notation by writting $R^{-1}(0)$, resp. $R^{-1}(\infty)$ to denote the set of zeros, resp. poles, of R, including the possible one at $\infty \in \overline{\mathbb{C}}$. Note that we do not count multiplicities. So we have

$$\mathcal{M}_n = \mathbb{C}(z) \cap \mathcal{F}_n \subset \mathcal{F}_n \ .$$

Theorem 2.17. *The group of transalgebraic functions $\mathcal{T}_n$ are the functions $f \in \mathcal{F}_n$ of the form*

$$f = R_0 e^{R_1}$$

where $R_0 \neq 0$ and R_1 are meromorphic functions with $R_1 = 0$ or

$$|R_0^{-1}(0) \cup R_0^{-1}(\infty)| + \deg R_1 \leq n \ . \tag{1}$$

In particular, the group of transalgebraic functions $\mathcal{T}$ is

$$\mathcal{T} = \{f = R_0 e^{R_1}; R_0, R_1 \in \mathbb{C}(z); R_0 \neq 0\}$$

Proof. For $n = 0$ the result is clear, so we assume $n \geq 1$. Such a function $f = R_0 e^{R_1}$ is clearly in $\mathcal{T}_n$. Conversely, given a function $f \in \mathcal{T}_n$, then the degree of the divisor $\mathrm{Div}_0(f)$ is 0 and we can choose a rational function K_0 such that

$$\mathrm{Div}\, K_0 = \mathrm{Div}_0(f) \ .$$

We can also choose a rational function K_1 with polar part matching the polar part of $d\log(f/K_0)$, in particular with the same integral residues. We consider the primitive

$$\exp\left(\int K_1\right)$$

which is of the form

$$\exp\left(\int K_1\right) = L_0 \exp(R_1)$$

where L_0 is a rational function coming from the integration of the order 1 polar part of K_1, and R_1 from the higher order polar part and both are rational functions since $\int K_1$ has no monodromy at the support of $\mathrm{Div}_\infty(f)$, and

$$R_1^{-1}(\infty) = \mathrm{Div}_\infty(f) \ .$$

Now, for $R_0 = L_0 K_0$ we have that $R_0 e^{R_1}/f$ is a meromorphic function with no zeros nor poles, thus it is a constant and we can multiply R_0 by a non-zero constant so that $f = R_0 e^{R_1}$. □

It is instructive to understand how there transalgebraic functions arise as a limit of rational functions. When $f = R_0 e^{R_1}$ we have when $k \to +\infty$,

$$f_k = R_0 \left(1 + \frac{R_1}{k}\right)^k \to f$$

and $f_k \in \mathcal{M}_n(\overline{\mathbb{C}})$ because of the inequality (1).

Conversely, let $f_k \to f \in \mathcal{F}_n(\overline{\mathbb{C}})$ with $f_k \in \mathcal{M}_n(\overline{\mathbb{C}})$. The zeros and poles collapse into the divisor of f. Only when zeros and poles cohalesce with the sum of residues becoming asymptotically constant that we can have the emergence of an exponential singularity of finite order for f.

We note a particular case of the previous theorem in the next Corollary.

Corollary 2.18. *For a positive integer $n \geq 0$, consider the space of non-zero polynomials $\mathcal{P}_n \subset \mathbb{C}[z]^*$ normalized having exactly n zeros, not counted with multiplicity. We endow this space with the topology of uniform convergence on compact sets off the zeros as before. Then we have*

$$\overline{\mathcal{P}}_n = \mathcal{P}_n \cup \mathcal{TP}_n \ ,$$

where $\mathcal{TP}_n$ is the space of functions of the form $f = P_0 e^{P_1}$ where $P_0 \in \mathbb{C}[z]^*$ is non-zero and $P_1 \in \mathbb{C}(z)$, with

$$|P_0^{-1}(0)| + \deg P_1 + 1 \leq n \ .$$

Proof. From the previous Theorem we get that all limits are of the form $f = R_0 e^{R_1}$ but the limit is holomorphic on $\mathbb{C}$ hence $R_0 = P_0$ and $R_1 = P_1$ are polynomials, and $f = P_0 e^{P_1}$ and P_0 is not identically 0. Each polynomial in $P \in \mathcal{P}_n$ is a rational function with

$$|P^{-1}(0) \cup P^{-1}(\infty)| = |P^{-1}(0)| \cup \{\infty\} = n + 1 \ .$$

We apply the general theorem getting

$$|P_0^{-1}(0) \cup P_0^{-1}(\infty)| + \deg P_1 + 1 = |P_0^{-1}(0)| + \deg P_1 + 2 \leq n + 1$$

and the result follows,

$$|P_0^{-1}(0)| + \deg P_1 + 1 \leq n$$

□

A particular case.

Let (P_k) be a sequence of polynomials with exactly n zeros, all escaping to ∞. If they are normalized conveniently in order to have a limit (for example, such that $P_k(0) = 1$, $P_k'(0) = 1$), then the limit has no finite zero, nor pole, thus the limit must be of the form $\exp(P_1)$ where P_1 is a polynomial of degree at most n.

Historical comments.

Fields generated by functions with exponential singularities of finite order on a compact Riemann surfaces X of genus $g > 0$ have been studied by P. Cutillas Ripoll in a series of remarkable papers starting with [10] (see also [11] and [12] and more recent articles). Cutillas proves the existence of minimal field of functions associated to the compact Riemann surface containing the space of meromorphic functions and realizing any divisor on $X - S$, S finite and non-empty. These minimal fields are all isomorphic and independent of S, so there is an abstract Cutillas field $\mathcal{C}(X)$ associated to any compact Riemann surface X. Moreover, he generates these function fields using functions in $\mathcal{T}(X, S)$, i.e. with exponential singularities located at S. This fantastic result seems to not be well known[1]. In [10] it is pointed out that these functions have been considered by Clebsch and Gordan, and also by Weierstrass. They appear in chapter VII of the treatise by Baker [1] where in footnote remarks there is some not very precise reference to Weierstrass work (see also [2]). Continuing the work of Baker, that was somewhat forgotten, functions with exponential singularities on hyperelliptic curves have been used, under the name of Baker-Akhiezer functions, by the russian school to construct explicit solutions of KP and KdV equations, see the surveys [13] and [3].

Transalgebraic functions on the Riemann sphere were studied by Nevanlinna ([19], [20]), and Taniguchi ([23], [24]). They also appear naturally as uniformizations of log-Riemann surfaces defined by K. Biswas and the author ([4], [5], [6]) which roughly speaking are Riemann surfaces with canonical flat charts. In particular, in [5] the Caratheodory convergence of log-Riemann surfaces is defined, and a generalization to this setting of Caratheodory's Kernel Convergence Theorem is proved (Theorems 1.1 and 1.2 in [5]). As Corollary (Corollary 1.3 in [5])) a purely geometric proof of Euler's limit (which is central to this article) is obtained,

$$e^z = \lim_{n\to+\infty} \left(1 + \frac{z}{n}\right)^n$$

Transalgebraic curves are defined in [6]. They generalize classical algebraic curves allowing infinite ramification points. More precisely, they are defined as log-Riemann surfaces having a finite number of finite (algebraic) and infinite (transcendental) ramification points. The Caratheodory closure of algebraic curves with uniformly bounded number of ramification points are proved to be transalgebraic curves (Theorem 2.11 in [6]). This result is the geometric counterpart of the previous Corollary 2.18. The more general closeness Theorem 2.13 is related to uniformizations of higher genus log-Riemann surfaces and more precisely to the main Theorem in [7]. More precisely, it is proved in [7] that any log-Riemann surface of finite topology (finitely generated fundamental group), is biholomorphic to a pointed compact Riemann surface $X - S$ equipped with a transalgebraic differential form $\omega \in \mathcal{T}\Omega^1(X)$, i.e. a differential form that is locally of the form $\omega = f(z)dz$ with f with exponential singularities, holomorphic out of S. Some of the transalgebraic properties of periods of transalgebraic curves are discussed in [8].

[1]Cutillas article [10] from 1984 has no citations according to Math Reviews, which shows how misguided is modern research.

3. Eñe product structure on the transalgebraic class of $\mathbb{P}^1(\mathbb{C})$.

In this section we extend the eñe product to the transalgebraic class of the Riemann sphere $\mathcal{T}(\overline{\mathbb{C}})$. The starting observation is the remarkable Convolution formula from [22] (Theorem 5.3)

$$e^{\frac{z}{1-z}} \star f(z) = \exp\left(\sum_{\alpha} \frac{z}{\alpha - z}\right) = \prod_{\alpha} e^{\frac{z}{\alpha - z}}$$

where (α) is the sequence of zeros of f. Observe that $f(z) = e^{\frac{z}{1-z}} \in \mathcal{T}^{0,1} \subset \mathcal{T}_1$ is a non-meromorphic transalgebraic function with a single point support for its divisor

$$\mathrm{Div}(f) = (1)_{\infty}$$

The function $f(z) = e^{\frac{z}{1-z}}$ is just the exponential function pre-composed by the Moebius transform mapping ∞ to $z = 1$ and tangent to the identity at 0. The convolution formula is what to expect if we consider the exponential singularity at 1 of $e^{\frac{z}{1-z}}$ as a zero of infinite order.

We define a sequence of rational functions $(R_k)_{\geq 1}$ appearing in Euler's computations in [15] p.85 (it is indeed $-z^{-1}R_k(-z)$ that Euler considers) for the summation of integer powers (see also [9] where the rational functions Φ_k are considered, and $R_k = -\Phi_{k-1}$),

$$R_1(z) = \frac{z}{z-1} = -\sum_{n=1}^{+\infty} z^n$$

and for $k \geq 0$,

$$R_k = R_1 \star_e \ldots \star_e R_1 = R_1^{\star_e k}$$

Therefore, for $k \geq 1$,

$$R_k(z) = -\sum_{n=1}^{+\infty} n^{k-1} z^n$$

and we can also define these rational functions by $R_k(0) = 0$ and

$$R_{k+1} = z\frac{dR_k}{dz} \tag{2}$$

There follow the first seven rational function listed by Euler in [15]

$$\begin{aligned}
R_1(z) &= -\frac{z}{1-z} \\
R_2(z) &= -\frac{z}{(1-z)^2} \\
R_3(z) &= -\frac{z(1+z)}{(1-z)^3} \\
R_4(z) &= -\frac{z(1+4z+z^2)}{(1-z)^4} \\
R_5(z) &= -\frac{z(1+11z+11z^2+z^3)}{(1-z)^5} \\
R_6(z) &= -\frac{z(1+26z+66z^2+26z^3+z^4)}{(1-z)^6} \\
R_7(z) &= -\frac{z(1+57z+302z^2+302z^3+57z^4+z^5)}{(1-z)^7}
\end{aligned}$$

In general we have,

Proposition 3.1. *For $k \geq 0$, we have*

$$R_k(z) = -\frac{zP_k(z)}{(1-z)^k} \tag{3}$$

where $P_k \in \mathbb{Z}[z]$ and for $k \geq 1$,

$$P_{k+1}(z) = (1 + (k-1)z)P_k(z) + z(1-z)P_k'(z) \tag{4}$$

and $P_1 = 1$. We have for $k \geq 2$, $\deg P_k = k-2$, $P_k(1) = (k-1)!$, $P_k(0) = 1$, and the functional equations,

$$R_k(z^{-1}) = (-1)^k R_k(z)$$
$$P_k(z^{-1}) = z^{2-k} P_k(z)$$

Hence, for $k \geq 2$, R_k vanishes at 0 and ∞, and has only one pole of order exactly k at 1.

Proof. By induction we get the recurrence (3), and the polynomial recurrence (4) follows from it. Making $z = 1$ in (4) we have for $k \geq 1$, $P_{k+1}(1) = kP_k(1)$ and $P_1(1) = 1$, hence $P_k(1) = (k-1)!$. Making $z = 0$ in (4) we have for $k \geq 1$, $P_{k+1}(0) = P_k(0) = P_1(0) = 1$. The two functional equations are equivalent. If we define $Q_k(z) = (-1)^{k+1} R_k(z^{-1})$, we check

$$Q_{k+1}(z) = zQ_k'(z)$$

and $Q_k(0) = 0$, $Q_2 = R_2$, thus $Q_k = R_k$.

Now, it is clear that for $k \geq 2$, R_k vanishes at 0 and ∞, and has only one pole of order exactly k at 1. □

Following Euler's intuition described in section 1, and the results from the previous section, it is natural to define a zero of infinite order *at a finite place* $z_0 \in \mathbb{C}$ "à la Euler":

Definition 3.2. *We define symbolically for $z_0 \in \mathbb{C}^*$*

$$\left(1 - \frac{z}{z_0}\right)^{\infty} \equiv \exp\left(\frac{z}{z_0 - z}\right) = e^{R_1(z/z_0)} \ .$$

With this notation, the convolution formula can be rewritten as,

$$\left(1 - \frac{z}{z_0}\right)^{\infty} \star f(z) = \prod_{\alpha} \left(1 - \frac{z}{z_0 \alpha}\right)^{\infty}$$

which is just distributivity with respect to infinite products and transalgebraic divisors

$$\begin{aligned}\left(1 - \frac{z}{z_0}\right)^{\infty} \star f(z) &= \left(1 - \frac{z}{z_0}\right)^{\infty} \star \prod_{\alpha} \left(1 - \frac{z}{\alpha}\right) \\ &= \prod_{\alpha} \left(1 - \frac{z}{z_0}\right)^{\infty} \star \left(1 - \frac{z}{\alpha}\right) \\ &= \prod_{\alpha} \left(1 - \frac{z}{z_0 \alpha}\right)^{\infty} \ .\end{aligned}$$

Now we define higher order zeros,

Definition 3.3. *For $z_0 \in \mathbb{C}^*$, we define symbolically,*

$$\left(1 - \frac{z}{z_0}\right)^{k.\infty} \equiv e^{R_k(z/z_0^k)} \in \mathcal{T}^{0,k} \cap \mathcal{T}_1 \subset \mathcal{T}$$

Note that $f_k(z) = \left(1 - \frac{z}{z_0}\right)^{k.\infty}$ is the function of exponential type with a single point transcendental divisor of order k at $z_0^k \in \mathbb{C}^*$,

$$\mathrm{Div}(f_k) = \mathrm{Div}_\infty(f_k) = k.(z_0^k)_\infty$$

Note the new fact that taking the eñe power of a simple infinite order zero at z_0, changes the support of the new infinite zero of order k to z_0^k. From the exponential form of the eñe product we get

Proposition 3.4. *We have for $k \geq 0$ and $z_0 \in \mathbb{C}^*$,*

$$\left(e^{\frac{z}{z_0 - z}}\right)^{\star k} = e^{\frac{z}{z_0 - z}} \star \ldots \star e^{\frac{z}{z_0 - z}} = e^{R_k(z/z_0)} \ ,$$

in particular

$$\left(e^{\frac{z}{1-z}}\right)^{\star 0} = 1 - z \ .$$

and this can be written à la Euler

$$\left(1 - \frac{z}{z_0}\right)^{k.\infty} = \left(\left(1 - \frac{z}{z_0^k}\right)^{\infty}\right)^{\star k} = e^{R_k(z/z_0^k)}$$

More generally, for $n \geq 1$ we have,

$$e^{R_{k_1}(z/z_1)} \star \ldots \star e^{R_{k_n}(z/z_n)} = e^{R_{k_1 + \ldots + k_n}(z/(z_1 \ldots z_n))} \ .$$

or, à la Euler,

$$\left(1 - \frac{z}{z_1}\right)^{k_1.\infty} \star \ldots \star \left(1 - \frac{z}{z_n}\right)^{k_n.\infty} = \left(1 - \frac{z}{z_1 \ldots z_n}\right)^{(k_1 + \ldots + k_n).\infty} \ .$$

The proof is clear from the definitions. It is satisfactory to check that this is the expected result from Euler heuristics. For example, we have the following formal computation à la Euler, for $z_1, z_2 \in \mathbb{C}^*$,

$$\begin{aligned}
\left(1 - \frac{z}{z_1}\right)^{\infty} \star \left(1 - \frac{z}{z_2}\right)^{\infty} &= \prod_{\infty} \left(1 - \frac{z}{z_1}\right) \star \prod_{\infty} \left(1 - \frac{z}{z_2}\right) \\
&= \prod_{\infty,\infty} \left(1 - \frac{z}{z_1}\right) \star \left(1 - \frac{z}{z_2}\right) \\
&= \prod_{\infty,\infty} \left(1 - \frac{z}{z_1 z_2}\right) \\
&= \left(1 - \frac{z}{z_1 z_2}\right)^{\infty.\infty} \\
&= \left(1 - \frac{z}{z_1 z_2}\right)^{2.\infty}
\end{aligned}$$

Now, it easy to see that these transalgebraic functions generate the multiplicative group of transalgebraic functions on the Riemann sphere:

Theorem 3.5. *The multiplicative group $(\mathcal{T}(\overline{\mathbb{C}}), .)$ is generated by the non-zero meromorphic functions $\mathcal{M}(\overline{\mathbb{C}})^*$, $e^{P(z)}$, $e^{P(1/z)}$, with $P \in \mathbb{C}[z]$, and for $z_0 \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, $k \geq 1$,*

$$e^{\alpha R_k(z/z_0)} = \left(\left(1 - \frac{z}{z_0}\right)^{k.\infty}\right)^{\alpha} = \left(1 - \frac{z}{z_0}\right)^{\alpha.(k.\infty)}$$

Proof. Using Theorem 2.17 any function $f \in \mathcal{T}(\overline{\mathbb{C}})$ is of the form $f = R_0 e^{R_1}$ with $R_0, R_1 \in \mathbb{C}$, $R_0 \neq 0$. By Proposition 3.1 the rational functions $R_k(z/z_0)$ can be used to reconstruct any polar part in $\mathbb{C}^*$, so we can find a finite linear combination of functions $R_k(z/z_0)$ such that

$$R_1 - \alpha_0 P_0(1/z) - \alpha_\infty P_\infty(z) - \sum_{z_0} \sum_{k=1}^{k(z_0)} \alpha_{k,z_0} R_k(z/z_0)$$

has no poles in the Riemann sphere, hence it is a constant. Therefore, we have that

$$f. \prod_{k,z_0} e^{-\alpha_{k,z_0} R_k(z/z_0)} e^{-\alpha_0 P_0(1/z)} e^{-\alpha_\infty P_\infty(z)} \in \mathbb{C}(z)^*$$

is a non-zero rational function and the result follows. □

Extension of the eñe product to $\mathcal{T}$.

In view of the previous factorization given by Theorem 3.5, and the definition of the eñe product from [22], we already have the eñe product in the subgroup of $\mathcal{T}$ of elements without a pole or singularity at 0. Using the projective invariance from Theorem 11.4 from [22] for rational functions,

$$f(1/z) \star g(1/z) = f \star g(1/z)$$

we can extend the exponential form of the eñe product to Laurent developments in the exponential at 0, i.e. if

$$f(z) = e^{F(z)}$$
$$g(z) = e^{G(z)}$$

with

$$F(z) = \sum_{k \in \mathbb{Z}} F_k z^k$$
$$G(z) = \sum_{k \in \mathbb{Z}} G_k z^k$$

then we have

$$f \star g(z) = e^{H(z)}$$

with

$$H(z) = -\sum_{k \in \mathbb{Z}} k F_k G_k z^k = F \star_e G(z)$$

where we denote $\star_e$ the extension of the linearized exponential form of the eñe product

$$F \star_e G(z) = -\sum_{k \in \mathbb{Z}} k F_k G_k \, z^k$$

With this definition we extend the eñe product to the full group $\mathcal{T}/\mathbb{C}^* = \bar{\mathcal{T}}$, i.e. $\mathcal{T}$ modulo non-zero constants. Note that the eñe product with the constant 1 is the constant 1 that plays the role of additive zero in the following ring structure,

Theorem 3.6. *$(\bar{\mathcal{T}}, ., \star)$ is a commutative ring.*

The eñe product is clearly continuous on the generators of the transalgebraic class for the topology defined in Section 1. We get

Theorem 3.7. *$(\bar{\mathcal{T}}, ., \star)$ is a topological commutative ring.*

Also it is clear that we have for $n, m \geq 0$,

$$\bar{\mathcal{T}}_n \star \bar{\mathcal{T}}_m \subset \bar{\mathcal{T}}_{nm}$$

hence we have

Theorem 3.8. *$(\bar{\mathcal{T}}, ., \star)$ is a graded topological commutative ring.*

Remark on further algebraization.

We can observe that if we consider a subfield $\mathbb{Q} \subset K \subset \mathbb{C}$ (for example a number field K), we can define $\mathcal{T}(\mathbb{P}^1(K))$ as the sub-group of $\mathcal{T}$ with functions with exponential singularities at places in $\mathbb{P}^1(K) \subset \mathbb{P}^1(\mathbb{C})$ such that $d \log f \in \Omega^1_{/K}(\mathbb{P}^1(K))$. Then, since the Euler rational functions have rational coeffidints, $R_k(z) \in \mathbb{Q}(z)$, we can check that the proof of Theorem 3.5 goes through (the coefficients (α_{k,z_0}) are elements of K), and the eñe product extends to $\bar{\mathcal{T}}(\mathbb{P}^1(K)) = \mathcal{T}(\mathbb{P}^1(K))/K^*$ and

$$(\bar{\mathcal{T}}(\mathbb{P}^1(K)), ., \star)$$

is a graded topological commutative ring.

For an algebraic curve X defined over a field K we can also define $\mathcal{T}(X, K)$, the transalgebraic class over K, and so on.

One of the magic in Euler's computations is its symbiotic analytic-algebraic content. Thus, it is not surprising that the eñe structures allow such algebraizations. We leave these rich algebraic extensions for future articles.

Eñe poles.

Once we have defined zeros of infinite order at finite places, it is natural to ask for the definition of poles of infinite order at finite places. Polylogarithm functions appear then naturally. Recall that polylogarithms are defined for $k \geq 1$ by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{+\infty} n^{-k} z^n \ .$$

We have $\mathrm{Li}_k(0) = 0$, the series has radius of convergence 1, and we have a singularity at 1 which is a branching point. On the other sheets of its log-Riemann surface we have branchings at 0 and 1 for $k \geq 2$, and only at 1 for $k = 1$ since $\mathrm{Li}_1(z) = -\log(1 - z)$ (see for example [21] for these geometric information and more properties of polylogarithms).

We can complete the sequence $(R_k)_{k \geq 1}$ of Euler rational function to indexes $k < 0$ using the differential recurrence 2 and the condition $R_k(0) = 0$, and we get

$$R_0(z) = \mathrm{Li}_1(z) = -\log(1 - z)$$

and for $k \leq 0$,

$$R_k(z) = \mathrm{Li}_k(z)$$

Proposition 3.9. *We have*

$$e^{R_k(z)} \star e^{-Li_{k+1}(z)} = 1 - z \ .$$

Proof. This follows directly from the exponential form, for $k \geq 0$,

$$R_k \star_e (-\mathrm{Li}_{k+1}) = \log(1 - z) \ .$$

□

So we have

$$\left(1-\frac{z}{1}\right)^{k.\infty} \star e^{-\operatorname{Li}_{k+1}(z)} = 1-z.$$

and it is natural to define the eñe-pole of infinite order $k \geq 1$ at 1 as

$$\left(1-\frac{z}{1}\right)^{-k.\infty} = e^{-\operatorname{Li}_{k+1}(z)}$$

More generally, we define the pole of infinite order k "at" a finite place $z_0 \neq 0, \infty$ as

Definition 3.10. *For $z_0 \in \mathbb{C}^*$, we symbolically define*

$$\left(1-\frac{z}{z_0}\right)^{-k.\infty} = e^{-\operatorname{Li}_{k+1}(zz_0^k)} \ ,$$

as the function with divisor a single pole of infite order k at $z_0 \in \mathbb{C}^*$ (but the singularity is at z_0^{-k}).

Note that this time these functions don't have exponential singularities, but branching singularities with non-trivial monodromy. The function with a single k-infinite pole at z_0 has a branching point located at z_0^{-k}. These functions take the value 1 at 0 (in their principal determination) and the eñe product is well defined through the exponential form. The important observation is that these type of singularities do appear naturally. Thus, it is natural to extend the eñe product to functions with singularities with non-trivial monodromy. This will be treated in subsequent articles.

With these definition we conclude, always à la Euler,

Theorem 3.11. *For $k, l \in \mathbb{Z}$, $z_1, z_2 \in \mathbb{C}$, $z_1, z_2 \neq 0, \infty$, we have*

$$\left(1-\frac{z}{z_1}\right)^{k.\infty} \star \left(1-\frac{z}{z_2}\right)^{l.\infty} = \left(1-\frac{z}{z_1^k z_2^l}\right)^{(k+l).\infty} \ .$$

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