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► **To cite this version:**

Raphaël Cerf, Nicolas Forien. Some toy models of self-organized criticality in percolation. *ALEA : Latin American Journal of Probability and Mathematical Statistics*, 2022, 10.30757/ALEA.v19-14 . hal-02417506v2

HAL Id: hal-02417506

<https://hal.science/hal-02417506v2>

Submitted on 18 Mar 2021 (v2), last revised 28 Jan 2022 (v3)

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Some toy models of self-organized criticality in percolation

Raphaël Cerf^{*†} Nicolas Forien^{*†‡}

March 18, 2021

Abstract

We consider the Bernoulli percolation model in a finite box and we introduce an automatic control of the percolation parameter, which is a function of the percolation configuration. For a suitable choice of this automatic control, the model is self-critical, i.e., the percolation parameter converges to the critical point p_c when the size of the box tends to infinity. We study here three simple examples of such models, involving the size of the largest cluster, the number of vertices connected to the boundary of the box, or the distribution of the cluster sizes.

Keywords: percolation, criticality, self-organized criticality

Mathematics Subject Classification (2020): 82B43, 60K35, 82B20, 82B27

1 Introduction

1.1 Self-organized criticality

Many physical models present a phenomenon called phase transition: there is a critical point or a critical curve in the parameter space separating two distinct regions characterized by very different macroscopic behaviours. In such systems, the behaviour of the model at criticality is of particular interest and presents some general features (e.g., fractal geometry or power-law temporal and spatial correlations) which are universal across a wide range of systems and do not depend much on the microscopic details of the system. In their seminal paper [BTW87], the physicists Bak, Tang and Wiesenfeld pointed out that these “critical features” are very common in nature, which is rather surprising because it seems that the parameters need to be finely tuned for a system to be critical. To explain this paradox, they showed that some systems tend to be naturally attracted by critical points, without any fine tuning of the parameters. They call this phenomenon self-organized criticality.

To illustrate this idea, they defined a simple model inspired by the dynamics of a sandpile. The balance between avalanches and accumulation of sand leads to a state where the system looks critical, with a self-similar distribution of the sizes of the avalanches and the slope self-adjusting to the critical slope, which is the slope at which large-scale avalanches appear. But despite a very simple dynamics, their model turns out to be very difficult to analyze mathematically [Dha06, Jár18, Hut20].

In [CG16], Cerf and Gorny constructed a self-critical model as a variant of the generalized Ising-Curie-Weiss model, by replacing the temperature with a function depending on the spin configuration. In this paper, we implement the same principle of a feedback from the configuration to the parameter, but within the framework of Bernoulli percolation. This technique to obtain self-organized criticality by “artificially” replacing the control parameter with a feedback function depending on the state of the model, which is explained in section 15.4.2 of [Sor06], was implemented by physicists to imagine self-critical variants of percolation in [Sor92, SWdA⁺00]. However, the understanding of such models often relies on computer simulations and few models are amenable to rigorous mathematical analysis.

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There have been several attempts to build models of self-organized criticality in the percolation setup. One strategy consists in modifying a process of dynamical percolation in order to freeze or burn down the large or infinite clusters. This approach was successfully implemented in the context of the Erdős-Rényi random graphs [RT09] and in Bernoulli percolation on the lattice by [KMS15, vdBN17]. The resulting models are dynamical in nature, in accordance with the general paradigm of self-organized criticality. Another very interesting model of self-organized criticality is invasion percolation [CCN87, HPS99, GPS18b, J03]. One starts directly on the infinite lattice and one defines an exploration process with the help of a parameter which is adjusted dynamically and ultimately converges towards the critical point. We present here a different approach, which is conceptually simpler, in that we define directly a probability measure on the percolation configurations in a finite box, which converges to the critical percolation as the size of the box tends to infinity. However, this somehow simpler model is already challenging and it presents a phenomenon of self-organized criticality.

1.2 Construction of the model and convergence result

Let $\Lambda(n)$ be the box of side n centered at 0 in \mathbb{Z}^d with $d \geq 2$, and let \mathbb{E}_n be the set of edges between nearest neighbours of $\Lambda(n)$. Consider a sequence of increasing functions $F_n : \{0, 1\}^{\mathbb{E}_n} \rightarrow \mathbb{N}$ and a parameter $a > 0$ and set, for $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ a percolation configuration on the edges of the box,

$$p_n(\omega) = \varphi_n(F_n(\omega)) \quad \text{where} \quad \varphi_n(x) = \exp\left(-\frac{x}{n^a}\right).$$

This function p_n will be our automatic control of the percolation parameter, and in this paper we will study three examples of such a control, involving different functions F_n . The model we consider is given by the following probability distribution on the configurations, which is obtained by replacing the parameter p of Bernoulli percolation with our feedback function p_n , with the appropriate normalization. Let

$$\mu_n : \omega \in \{0, 1\}^{\mathbb{E}_n} \mapsto \frac{1}{Z_n} \mathbb{P}_{p_n(\omega)}(\omega)$$

where

$$Z_n = \sum_{\omega \in \{0, 1\}^{\mathbb{E}_n}} \mathbb{P}_{p_n(\omega)}(\omega)$$

will be called the partition function, and \mathbb{P}_p is the Bernoulli percolation measure with parameter p , namely

$$\forall \omega \in \{0, 1\}^{\mathbb{E}_n} \quad \mathbb{P}_p(\omega) = \prod_{e \in \mathbb{E}_n} p^{\omega(e)} (1-p)^{1-\omega(e)}.$$

For $x \in \Lambda(n)$ and $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$, we write

$$C(x, \omega) = \left\{ y \in \Lambda(n) : x \overset{\omega}{\longleftrightarrow} y \right\}$$

for the open cluster of x in the configuration ω . We show the following convergence result, valid in any dimension $d \geq 2$. The critical point of the Bernoulli percolation model is denoted by p_c .

Theorem 1. *If F_n is one of the following sequences of functions:*

- $F_n : \omega \mapsto |C_{max}(\omega)| = \max_{x \in \Lambda(n)} |C(x, \omega)|$ with $0 < a < d$;
- $F_n : \omega \mapsto |\mathcal{M}_n(\omega)| = \left| \left\{ x \in \Lambda(n) : x \overset{\omega}{\longleftrightarrow} \partial\Lambda(n) \right\} \right|$ with $d-1 < a < d$;
- $F_n : \omega \mapsto |B_n^b(\omega)| = \left| \left\{ x \in \Lambda(n) : |C(x, \omega)| \geq n^b \right\} \right|$ with $0 < b < a < d$,

then the law of p_n under μ_n converges to δ_{p_c} when $n \rightarrow \infty$, and we have the following control:

$$\forall \varepsilon > 0 \quad -\infty < \liminf_{n \rightarrow \infty} \frac{1}{(\ln n)n^v} \ln \mu_n(|p_n - p_c| > \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^v} \ln \mu_n(|p_n - p_c| > \varepsilon) < 0,$$

where the $(\ln n)$ factor can be dropped in the case of the first model, and where the exponent v is given by

$$\begin{cases} v = a \wedge (d - \frac{a}{d}) & \text{if } F_n = |C_{max}| ; \\ v = d - 1 & \text{if } F_n = |\mathcal{M}_n| ; \\ v = a \wedge (d - \frac{b}{d}) & \text{if } F_n = |B_n^b| . \end{cases}$$

Concerning the third model, an estimate on the convergence speed can be obtained, provided that we assume the existence of the critical exponents β and γ . The existence of these exponents was proven for dimension 2 in the case of the triangular lattice in [SW01], with $\beta = 5/36$ and $\gamma = 43/18$.

Theorem 2. *Take $F_n = |B_n^b|$. Assume that there exist real constants $\beta, \gamma > 0$ such that*

$$\limsup_{\substack{p \rightarrow p_c \\ p > p_c}} \frac{\ln \theta(p)}{\ln(p - p_c)} \leq \beta \quad \text{and} \quad \liminf_{\substack{p \rightarrow p_c \\ p < p_c}} \frac{\ln \chi(p)}{\ln(p_c - p)} \geq -\gamma .$$

Then, for any real parameters a, b and c , we have

$$0 < b < a < d \quad \text{and} \quad c < \min\left(\frac{b}{2\gamma}, \frac{a-b}{2\gamma}, \frac{d-a}{\beta}, \frac{d-bd-b}{\beta}\right) \implies n^c(p_n - p_c) \xrightarrow{\mathcal{L}} 0 .$$

There, we do not believe the condition on c to be optimal, since the term $(d - bd - b)/\beta$ comes from a quite rough estimate (see lemma 19), and it does not allow to deal with $b \geq d/(d+1)$. It may be possible to improve our technique to get rid of this limitation, and to obtain a similar estimate on the convergence speed for the two first models.

1.3 Motivation to consider this model

Let us explain the heuristics which lead to the choice of the sequences F_n . The function p_n introduces a negative feedback which assigns low values $p_n(\omega) \ll p_c$ to percolation configurations which are ‘‘typical’’ of the supercritical phase $p > p_c$, and high values $p_n(\omega) \gg p_c$ to configurations which are ‘‘typical’’ of the subcritical phase $p < p_c$. For example, if $F_n = |C_{max}|$, a configuration ω with a largest cluster containing a number of vertices of order n^d will be assigned a very low value $p_n(\omega) \ll p_c$. Yet, for this value of the parameter p in Bernoulli percolation, it is very unlikely to have such a large cluster, which will give ω a very low weight in the measure μ_n . Indeed, we will show that under μ_n , configurations which are either ‘‘typically subcritical’’ or ‘‘typically supercritical’’ have a very low probability. Therefore, the mass of μ_n must concentrate on configurations ω with $p_n(\omega)$ sufficiently close to p_c , hence the self-critical behaviour of our model.

Note that our parameter a does not need to be finely tuned for our result to hold, showing the robustness of the construction. Indeed, one could expect a different behaviour depending on whether a is smaller or larger than the so-called fractal dimension d_f of the incipient infinite cluster (see for example [BCKS01]), but p_n tends to p_c regardless of a . In fact, one can conjecture that, if $a > d_f$, then our p_n will tend to p_c ‘‘from above’’, and the configurations in our model might look slightly supercritical, while they might look slightly subcritical when $a < d_f$. This can be understood because the definition of our model more or less amounts to forcing the size of the largest cluster (or $|\mathcal{M}_n|$, or $|B_n^b|$) to be of order n^a .

Our list of three models is of course not comprehensive, since many variants could be defined using the same approach. For example, the case of the largest cluster can be extended to the largest cluster in the torus, which means we can set periodic boundary conditions on the box $\Lambda(n)$. In the model defined with B_n^b , one could consider the distribution of the cluster diameters instead of the cluster sizes, by setting

$$\tilde{B}_n^b(\omega) = \left| \left\{ x \in \Lambda(n) : \text{diam } C(x, \omega) \geq n^b \right\} \right| , \tag{1}$$

which gives exactly the same convergence result, under the same conditions for a and b , and with a similar estimate on the convergence speed.

1.4 Outline of the article

This article is organized as follows. First, we give some standard definitions and notations in section 2. We then prove a general geometric inequality for finite subgraphs of \mathbb{Z}^d in section 3. The last three sections 4, 5 and 6 are devoted to the proofs of the three cases of theorem 1, theorem 2 being proved at the end of section 6. In each of these three sections, the convergence result of theorem 1 is obtained in two main steps, which we present here.

1.4.1 Exponential decay estimates in the subcritical and supercritical phases

Let ε be such that $0 < \varepsilon < \min(p_c, 1 - p_c)$. We start with an upper bound on the left tail of the law of p_n . To this end, we define

$$t_n^+ = \lceil n^a (-\ln(p_c + \varepsilon)) \rceil. \quad (2)$$

Grouping the configurations according to the value of F_n , we can write

$$\begin{aligned} \mu_n(p_n > p_c + \varepsilon) &= \frac{1}{Z_n} \sum_{\omega \in \{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{p_n(\omega) > p_c + \varepsilon\}} \mathbb{P}_{p_n(\omega)}(\omega) = \frac{1}{Z_n} \sum_{\omega \in \{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{F_n(\omega) < t_n^+\}} \mathbb{P}_{p_n(\omega)}(\omega) \\ &= \frac{1}{Z_n} \sum_{t=0}^{t_n^+-1} \sum_{\omega \in \{0,1\}^{\mathbb{E}_n}} \mathbb{1}_{\{F_n(\omega)=t\}} \mathbb{P}_{p_n(\omega)}(\omega) = \frac{1}{Z_n} \sum_{t=0}^{t_n^+-1} \mathbb{P}_{\varphi_n(t)}(F_n = t) \\ &\leq \frac{1}{Z_n} \sum_{t=0}^{t_n^+-1} \mathbb{P}_{\varphi_n(t)}(F_n < t_n^+). \end{aligned} \quad (3)$$

Yet, the variables F_n are increasing, whence

$$\mu_n(p_n > p_c + \varepsilon) \leq \frac{n^d}{Z_n} \mathbb{P}_{p_c + \varepsilon}(F_n < (-\ln(p_c + \varepsilon))n^a). \quad (4)$$

Similarly, we can show that

$$\mu_n(p_n < p_c - \varepsilon) \leq \frac{n^d}{Z_n} \mathbb{P}_{p_c - \varepsilon}(F_n > (-\ln(p_c - \varepsilon))n^a). \quad (5)$$

Therefore, the first step is to show exponential decay estimates for

$$\mathbb{P}_{p_c + \varepsilon}(F_n < (-\ln(p_c + \varepsilon))n^a) \quad \text{and} \quad \mathbb{P}_{p_c - \varepsilon}(F_n > (-\ln(p_c - \varepsilon))n^a) \quad (6)$$

with $\varepsilon > 0$ fixed. This is done in subsections 4.1 and 4.2 in the case of $F_n = |C_{max}|$, 5.1 and 5.2 for $F_n = |\mathcal{M}_n|$ and 6.1 and 6.2 with $F_n = B_n^b$. The estimates we obtain there are quite standard and follow from classical results in the literature about the behaviour of C_{max} and \mathcal{M}_n in the subcritical and supercritical phases.

1.4.2 The key ingredient : a lower bound on the partition function

The second step, which is the crucial and more interesting step, is to obtain a lower bound on the partition function Z_n . Indeed, to show that (4) and (5) tend to 0 as n tends to infinity, one must not only show that the two terms in (6) are small enough, but also that Z_n is not too small. To obtain this lower bound, we rewrite the partition function as

$$Z_n = \sum_{\omega \in \{0,1\}^{\mathbb{E}_n}} \mathbb{P}_{p_n(\omega)}(\omega) = \sum_{t=0}^{n^d} \sum_{\substack{\omega \in \{0,1\}^{\mathbb{E}_n} \\ F_n(\omega)=t}} \mathbb{P}_{\varphi_n(t)}(\omega) = \sum_{t=0}^{n^d} \mathbb{P}_{\varphi_n(t)}(F_n = t).$$

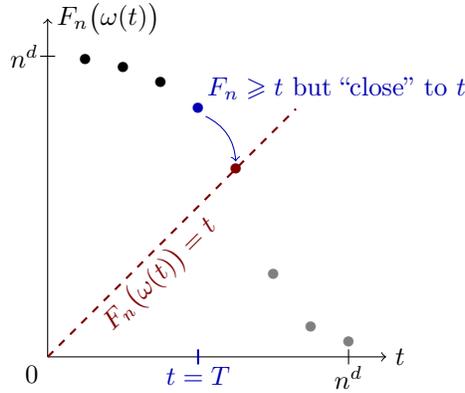


Figure 1: The partition function Z_n may be expressed as the probability that the random function $t \mapsto F_n(\omega(t))$ admits a fixed point. This allows for the construction of a scenario where we can force such a fixed point to appear, with a reasonable probabilistic cost.

To make this expression more concrete, we construct a decreasing coupling $\omega(0) \geq \omega(1) \geq \dots \geq \omega(n^d)$ of percolation configurations, such that for every $t \in \{0, \dots, n^d\}$, we have $\omega(t) \sim \mathbb{P}_{\varphi_n(t)}$. Then Z_n rewrites as

$$Z_n = \sum_{t=0}^{n^d} \mathbb{P}(F_n(\omega(t)) = t) = \mathbb{P}(\exists t \in \{0, \dots, n^d\} \quad F_n(\omega(t)) = t). \quad (7)$$

Hence, the partition function Z_n is equal to the probability that the random non-increasing function $t \mapsto F_n(\omega(t))$ admits a fixed point. This leads us to build the coupling step by step, and to consider a (random) stopping time T situated just before this function goes under the first bisector (see figure 1). We then show a lower bound on the probability that the next steps of the coupling lead to a fixed point.

Because this instant T when we try to force a fixed point corresponds to a percolation parameter close to p_c , the classical estimates available in subcritical or supercritical percolation are of no use. Indeed, we need to study the behaviour of F_n as p decreases towards p_c , and to show that F_n does not vary too abruptly close to the critical point. Our problem is therefore closely related to the study of finite-size scaling, i.e., the behaviour of the model when one takes $n \rightarrow \infty$ and $p \rightarrow p_c$ simultaneously (see [BCKS01, GPS18a]).

Yet, we are able to bypass the use of (unproven) scaling laws thanks to the geometric argument of section 3, which is more general and does not rely on the near-critical behaviour of F_n . Roughly speaking, this geometric result indicates that to cut a piece of a precise size out of a subgraph of \mathbb{Z}^d of size N , one only needs to close $O(N^{(d-1)/d})$ edges. This geometric argument allows us to implement a surgery procedure on the configuration $\omega(T)$ which leads to a fixed point by forcing a reasonable number of edges to be closed in the following steps of the coupling. The surgery procedure is different for each of the three considered models, but the core ingredient is always this graph separation result.

1.4.3 Adapting our technique to the random-cluster model?

An important goal is to build a similar model of self-organized criticality associated with the Ising model. A natural strategy consists in adapting the results presented here to the FK percolation model. However, a major complication arises with the FK model. Indeed, in a dynamical coupling of the FK processes, there is already a phenomenon of self-organized criticality in the way the edges become open when one approaches the critical point from below [DCGP14]. As a consequence, our proof for the lower bound on the partition function is not valid in this context.

2 Definitions and notations

2.1 The box

We fix an integer $d \geq 2$ for the whole article. Let \mathbb{E}^d be the set of edges between nearest neighbours of \mathbb{Z}^d :

$$\mathbb{E}^d = \left\{ \{x, y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1 \right\}.$$

Let $n \geq 1$. Let us consider the box centered at 0 and containing n^d vertices,

$$\Lambda(n) = \left[-\frac{n}{2}, \frac{n}{2} \right]^d \cap \mathbb{Z}^d = \left\{ -\left\lfloor \frac{n}{2} \right\rfloor, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\}^d.$$

For $V \subset \mathbb{Z}^d$ a set of vertices, we write

$$\mathbb{E}[V] = \left\{ \{x, y\} \subset V : \|x - y\|_1 = 1 \right\}$$

for the set of edges in \mathbb{E}^d connecting two vertices of V , and we write in particular $\mathbb{E}_n = \mathbb{E}[\Lambda(n)]$. The boundary of the box $\Lambda(n)$ will be denoted

$$\partial\Lambda(n) = \left\{ x \in \Lambda(n) : \exists y \in \mathbb{Z}^d \setminus \Lambda(n) \quad \|x - y\|_1 = 1 \right\}.$$

2.2 Bernoulli percolation

For $0 \leq p \leq 1$, on the space $\{0, 1\}^{\mathbb{E}^d}$ equipped with the σ -field generated by events depending on finitely many edges, let \mathbb{P}_p be the product measure such that the state of each edge follows a Bernoulli law of parameter p . An element $\omega : \mathbb{E}^d \rightarrow \{0, 1\}$ is called a percolation configuration. Edges $e \in \mathbb{E}^d$ such that $\omega(e) = 1$ are said open in ω , and the other edges are said closed in ω . Under the law \mathbb{P}_p , each edge is open with probability p and the states of different edges are independent of each other. For any configuration $\omega : \mathbb{E}^d \rightarrow \{0, 1\}$ and any edge $e \in \mathbb{E}^d$, we will write

$$\omega_e : f \in \mathbb{E}^d \mapsto \begin{cases} 0 & \text{if } f = e, \\ \omega(f) & \text{otherwise} \end{cases}$$

for the configuration obtained from ω by closing the edge e . Similarly, for any configuration $\omega : \mathbb{E}^d \rightarrow \{0, 1\}$ and any set of edges $H \subset \mathbb{E}^d$, we will write

$$\omega^H : f \in \mathbb{E}^d \mapsto \begin{cases} 1 & \text{if } f \in H, \\ \omega(f) & \text{otherwise} \end{cases} \quad \text{and} \quad \omega_H : f \in \mathbb{E}^d \mapsto \begin{cases} 0 & \text{if } f \in H, \\ \omega(f) & \text{otherwise} \end{cases}$$

for the configurations obtained from ω by opening or closing all the edges of H . These notations naturally extend to configurations $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ on the edges of the box $\Lambda(n)$. We will also write \mathbb{P}_p for the induced probability distribution on these configurations.

2.3 Clusters

Let $\omega : \mathbb{E}^d \rightarrow \{0, 1\}$ be a percolation configuration on \mathbb{Z}^d . For $x, y \in \mathbb{Z}^d$, we write $x \xleftrightarrow{\omega} y$ if there exists a path of open edges in the configuration ω joining x and y . For $x \in \mathbb{Z}^d$, we will write

$$C(x) = C(x, \omega) = \left\{ y \in \mathbb{Z}^d : x \xleftrightarrow{\omega} y \right\}$$

for the connected component of x , which is called the cluster of x in ω . If $x \in \mathbb{Z}^d$ and $Y \subset \mathbb{Z}^d$, we write

$$x \xleftrightarrow{\omega} Y \iff \exists y \in Y \quad x \xleftrightarrow{\omega} y.$$

All these notations naturally extend to percolation configurations on the box $\Lambda(n)$. Thus, for $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ and $x \in \Lambda(n)$, we will write $C(x, \omega)$ (or $C(x)$) for the set of vertices in $\Lambda(n)$ which are connected to x in $\Lambda(n)$ by an open path in the configuration ω . When it is not clear whether we consider paths which stay in the box or not, for example if ω is defined on \mathbb{E}^d , we will specify $C_{\Lambda(n)}(x)$ to denote the set of vertices which are connected to x by an open path with all its intermediate vertices belonging to $\Lambda(n)$, i.e., the cluster of x in the configuration restricted to \mathbb{E}_n .

For a percolation configuration $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ in the box $\Lambda(n)$, we will denote by $C_{max}(\omega)$, or sometimes $C_{max}(\Lambda(n))$, the largest cluster in ω , speaking in terms of number of vertices. In case of equality between several maximal clusters, we choose one of them with an arbitrary order on subsets of $\Lambda(n)$. For $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$, we also define

$$\mathcal{M}_n(\omega) = \left\{ x \in \Lambda(n) : x \overset{\omega}{\longleftrightarrow} \partial\Lambda(n) \right\} \quad \text{and} \quad B_n^b(\omega) = \left\{ x \in \Lambda(n) : |C_{\Lambda(n)}(x, \omega)| \geq n^b \right\},$$

where $b > 0$ is a fixed parameter. Given $p \in [0, 1]$, let

$$\theta(p) = \mathbb{P}_p(|C(0)| = \infty)$$

be the probability that the origin lies in an infinite open cluster in a percolation configuration drawn according to \mathbb{P}_p . We will write p_c for the critical point of Bernoulli percolation in dimension d , defined by

$$p_c = \inf \left\{ p \in [0, 1] : \theta(p) > 0 \right\},$$

and which is such that $0 < p_c < 1$ (see theorem 1.10 of [Gri99]).

3 Geometrical interlude

3.1 Main result

The purpose of this section is to show the following geometric inequality, which one could sum up as “separating a cluster of a given size in a graph (V, E) requires $O(|V|^{(d-1)/d})$ edges”.

Lemma 1. *There exists a constant $K = K(d)$ such that, for any finite connected subgraph $G = (V, E)$ of $(\mathbb{Z}^d, \mathbb{E}^d)$, for any vertex $x \in V$ and for any integer m such that $1 \leq m \leq |V|$, there exists a subset $E_0 \subset E$ of edges of G with cardinality*

$$|E_0| \leq K |V|^{\frac{d-1}{d}}$$

such that the connected component of x in the graph $(V, E \setminus E_0)$ contains exactly m vertices.

We decompose the proof of this lemma in two steps. In section 3.2, we prove “the butcher’s lemma”, which allows to cut a graph into small components, which may be too small, in particular the component of x might have a cardinality strictly smaller than the goal m . In section 3.3, we prove “the surgeon’s lemma”, which involves an adequate algorithm to reopen some of the edges closed by the butcher’s lemma in order to reach the goal size m for the cluster of x .

3.2 “The butcher’s lemma”

We start with an upper bound on the number of edges that one needs to remove from a connected graph to divide it into pieces which are smaller than half of the initial graph.

Lemma 2 (The butcher’s lemma). *For every finite subgraph $G = (V, E)$ of $(\mathbb{Z}^d, \mathbb{E}^d)$, there exists a subset $E_0 \subset E$ of edges of G with cardinality*

$$|E_0| \leq 4^{d+1} d^2 |V|^{\frac{d-1}{d}}$$

such that any connected component of the graph $(V, E \setminus E_0)$ contains at most $\lceil |V|/2 \rceil$ vertices.

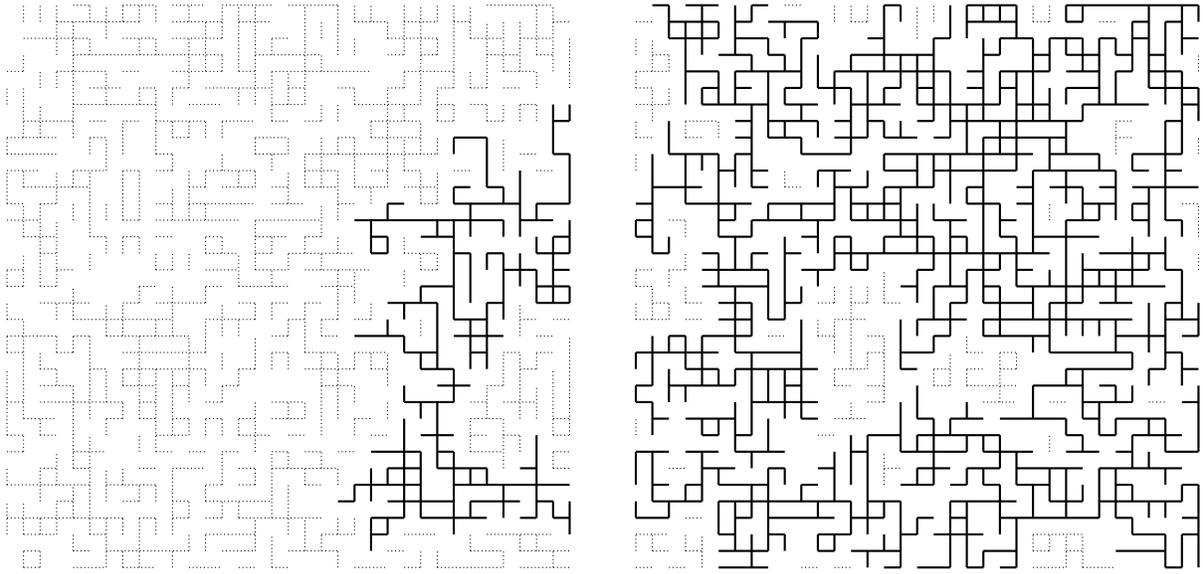


Figure 2: Percolation in the box $\Lambda(35)$ with, left, $p = 0.48$ and right, $p = 0.52$. Open edges belonging to the largest cluster are drawn in solid lines, while other open edges are in dotted lines.

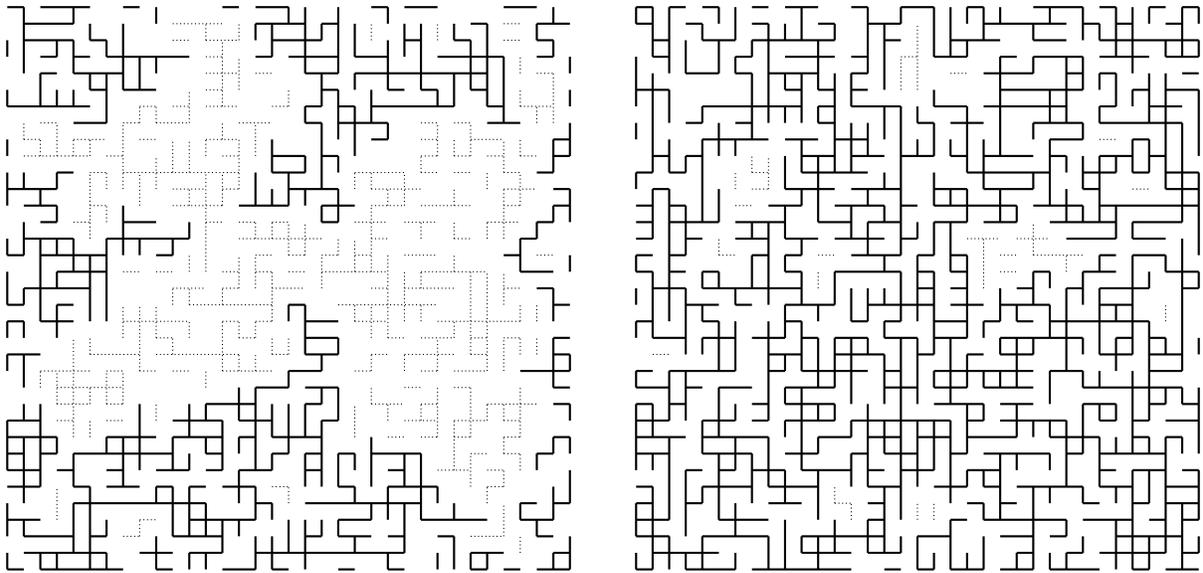


Figure 3: Percolation in the box $\Lambda(35)$ with, left, $p = 0.48$ and right, $p = 0.52$. Open edges connected to the boundary of the box by an open path are drawn in solid lines, while other open edges are in dotted lines.

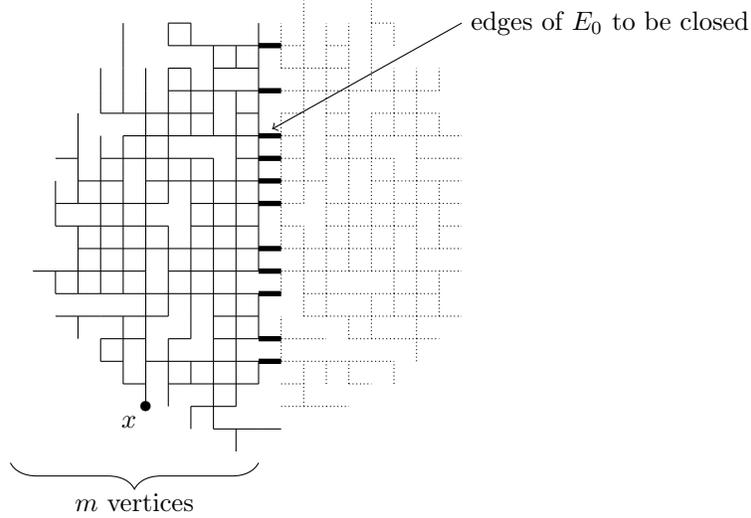


Figure 4: Closing the edges of E_0 (drawn in thick lines) cuts the graph in several connected components, such that x lies in a component (drawn in normal lines) containing the required number of vertices. Lemma 1 states that, in dimension 2, the subset E_0 can be chosen containing $O(\sqrt{|V|})$ edges.

This separation lemma, which can be summarized by “cutting a graph in two requires $O(|V|^{(d-1)/d})$ edges”, was proved in [BST12], corollary 3.3. For completeness, we present here a self-contained proof of this geometric result for the case of \mathbb{Z}^d . The more general technique of [BST12] would make it possible to extend our result to more general graphs, but we choose here to restrict our presentation to the d -dimensional square grid. For $x \in \mathbb{Z}^d$, we will write its coordinates $x = (x_1, \dots, x_d)$. For any finite non-empty subset $V \subset \mathbb{Z}^d$ and any $i \in \{1, \dots, d\}$, we define

$$\text{diam}_i V = \max_{x \in V} x_i - \min_{x \in V} x_i \quad \text{and} \quad \text{diam} V = \max_{1 \leq i \leq d} \text{diam}_i V.$$

If $i \in \{1, \dots, d\}$ and $m \in \mathbb{Z}$, then

$$T_{i,m} = \left\{ e = \{x, y\} \in \mathbb{E}^d : x_i = m \text{ and } y_i = m + 1 \right\}$$

will denote the slice of edges cutting \mathbb{Z}^d in two parts in the direction i between abscissa m and $m + 1$. We first prove an auxiliary lemma.

Lemma 3. *For every $k \in \mathbb{N}$ and for any real number $A \geq 4$, given a subgraph $G = (V, E)$ of $(\mathbb{Z}^d, \mathbb{E}^d)$ such that $|V| \leq A^d$ and*

$$\text{diam} V \leq \left(\frac{3}{2}\right)^k (A - 1),$$

there exists a subset $E_0 \subset E$ of edges of G with cardinality

$$|E_0| \leq 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^k\right) A^{d-1}$$

such that any connected component of the graph $(V, E \setminus E_0)$ contains at most $\lceil A^d/2 \rceil$ vertices.

Remark 1. *In the sequel, this lemma will only be used with $A = |V|^{1/d}$ but it will be helpful for the proof to keep this parameter A fixed rather than have it depending on the graph.*

Proof. Fix $A \geq 4$. We will proceed by induction on k , and therefore we start with the case $k = 0$. Let $G = (V, E)$ be a subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ such that $|V| \leq A^d$ and $\text{diam } V \leq A - 1$. Without loss of generality, we can assume that $V \subset \Lambda(\text{diam } V + 1)$. Let us choose

$$E_0 = E \cap (T_{1,-1} \cup T_{1,0}),$$

whose cardinality is

$$|E_0| \leq 2(\text{diam } V + 1)^{d-1} \leq 2A^{d-1}.$$

If $C \subset V$ is a connected component of $(V, E \setminus E_0)$, then we have

$$|C| \leq \max\left(\left\lfloor \frac{\text{diam } V}{2} \right\rfloor, \left\lfloor \frac{\text{diam } V + 1}{2} \right\rfloor\right) (\text{diam } V + 1)^{d-1} \leq \frac{(\text{diam } V + 1)^d}{2} \leq \frac{A^d}{2}.$$

We now perform the induction step. Take $k \geq 1$ such that the result holds for $k - 1$. Let $G = (V, E)$ be a subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ such that $|V| \leq A^d$ and

$$\text{diam } V \leq \left(\frac{3}{2}\right)^k (A - 1).$$

We are going to trim the graph G to decrease its diameter by a factor $2/3$. To this end, we will remove slices of edges in directions i in which the diameter is “too big”. Consider

$$\mathcal{I} = \left\{ i \in \{1, \dots, d\} : \text{diam}_i V > \left(\frac{3}{2}\right)^{k-1} (A - 1) \right\},$$

and take $i \in \mathcal{I}$. Without loss of generality, one can assume that $\min_{x \in V} x_i = 0$. By the pigeonhole principle, there exists an integer k_i satisfying

$$\left\lfloor \frac{\text{diam}_i V}{3} \right\rfloor < k_i \leq 2 \left\lfloor \frac{\text{diam}_i V}{3} \right\rfloor \quad \text{and} \quad |E \cap T_{i,k_i}| \leq \frac{|E|}{\left\lfloor \frac{\text{diam}_i V}{3} \right\rfloor}.$$

We choose such a k_i and we write, recalling that $A \geq 4$,

$$\begin{aligned} \left\lfloor \frac{\text{diam}_i V}{3} \right\rfloor &\geq \frac{\text{diam}_i V}{3} - \frac{2}{3} \\ &\geq \frac{1}{3} \left(\frac{3}{2}\right)^{k-1} (A - 1) - \frac{2}{3} \\ &= \frac{1}{9} \left(\frac{3}{2}\right)^{k-1} (A - 1) + \frac{2}{9} \left(\left(\frac{3}{2}\right)^{k-1} (A - 1) - 3 \right) \\ &\geq \frac{1}{9} \left(\frac{3}{2}\right)^{k-1} (A - 1) \\ &\geq \frac{1}{9} \left(\frac{3}{2}\right)^{k-1} \frac{3}{4} A \\ &= \frac{1}{12} \left(\frac{3}{2}\right)^{k-1} A. \end{aligned}$$

Noting that $|E| \leq d|V| \leq dA^d$, we get

$$|E \cap T_{i,k_i}| \leq \left(\frac{2}{3}\right)^{k-1} \frac{12|E|}{A} \leq \left(\frac{2}{3}\right)^{k-1} \frac{12dA^d}{A} = 12d \left(\frac{2}{3}\right)^{k-1} A^{d-1}.$$

Consider now

$$E_1 = \bigcup_{i \in \mathcal{I}} (E \cap T_{i,k_i}),$$

whose cardinality is

$$|E_1| \leq 12d^2 \left(\frac{2}{3}\right)^{k-1} A^{d-1}.$$

Let $G' = (V', E')$ be a maximal connected component of the graph $(V, E \setminus E_1)$, in terms of number of vertices. By construction, we have that, for $i \in \mathcal{I}$,

$$\text{diam}_i(V') \leq \max\left(k_i, \text{diam}_i V - (k_i + 1)\right) \leq \frac{2}{3} \text{diam}_i V \leq \left(\frac{3}{2}\right)^{k-1} (A - 1),$$

while for $i \notin \mathcal{I}$, the definition implies

$$\text{diam}_i(V') \leq \text{diam}_i V \leq \left(\frac{3}{2}\right)^{k-1} (A - 1).$$

Taking the maximum over i yields

$$\text{diam}(V') \leq \left(\frac{3}{2}\right)^{k-1} (A - 1).$$

Besides, note that $|V'| \leq |V| \leq A^d$. Hence, by the induction hypothesis applied to G' , there exists $E_2 \subset E'$ such that

$$|E_2| \leq 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^{k-1}\right) A^{d-1},$$

and all connected components of the graph $(V', E' \setminus E_2)$ contain at most $\lceil A^d/2 \rceil$ vertices. Now take $E_0 = E_1 \cup E_2$. We have

$$\begin{aligned} |E_0| &= |E_1| + |E_2| \\ &\leq 12d^2 \left(\frac{2}{3}\right)^{k-1} A^{d-1} + 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^{k-1}\right) A^{d-1} \\ &= 2A^{d-1} + 36d^2 \left(1 - \left(\frac{2}{3}\right)^k\right) A^{d-1}. \end{aligned}$$

If C is a connected component of the graph $(V, E \setminus E_0)$, then either $C \subset V \setminus V'$ which, by maximality of V' , entails $|C| \leq |V|/2 \leq A^d/2$, or $C \subset V'$ in which case C turns out to be a connected component of the graph $(V', E' \setminus E_2)$, which implies $|C| \leq \lceil A^d/2 \rceil$. \square

We can now prove the butcher's lemma, which is a mere rephrasing of lemma 3.

Proof of lemma 2. If $|V| \geq 4^d$, this is a straightforward consequence of lemma 3 with

$$A = |V|^{1/d} \quad \text{and} \quad k = \left\lceil \frac{d \ln A - \ln(A - 1)}{\ln 3 - \ln 2} \right\rceil$$

because we then have

$$\text{diam } V \leq |V| = \frac{A^d}{A - 1} (A - 1) \leq \left(\frac{3}{2}\right)^k (A - 1)$$

and the lemma provides us with a subset $E_0 \subset E$ with cardinality

$$|E_0| \leq (2 + 36d^2) A^{d-1} \leq 4^{d+1} d^2 |V|^{\frac{d-1}{d}}$$

such that all connected components of $(V, E \setminus E_0)$ contain at most $\lceil A^d/2 \rceil = \lceil |V|/2 \rceil$ vertices. Otherwise, if we have $|V| < 4^d$, then $E_0 = E$ is solution of the problem. \square

3.3 “The surgeon’s lemma”

The application of the butcher’s lemma allows us to separate a graph into connected components which are at least twice smaller than the original graph. If the connected component of x in the remaining graph still contains more vertices than the goal size m , one can apply again the butcher’s lemma to this component of x , to obtain a connected component which contains at most a fourth of the initial number of vertices. This operation can be repeated until the connected component of x contains strictly less than m edges, which means that we have closed too many edges. The surgeon’s lemma will fix this problem, by reopening some of the edges closed by the butcher’s lemma.

Lemma 4 (The surgeon’s lemma). *Let $k \in \mathbb{N}$ and let $G = (V, E)$ be a connected subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ with $|V| \leq 2^k$. Let $x \in V$ and let m be an integer such that $1 \leq m \leq |V|$. There exists a subset $E_0 \subset E$ of edges of G with cardinality*

$$|E_0| \leq \frac{1 - a^k}{1 - a} 4^{d+1} d^2 |V|^{\frac{d-1}{d}}, \quad \text{where} \quad a = \frac{1}{2^{\frac{d-1}{d}}},$$

such that, in the graph $(V, E \setminus E_0)$, the connected component of x contains exactly m vertices.

Proof. We proceed by induction on k . The result is trivial if $k = 0$, so we perform next the induction step. Take $k \geq 1$ such that the result holds for $k - 1$. Let $G = (V, E)$ be a connected subgraph of $(\mathbb{Z}^d, \mathbb{E}^d)$ with $2^{k-1} < |V| \leq 2^k$, let $x \in V$ and let m be an integer such that $1 \leq m \leq |V|$. According to lemma 2, we can choose a subset $E_0 \subset E$ of cardinality

$$|E_0| \leq 4^{d+1} d^2 |V|^{\frac{d-1}{d}}$$

such that any connected component of the graph $(V, E \setminus E_0)$ contains at most 2^{k-1} vertices. The idea is to reopen the edges of E_0 one by one starting from the cluster of x , in order to make this cluster grow until it reaches the size m . Then we will apply the induction hypothesis on the last piece added, which contains at most 2^{k-1} vertices.

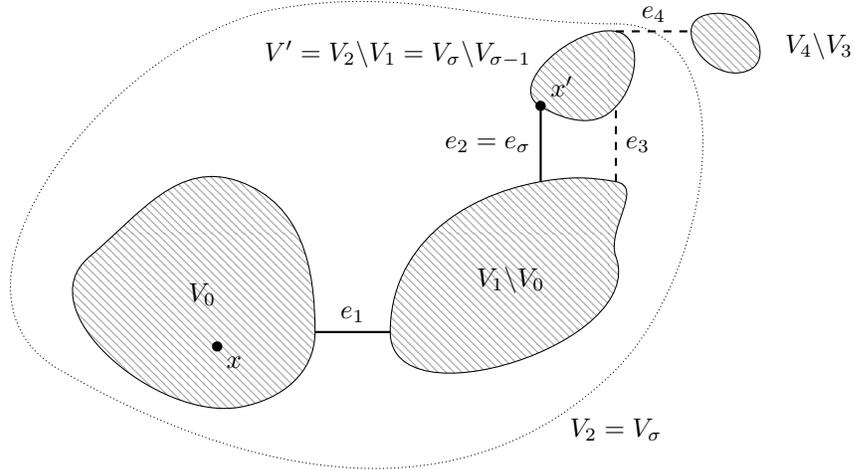


Figure 5: Illustration of the proof of lemma 4: closing the edges of $E_0 = \{e_1, e_2, e_3, e_4\}$ cuts the graph in pieces containing at most 2^{k-1} vertices. We reopen the edges e_i in this order until the number of vertices in the cluster of x reaches or exceeds m . In the case drawn here, $\sigma = 2$, and $V_3 = V_2$ because the edge e_3 connects two vertices which already belong to V_2 .

We are going to order the edges of E_0 by exploring them one by one starting from the cluster of x . We start by writing V_0 for the connected component of x in the graph $(V, E \setminus E_0)$. We have that $|V_0| \leq 2^{k-1} < |V|$,

hence $V_0 \subsetneq V$. Yet the graph (V, E) is connected, therefore we can choose an edge $e_1 \in E_0$ incident to this cluster V_0 . Assume now that we have defined $e_1, \dots, e_s \in E_0$ for some $s \geq 1$. Let V_s be the connected component of x in the graph

$$(V, E \setminus (E_0 \setminus \{e_1, \dots, e_s\})).$$

If $s < |E_0|$, then we can choose an edge $e_{s+1} \in E_0$ incident to V_s . Such an edge exists because (V, E) is connected. We proceed with this construction until all the edges of E_0 are ordered in a sequence e_1, \dots, e_r where $r = |E_0|$. We have then

$$x \in V_0 \subset V_1 \subset \dots \subset V_r = V.$$

If we close all the edges of E_0 and then reopen these edges one by one in the order e_1, \dots, e_r , then after having reopened s edges, the cluster of x is V_s . Therefore, we introduce

$$\sigma = \min \left\{ s \in \{0, \dots, r\} : |V_s| \geq m \right\}$$

which is the number of reopened edges at which the size of the cluster of x reaches or exceeds the desired size m . This number σ is well-defined because $|V_r| = |V| \geq m$. Assume that $\sigma \geq 1$. By minimality of σ , we have $|V_{\sigma-1}| < m \leq |V_\sigma|$, hence $V_\sigma \neq V_{\sigma-1}$. In that case, the edge e_σ must connect a vertex of $V_{\sigma-1}$ to a vertex $x' \in V_\sigma \setminus V_{\sigma-1}$. Letting $m' = m - |V_{\sigma-1}|$, we have that

$$1 \leq m' \leq |V_\sigma| - |V_{\sigma-1}| = |V_\sigma \setminus V_{\sigma-1}|.$$

Otherwise, if $\sigma = 0$, we set $x' = x$ and $m' = m$, which entails $1 \leq m' \leq |V_0|$.

Let us consider the graph $G' = (V', E')$ of the connected component of x' in $(V, E \setminus E_0)$. The choice of E_0 ensures that $|V'| \leq 2^{k-1}$. What's more, we have that $V' = V_\sigma \setminus V_{\sigma-1}$ if $\sigma \geq 1$ and $V' = V_0$ otherwise, which in both cases leads to $1 \leq m' \leq |V'|$. The induction hypothesis applied to the graph $G' = (V', E')$ gives us a subset $E'_0 \subset E'$ satisfying

$$|E'_0| \leq \frac{1 - a^{k-1}}{1 - a} 4^{d+1} d^2 |V'|^{\frac{d-1}{d}} \leq \frac{1 - a^{k-1}}{1 - a} 4^{d+1} d^2 a |V|^{\frac{d-1}{d}}$$

and such that the connected component of x' in $(V', E' \setminus E'_0)$, which will be denoted $V'_{x'}$, contains exactly m' vertices. Now, we consider the set

$$E''_0 = \{e_{\sigma+1}, \dots, e_r\} \cup E'_0,$$

which is such that

$$\begin{aligned} |E''_0| &= (r - \sigma) + |E'_0| \\ &\leq 4^{d+1} d^2 |V|^{\frac{d-1}{d}} + \frac{a - a^k}{1 - a} 4^{d+1} d^2 |V|^{\frac{d-1}{d}} \\ &= \frac{1 - a^k}{1 - a} 4^{d+1} d^2 |V|^{\frac{d-1}{d}}. \end{aligned}$$

If $\sigma = 0$, then the connected component of x in the graph $(V, E \setminus E''_0)$ is $V'_{x'}$ and thus it contains exactly $m' = m$ vertices. Otherwise, if $\sigma \geq 1$, then this connected component is $V_{\sigma-1} \cup V'_{x'}$, which contains $|V_{\sigma-1}| + m' = m$ vertices. \square

4 The largest cluster

This section is devoted to the proof of theorem 1 for the case of the first model, i.e., the one defined with the largest cluster in the box $\Lambda(n)$. As explained in the introduction, the first step is to show the exponential decay of the distribution of $|C_{max}|$ in the subcritical and supercritical phases.

4.1 Exponential decay in the subcritical phase

We first present a classical estimate about the size of the largest cluster below p_c :

Lemma 5. *For any $a \in (0, d)$, for $p < p_c$ and $A > 0$, we have*

$$-\infty < \liminf_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| > An^a \right) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| > An^a \right) < 0.$$

Proof. Let $a > 0$, $p < p_c$ and $A > 0$. For all $n \geq 1$, we have that

$$\begin{aligned} \mathbb{P}_p \left(|C_{max}(\Lambda(n))| > An^a \right) &= \mathbb{P}_p \left(\max_{v \in \Lambda(n)} |C_{\Lambda(n)}(v)| > An^a \right) \leq \mathbb{P}_p \left(\max_{v \in \Lambda(n)} |C(v)| > An^a \right) \\ &\leq n^d \mathbb{P}_p \left(|C(0)| > An^a \right). \end{aligned}$$

According to theorem 6.75 in [Gri99], there exists a constant $\lambda(p) > 0$ such that, for all $m \geq 1$,

$$\mathbb{P}_p \left(|C(0)| \geq m \right) \leq e^{-m\lambda(p)}. \quad (8)$$

It follows that, for all $n \geq 1$,

$$\mathbb{P}_p \left(|C_{max}(\Lambda(n))| > An^a \right) \leq n^d e^{-A\lambda(p)n^a},$$

which implies the desired upper bound. To create a cluster of size more than An^a , one may simply open a self-avoiding path of $\lfloor An^a \rfloor$ edges and $\lfloor An^a \rfloor + 1$ vertices, hence

$$\mathbb{P}_p \left(|C_{max}(\Lambda(n))| > An^a \right) \geq p^{\lfloor An^a \rfloor},$$

which shows the lower bound. □

4.2 Exponential decay in the supercritical phase

We establish a similar result in the supercritical regime:

Lemma 6. *For all $a \in (0, d)$, for $p > p_c$ and $A > 0$, we have*

$$-\infty < \liminf_{n \rightarrow \infty} \frac{1}{n^{d-a/d}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| < An^a \right) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^{d-a/d}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| < An^a \right) < 0.$$

The upper bound is a consequence of the following result, which easily follows from the classical literature:

Lemma 7. *For all $p > p_c$, we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| \leq \frac{\theta(p)n^d}{8} \right) < 0.$$

Proof. Assume first that $d \geq 3$. From theorem 1.2 of [Pis96], it follows that, for $d \geq 3$, for all $p > \hat{p}_c$ (where \hat{p}_c denotes the slab-percolation threshold),

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| \leq \frac{\theta(p)n^d}{2} \right) < 0.$$

In addition, Grimmett and Marstrand proved the identity $p_c = \hat{p}_c$ for $d \geq 3$ in [GM90]. The claim for $d \geq 3$ thus follows immediately.

Consider now the case $d = 2$. Theorem 6.1 of [ACC90] implies that, for all $p > p_c$, if we consider a percolation configuration on \mathbb{Z}^d and write $C_\infty \subset \mathbb{Z}^d$ for the unique infinite cluster of the configuration, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}_p \left(|C_\infty \cap \Lambda(n)| \leq \frac{\theta(p)n^2}{2} \right) < 0.$$

Thereby, there exists $L > 0$ such that, for all $n \geq 1$,

$$\mathbb{P}_p \left(|C_\infty \cap \Lambda(n)| \leq \frac{\theta(p)n^2}{2} \right) \leq e^{-Ln}.$$

Besides, if we set, for $m \geq k \geq 1$,

$$L_{k,m} = \left\{ \text{The rectangle } \{0, \dots, k\} \times \{0, \dots, m\} \text{ is crossed by an open path in its long direction} \right\},$$

then it follows from equation (7.110) in [Gri99] that there exist positive constants $C_2(p)$ and $C_3(p)$ such that, for all $m \geq k \geq 1$,

$$\mathbb{P}_p(L_{k,m}) \geq 1 - C_2 m e^{-C_3 k}. \quad (9)$$

Define the rectangles

$$\begin{aligned} R_1 &= \mathbb{Z}^2 \cap \left] \frac{n}{2}, n[\times [-n, n[, & R_2 &= \mathbb{Z}^2 \cap [-n, n[\times \left] \frac{n}{2}, n[, \\ R_3 &= \mathbb{Z}^2 \cap \left[-n, -\frac{n}{2}[\times [-n, n[, & R_4 &= \mathbb{Z}^2 \cap [-n, n[\times \left[-n, -\frac{n}{2}[, \end{aligned}$$

which are represented in figure 6. Following a classical argument (see the proof of theorem 7.61 in [Gri99]), we

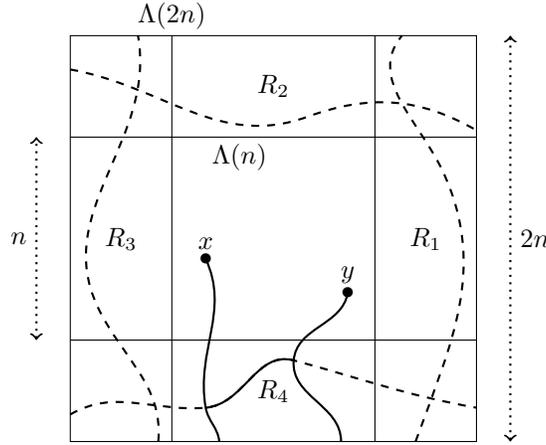


Figure 6: If each of the four rectangles is crossed by an open path in its long direction, then $\Lambda(n)$ is surrounded by an open path in $\Lambda(2n)$, and thus any two vertices x and y in the box $\Lambda(n)$ cannot be connected to $\partial\Lambda(2n)$ without being connected to each other by an open path inside $\Lambda(2n)$.

consider the events

$$\mathcal{E}_n = \left\{ \text{There exists an open path in } \Lambda(2n) \setminus \Lambda(n) \text{ containing } \Lambda(n) \text{ in its interior} \right\}$$

and

$$\mathcal{F}_n = \left\{ \text{Each of the rectangles } R_1, R_2, R_3, R_4 \text{ is crossed by an open path in its long direction} \right\}.$$

As illustrated on figure 6, we have the inclusion $\mathcal{E}_n \supset \mathcal{F}_n$. In addition, by the FKG inequality, we have that

$$\mathbb{P}_p(\mathcal{F}_n) \geq \mathbb{P}_p(L_{\lfloor n/2 \rfloor, 2n})^4.$$

In combination with (9), this yields

$$\mathbb{P}_p(\mathcal{E}_n) \geq \mathbb{P}_p(\mathcal{F}_n) \geq \mathbb{P}_p(L_{\lfloor n/2 \rfloor, 2n})^4 \geq \left(1 - 2C_2 n e^{-C_3 \lfloor n/2 \rfloor}\right)^4 \geq 1 - 8C_2 n e^{-C_3 \lfloor n/2 \rfloor}.$$

Yet if the event \mathcal{E}_n occurs, then all the vertices of $\Lambda(n)$ which are connected by an open path to the boundary of $\Lambda(2n)$ must be connected to each other inside $\Lambda(2n)$, which implies that $|C_{max}(\Lambda(2n))| \geq |C_\infty \cap \Lambda(n)|$. Therefore, we have the inclusion

$$\mathcal{E}_n \cap \left\{ |C_\infty \cap \Lambda(n)| > \frac{\theta(p)n^2}{2} \right\} \subset \left\{ |C_{max}(\Lambda(2n))| > \frac{\theta(p)n^2}{2} \right\}.$$

Considering complementary events leads to

$$\begin{aligned} \mathbb{P}_p \left(|C_{max}(\Lambda(2n))| \leq \frac{\theta(p)n^2}{2} \right) &\leq 1 - \mathbb{P}_p(\mathcal{E}_n) + \mathbb{P}_p \left(|C_\infty \cap \Lambda(n)| \leq \frac{\theta(p)n^2}{2} \right) \\ &\leq 8C_2 n e^{-C_3 \lfloor n/2 \rfloor} + e^{-Ln} \\ &\leq e^{-L'n} \end{aligned}$$

for a certain constant $L' > 0$, which concludes the proof. \square

We now briefly explain how to deduce lemma 6 from lemma 7:

Proof of lemma 6. We divide the box $\Lambda(n)$ into smaller boxes of side

$$N_n = \left\lceil \left(\frac{8An^a}{\theta(p)} \right)^{1/d} \right\rceil.$$

The box $\Lambda(n)$ contains at least $\lfloor n/N_n \rfloor^d$ disjoint boxes of side N_n , so that we have

$$\mathbb{P}_p \left(|C_{max}(\Lambda(n))| < An^a \right) \leq \mathbb{P}_p \left(|C_{max}(\Lambda(N_n))| < An^a \right)^{\lfloor n/N_n \rfloor^d} \leq \mathbb{P}_p \left(|C_{max}(\Lambda(N_n))| < \frac{\theta(p)N_n^d}{8} \right)^{\lfloor n/N_n \rfloor^d},$$

which implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n^{d-a/d}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| < An^a \right) &\leq \limsup_{n \rightarrow \infty} \frac{n^d}{N_n^d n^{d-a/d}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(N_n))| < \frac{\theta(p)N_n^d}{8} \right) \\ &= \limsup_{N \rightarrow \infty} \frac{1}{N^{d-1}} \ln \mathbb{P}_p \left(|C_{max}(\Lambda(N))| < \frac{\theta(p)N^d}{8} \right) < 0, \end{aligned}$$

using lemma 7. To obtain the lower bound, divide the box $\Lambda(n)$ into boxes of side $N_n = \lceil (An^a)^{1/d} \rceil - 1$, which all contain strictly less than An^a vertices, and consider the event that all the edges between two neighbouring boxes are closed, to obtain :

$$\ln \mathbb{P}_p \left(|C_{max}(\Lambda(n))| < An^a \right) \geq dn^{d-1} \left\lceil \frac{n}{N_n} \right\rceil \ln(1-p) \stackrel{n \rightarrow \infty}{\sim} \frac{d \ln(1-p)}{A^{1/d}} n^{d-a/d},$$

which shows that $d - a/d$ is indeed the correct exponent. \square

4.3 Lower bound on the partition function

We show the following inequality on the normalization constant Z_n of our model:

Lemma 8. *For any real number a such that $0 < a < d$, we have*

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^{a(d-1)/d}} > -\infty.$$

Proof. Construction of the coupling: As explained in the introduction, we define a monotone coupling of the distributions $\mathbb{P}_{\varphi_n(t)}$ for $t \in \{0, \dots, n^d\}$. Write $\mathbb{E}_n = \{e_1, \dots, e_r\}$ with $r = |\mathbb{E}_n|$, and consider a collection of i.i.d. random variables

$$(X_{t,e})_{t \in \{0, \dots, n^d-1\}, e \in \mathbb{E}_n}$$

with Bernoulli law of parameter $\exp(-1/n^a)$. For $t_0 \in \{0, \dots, n^d\}$, define a random configuration

$$\omega(t_0) : e \in \mathbb{E}_n \mapsto \min_{0 \leq t < t_0} X_{t,e}.$$

Hence, for $t_0 \in \{0, \dots, n^d\}$ and $e \in \mathbb{E}_n$, we see that

$$\mathbb{P}(\omega(t_0)(e) = 1) = \prod_{t=0}^{t_0-1} \mathbb{P}(X_{t,e} = 1) = \exp\left(-\frac{t_0}{n^a}\right) = \varphi_n(t_0),$$

therefore the configuration $\omega(t_0)$ has distribution $\mathbb{P}_{\varphi_n(t_0)}$. What's more, configurations are coupled in such a way that

$$\mathbb{1}_{\mathbb{E}_n} = \omega(0) \geq \omega(1) \geq \dots \geq \omega(n^d).$$

When going from the configuration $\omega(t)$ to the configuration $\omega(t+1)$, a certain number of edges are closed (these are the edges e such that $\omega(t)(e) = 1$ and $X_{t,e} = 0$). In order to control the edge closures one by one, we define intermediate configurations. For $t \in \{0, \dots, n^d - 1\}$ and $s_0 \in \{0, \dots, r\}$, we set

$$\omega(t, s_0) : e_s \in \mathbb{E}_n \mapsto \begin{cases} \omega(t+1)(e_s) & \text{if } s \leq s_0, \\ \omega(t)(e_s) & \text{otherwise.} \end{cases}$$

In this way, we have $\omega(t, 0) = \omega(t)$ and for $s \geq 1$, the configuration $\omega(t, s)$ is obtained from the configuration $\omega(t, s-1)$ by closing the edge e_s if $X_{t,e_s} = 0$, and by keeping everything unchanged if $X_{t,e_s} = 1$. For $s = r = |\mathbb{E}_n|$, all edges have been updated, so $\omega(t, r) = \omega(t+1)$. The configurations are therefore coupled in such a way that

$$(t, s) \leq (t', s') \implies \omega(t, s) \geq \omega(t', s'),$$

where we use the lexicographic order on $\{0, \dots, n^d - 1\} \times \{0, \dots, r\}$. As we have shown in (7), the partition function Z_n is equal to the probability that the non-increasing function $t \mapsto |C_{max}(\omega(t))|$ admits a fixed point. Thus, we now look for an instant $t = T$ situated before this function goes under the first bisector, and we will see what is needed on the variables $X_{t,e}$ for this function to actually cross the bisector at the instant $t = T+2$.

Definition of the instant T : Still considering the lexicographic order, we define a pair of random variables

$$(T, S) = \min \left\{ (t, s) \in \{0, \dots, n^d - 2\} \times \{0, \dots, r\} : \exists e \in \mathbb{E}_n \quad |C_{max}(\omega(t, s)_e)| \leq t + 2 \right\}.$$

This minimum is well defined because one always has $|C_{max}(\omega(n^d - 2, 0))| \leq n^d$. In addition, for every (t_0, s_0) , the event $\{(T, S) = (t_0, s_0)\}$ only depends on the variables X_{t,e_s} for $(t, s) \leq (t_0, s_0)$, which means that (T, S) is a stopping time for the filtration generated by the variables X_{t,e_s} . Also, closing one single edge cannot divide the size of the largest cluster by more than two, whence

$$|C_{max}(\omega(T, S))| \leq 2(T+2). \tag{10}$$

Let us prove that we also have

$$|C_{max}(\omega(T, S))| \geq T+2. \tag{11}$$

We distinguish several cases.

- If $S \geq 1$, then the minimality of (T, S) ensures that, for all $e \in \mathbb{E}_n$,

$$|C_{max}(\omega(T, S-1)_e)| > T+2.$$

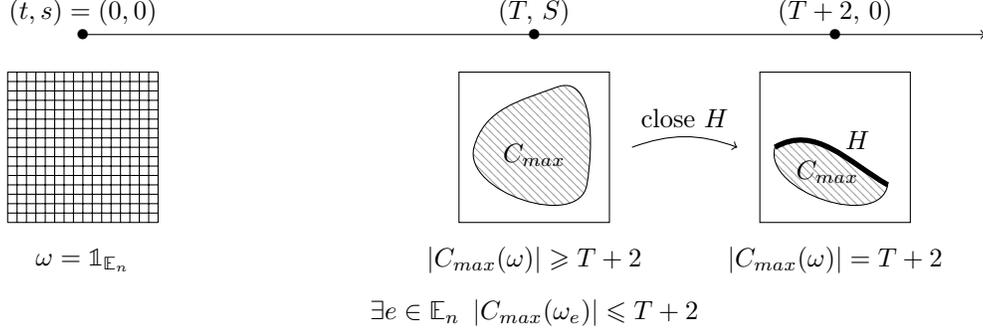


Figure 7: Sketch of the proof: if \mathcal{E} occurs, i.e., between the instants (T, S) and $(T + 2, 0)$, the edges H are closed but no other edges of C_{max} is closed, then the largest cluster in the configuration $\omega(T + 2, 0)$ contains $T + 2$ vertices.

Yet the configuration $\omega(T, S)$ is obtained from $\omega(T, S - 1)$ by closing at most one edge, whence (11).

- If $S = 0$ and $T > 0$, then, (T, S) being minimal, we have that

$$|C_{max}(\omega(T - 1, r))| > T - 1 + 2 = T + 1.$$

The configurations $\omega(T - 1, r)$ and $\omega(T, 0)$ being identical, inequality (11) is also satisfied.

- The case $(T, S) = (0, 0)$ does not happen because all edges are open in the configuration $\omega(0, 0)$.

Construction of the happy event: Let (V, E) be the graph of the largest cluster in $\omega(T, S)$, that is to say $V = C_{max}(\omega(T, S))$ and E is the set of edges between two vertices of V which are open in $\omega(T, S)$. Given (11), it follows from lemma 1 that there exists a (random) set of edges

$$H = H(T, \omega(T, S)) \subset E,$$

satisfying

$$|H| \leq K |V|^{\frac{d-1}{d}} \quad (12)$$

and such that the largest connected component of the graph $(V, E \setminus H)$ contains exactly $T + 2$ vertices. Note that we have defined $H = H(T, \omega(T, S))$ as a deterministic function of the variables T and $\omega(T, S)$, this will be useful later. The existence of an edge $e \in E_n$ such that

$$|C_{max}(\omega(T, S)_e)| \leq T + 2$$

entails that, in $\omega(T, S)$, there is at most one cluster containing strictly more than $T + 2$ vertices. Thus, closing the edges of H is enough for the remaining largest cluster to contain $T + 2$ vertices, i.e.

$$|C_{max}(\omega(T, S)_H)| = T + 2.$$

Hence, closing the edges of H and no other edge of $\mathbb{E}[C_{max}(\omega(T, S))]$ between the instants (T, S) and $(T + 2, 0)$ ensures that $|C_{max}(\omega(T + 2))| = T + 2$. But the edges $e_s \in H$ are not necessarily labeled with numbers $s > S$. It is therefore not generally possible to close all the edges of H between the instants (T, S) and $(T + 1, 0)$. For this reason, the event we consider is the one in which no edge of $\mathbb{E}[C_{max}(\omega(T, S))]$ is closed between (T, S) and $(T + 1, 0)$, and the edges of $\mathbb{E}[C_{max}(\omega(T, S))]$ which are closed between $(T + 1, 0)$ and $(T + 2, 0)$ are precisely the edges of H , that is to say

$$\mathcal{E} = \left\{ \begin{array}{l} \forall s > S \quad e_s \in \mathbb{E}[C_{max}(\omega(T, S))] \Rightarrow X_{T, e_s} = 1 \\ \forall e \in H \quad X_{T+1, e} = 0 \\ \forall e \in \mathbb{E}[C_{max}(\omega(T, S))] \setminus H \quad X_{T+1, e} = 1 \end{array} \right\}.$$

If this event occurs, then in $\omega(T+2)$, all the edges of H are closed, the other edges of $\mathbb{E}[C_{max}(\omega(T, S))]$ which were open in the configuration $\omega(T, S)$ remain open, and all the other clusters contain at most $T+2$ vertices, whence

$$\mathcal{E} \subset \left\{ |C_{max}(\omega(T+2))| = |C_{max}(\omega(T, S)_H)| = T+2 \right\}.$$

Conditional probability of the happy event: Coming back to the expression (7) of the partition function, we find that

$$Z_n \geq \mathbb{P}\left(|C_{max}(\omega(T+2))| = T+2\right) \geq \mathbb{P}(\mathcal{E}). \quad (13)$$

Let (t_0, s_0) and $\omega_0 : \mathbb{E}_n \rightarrow \{0, 1\}$ be such that

$$\mathbb{P}(\mathcal{C}_{t_0, s_0, \omega_0}) > 0 \quad \text{where} \quad \mathcal{C}_{t_0, s_0, \omega_0} = \left\{ (T, S) = (t_0, s_0) \text{ and } \omega(T, S) = \omega_0 \right\}.$$

Having defined H as a deterministic function of T and $\omega(T, S)$, we can consider the event

$$\tilde{\mathcal{E}}_{t_0, s_0, \omega_0} = \left\{ \begin{array}{ll} \forall s > s_0 & e_s \in \mathbb{E}[C_{max}(\omega_0)] \Rightarrow X_{t_0, e_s} = 1 \\ \forall e \in H(t_0, \omega_0) & X_{t_0+1, e} = 0 \\ \forall e \in \mathbb{E}[C_{max}(\omega_0)] \setminus H(t_0, \omega_0) & X_{t_0+1, e} = 1 \end{array} \right\},$$

which satisfies

$$\mathbb{P}(\mathcal{E} | \mathcal{C}_{t_0, s_0, \omega_0}) = \mathbb{P}(\tilde{\mathcal{E}}_{t_0, s_0, \omega_0} | \mathcal{C}_{t_0, s_0, \omega_0}). \quad (14)$$

Now note that this event $\tilde{\mathcal{E}}_{t_0, s_0, \omega_0}$ depends only on the variables X_{t, e_s} with $(t, s) > (t_0, s_0)$, whereas the event $\mathcal{C}_{t_0, s_0, \omega_0}$ depends only on the variables X_{t, e_s} with $(t, s) \leq (t_0, s_0)$. Thus, these two events are independent of each other, which allows us to write

$$\begin{aligned} \mathbb{P}(\tilde{\mathcal{E}}_{t_0, s_0, \omega_0} | \mathcal{C}_{t_0, s_0, \omega_0}) &= \mathbb{P}(\tilde{\mathcal{E}}_{t_0, s_0, \omega_0}) \\ &= \prod_{\substack{s > s_0 \\ e_s \in \mathbb{E}[C_{max}(\omega_0)]}} \mathbb{P}(X_{t_0, e_s} = 1) \times \prod_{e \in H(t_0, \omega_0)} \mathbb{P}(X_{t_0+1, e} = 0) \\ &\quad \times \prod_{e \in \mathbb{E}[C_{max}(\omega_0)] \setminus H(t_0, \omega_0)} \mathbb{P}(X_{t_0+1, e} = 1) \\ &\geq \left(e^{-1/n^a}\right)^{2|\mathbb{E}[C_{max}(\omega_0)]|} \left(1 - e^{-1/n^a}\right)^{|H(t_0, \omega_0)|}. \end{aligned}$$

Combining this with (14) yields

$$\mathbb{P}(\mathcal{E} | (T, S, \omega(T, S))) \geq \left(e^{-1/n^a}\right)^{2|\mathbb{E}[C_{max}(\omega(T, S))]|} \left(1 - e^{-1/n^a}\right)^{|H(T, \omega(T, S))|}. \quad (15)$$

Yet, according to (10), we have

$$|\mathbb{E}[C_{max}(\omega(T, S))]| \leq d|C_{max}(\omega(T, S))| \leq 2d(T+2).$$

Furthermore, by concavity of $x \mapsto 1 - e^{-x}$, we get

$$1 - e^{-1/n^a} \geq \frac{1}{n^a} (1 - e^{-1}) \geq \frac{1}{2n^a}.$$

In addition, combining (12) and (10) leads to

$$|H| \leq K|C_{max}(\omega(T, S))|^{\frac{d-1}{d}} \leq K(2(T+2))^{\frac{d-1}{d}} \leq 2K(T+2)^{\frac{d-1}{d}}.$$

Replacing all this in equation (15), we obtain

$$\mathbb{P}(\mathcal{E} \mid (T, S, \omega(T, S))) \geq \exp\left(-\frac{4d(T+2)}{n^a}\right) \left(\frac{1}{2n^a}\right)^{2K(T+2)\frac{d-1}{d}}.$$

We take the conditional expectation with respect to T , and we deduce that

$$\mathbb{P}(\mathcal{E} \mid T) \geq \exp\left(-\frac{4d(T+2)}{n^a}\right) \left(\frac{1}{2n^a}\right)^{2K(T+2)\frac{d-1}{d}}. \quad (16)$$

Upper bound on T : We need a control on T in order to obtain a lower bound on $\mathbb{P}(\mathcal{E})$. Define

$$\tau_n^+ = \left\lceil n^a \left(-\ln\left(\frac{p_c}{2}\right)\right) \right\rceil. \quad (17)$$

Lemma 6 implies that

$$\mathbb{P}_{p_c/2}(|C_{max}| \leq \tau_n^+) \xrightarrow{n \rightarrow \infty} 1.$$

This entails that, for n large enough,

$$\mathbb{P}_{p_c/2}(|C_{max}| \leq \tau_n^+) \geq \frac{1}{2}.$$

Given that

$$\varphi_n(\tau_n^+) \leq \varphi_n\left(n^a \left(-\ln\left(\frac{p_c}{2}\right)\right)\right) = \frac{p_c}{2},$$

we deduce that, for n large enough,

$$\mathbb{P}(T \leq \tau_n^+) \geq \mathbb{P}(|C_{max}(\omega(\tau_n^+))| \leq \tau_n^+ + 2) = \mathbb{P}_{\varphi_n(\tau_n^+)}(|C_{max}| \leq \tau_n^+ + 2) \geq \mathbb{P}_{p_c/2}(|C_{max}| \leq \tau_n^+ + 2) \geq \frac{1}{2}.$$

Therefore, we can find $\kappa \geq 2$ such that, for all $n \geq 1$,

$$\mathbb{P}(T \leq \kappa n^a) \geq \frac{1}{2}. \quad (18)$$

Conclusion: Combining (18) with (16) gives

$$\begin{aligned} \mathbb{P}(\mathcal{E}) &\geq \mathbb{P}(\mathcal{E} \cap \{T \leq \kappa n^a\}) \\ &= \mathbb{P}(T \leq \kappa n^a) \mathbb{P}(\mathcal{E} \mid T \leq \kappa n^a) \\ &\geq \frac{1}{2} \exp\left(-\frac{4d(\kappa n^a + 2)}{n^a} - 2K(\kappa n^a + 2)^{\frac{d-1}{d}} \ln(2n^a)\right) \\ &\geq \frac{1}{2} \exp\left(-8d\kappa - 4K\kappa(\ln 2)n^{a(d-1)/d} - 4K\kappa a(\ln n)n^{a(d-1)/d}\right). \end{aligned}$$

Now recall inequality (13) to deduce that

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^{a(d-1)/d}} \geq -4K\kappa a > -\infty,$$

which is the required lower bound. \square

4.4 Proof of the convergence result

We are now in position to prove theorem 1 in the case of the first model.

Proof of theorem 1, case $F_n = |C_{max}|$. Let $\varepsilon > 0$ and $a \in (0, d)$. Given that $a(d-1)/d < a$, the lower bound on Z_n we have obtained in lemma 8 implies that $\ln Z_n/n^a \rightarrow 0$. Combining this with the result of lemma 5 and plugging it into the inequality (5) then leads to

$$\limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mu_n(p_n < p_c - \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_{p_c - \varepsilon}(|C_{max}| > (-\ln(p_c - \varepsilon))n^a) < 0. \quad (19)$$

Similarly, using lemma 6 and the fact that $a(d-1)/d < d - a/d$, we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-a/d}} \ln \mu_n(p_n > p_c + \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^{d-a/d}} \ln \mathbb{P}_{p_c + \varepsilon}(|C_{max}| < (-\ln(p_c + \varepsilon))n^a) < 0. \quad (20)$$

It remains to show that the exponent $v = a \wedge (d - a/d)$ is optimal. To this end, we go back to our computation (3) and we recall that Z_n was expressed as a probability in (7), whence $Z_n \leq 1$. Therefore, with t_n^+ as defined in (2) we have

$$\mu_n(p_n > p_c + \varepsilon) = \frac{1}{Z_n} \sum_{t=0}^{t_n^+ - 1} \mathbb{P}_{\varphi_n(t)}(|C_{max}| = t) \geq \sum_{t=0}^{t_n^+ - 1} \mathbb{P}_{\varphi_n(t)}(|C_{max}| = t).$$

Using the notations of the last subsection, this becomes

$$\mu_n(p_n > p_c + \varepsilon) \geq \mathbb{P}(\exists t \in \{0, \dots, t_n^+ - 1\} : |C_{max}(\omega(t))| = t) \geq \mathbb{P}(\mathcal{E} \cap \{T \leq t_n^+ - 3\}), \quad (21)$$

since the occurrence of the event \mathcal{E} implies that $|C_{max}(\omega(T+2))| = T+2$. As we did in the proof of lemma 8, we can write

$$\mathbb{P}(T \leq t_n^+ - 3) \geq \mathbb{P}(|C_{max}(\omega(t_n^+ - 3))| < t_n^+) = \mathbb{P}_{\varphi_n(t_n^+ - 3)}(|C_{max}| < t_n^+).$$

Now notice that $\varphi_n(t_n^+ - 3) \rightarrow p_c + \varepsilon$, whence $\varphi_n(t_n^+ - 3) \leq p_c + 2\varepsilon$ for n large enough. Thus, for n large enough, we have

$$\mathbb{P}(T \leq t_n^+ - 3) \geq \mathbb{P}_{p_c + 2\varepsilon}(|C_{max}| < t_n^+).$$

Plugging this into (21) and using our lower bound (16) on the conditional probability of \mathcal{E} with respect to T then leads to

$$\begin{aligned} \frac{1}{n^{d-a/d}} \ln \mu_n(p_n > p_c + \varepsilon) &\geq -\frac{4d(t_n^+ - 1)}{n^{a+d-a/d}} - \frac{2K(t_n^+ - 1)^{\frac{d-1}{d}} \ln(2n^a)}{n^{d-a/d}} + \frac{1}{n^{d-a/d}} \ln \mathbb{P}_{p_c + 2\varepsilon}(|C_{max}| < t_n^+) \\ &= O\left(\frac{1}{n^{d-a/d}}\right) + O\left(\frac{\ln n}{n^{d-a}}\right) + \frac{1}{n^{d-a/d}} \ln \mathbb{P}_{p_c + 2\varepsilon}(|C_{max}| < t_n^+). \end{aligned}$$

Taking the limit inferior and using the lower bound given by lemma 6, we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{d-a/d}} \ln \mu_n(p_n > p_c + \varepsilon) > -\infty. \quad (22)$$

To handle the other tail, we choose a' such that

$$a < a' < d \wedge \frac{da}{d-1},$$

we define $t_n^- = \lfloor n^a(-\ln(p_c - \varepsilon)) \rfloor$ and we write

$$\begin{aligned} \mu_n(p_n < p_c - \varepsilon) &\geq \mathbb{P}(\exists t \in \{t_n^- + 1, \dots, n^d\} : |C_{max}(\omega(t))| = t) \\ &\geq \mathbb{P}(\mathcal{E} \cap \{T \geq t_n^- - 1\}) \geq \mathbb{P}(\mathcal{E} \cap \{t_n^- - 1 \leq T \leq n^{a'}\}). \end{aligned} \quad (23)$$

We now have

$$\begin{aligned}
\mathbb{P}(t_n^- - 1 \leq T \leq n^{a'}) &= \mathbb{P}(T \geq t_n^- - 1) - \mathbb{P}(T > n^{a'}) \\
&\geq \mathbb{P}\left(|C_{max}(\omega(t_n^- - 1))| > 2(t_n^- + 1)\right) - \mathbb{P}\left(|C_{max}(\omega(\lfloor n^{a'} \rfloor))| > n^{a'}\right) \\
&\geq \mathbb{P}_{p_c - 2\varepsilon}\left(|C_{max}| > 2(t_n^- + 1)\right) - \mathbb{P}_{p_c/2}\left(|C_{max}| > n^{a'}\right) \\
&\geq e^{-Cn^a} - e^{-C'n^{a'}} \geq \frac{e^{-Cn^a}}{2},
\end{aligned}$$

with $C, C' > 0$, using the exponential estimate of lemma 5. Plugging this into (23) and using again (16), we now obtain

$$\frac{1}{n^a} \ln \mu_n(p_n < p_c - \varepsilon) \geq O\left(\frac{1}{n^{2a-a'}}\right) + O\left(\frac{\ln n}{n^{a-a'+a'/d}}\right) - C - \frac{\ln 2}{n^a} \xrightarrow{n \rightarrow \infty} -C > -\infty. \quad (24)$$

The first case of theorem 1 then follows from (19), (20), (22) and (24). \square

4.5 A variant on the torus

One can define a similar model on the torus of side n , which boils down to considering periodic boundary conditions on the box $\Lambda(n)$. Clusters on the torus are at least as big as in the box, so the exponential decay in the supercritical phase for the model defined on the torus immediately follows from lemma 6. The analog of lemma 5 can be proved by noting that the size of the cluster of the origin in the torus is stochastically dominated by the size of the cluster of the origin in a configuration on all \mathbb{Z}^d . The same proof for the lower bound on the partition function applies in the case of the torus, by adapting our geometrical lemma to extend it to subgraphs of the torus. We therefore have the same convergence of p_n to p_c when $n \rightarrow \infty$ for this alternative model.

5 The number of vertices connected to the boundary

We prove here theorem 1 in the case of the model defined with $|\mathcal{M}_n|$, which is the number of vertices connected by an open path to the boundary of the box $\partial\Lambda(n)$.

5.1 Exponential decay in the subcritical phase

Following the same method as for the first model, we start with an upper bound on the law of $|\mathcal{M}_n|$ in the subcritical regime (the lower bound is straightforward, but we will not need it).

Lemma 9. *For any $a > d - 1$, for $p < p_c$ and $A > 0$, we have the upper bound*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_p\left(|\mathcal{M}_n| > An^a\right) < 0.$$

Proof. Take $a > d - 1$, $p < p_c$ and $A > 0$. Write $\partial\Lambda(n) = \{x_1, \dots, x_t\}$ with $t = |\partial\Lambda(n)|$. If A and T are two events, then $A \circ T$ denotes the disjoint occurrence of these two events, which is defined in section 2.3 of [Gri99]. Let $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ be a configuration such that $|\mathcal{M}_n(\omega)| > An^a$. Define, for $i \in \{1, \dots, t\}$,

$$n_i = \left| C_{\Lambda(n)}(x_i) \setminus \bigcup_{j < i} C_{\Lambda(n)}(x_j) \right| = \begin{cases} 0 & \text{if there exists } j < i \text{ such that } x_i \xleftarrow{\omega} x_j, \\ |C_{\Lambda(n)}(x_i)| & \text{otherwise.} \end{cases}$$

We have that

$$\sum_{i=1}^t n_i = \left| \bigcup_{i=1}^t C_{\Lambda(n)}(x_i) \right| = |\mathcal{M}_n(\omega)| > An^a,$$

and

$$\omega \in \{ |C_{\Lambda(n)}(x_1)| \geq n_1 \} \circ \dots \circ \{ |C_{\Lambda(n)}(x_t)| \geq n_t \}.$$

Indeed, if $n_i = 0$, then the event $\{|C_{\Lambda(n)}(x_i)| \geq n_i\}$ is trivial, whereas if we have $n_i > 0$ and $n_j > 0$ for some $i \neq j$, then the vertices x_i and x_j must belong to disjoint clusters. Whence the inclusion

$$\left\{ |\mathcal{M}_n| > An^a \right\} \subset \bigcup_{\substack{0 \leq n_1, \dots, n_t \leq n^d \\ n_1 + \dots + n_t > An^a}} \left\{ |C_{\Lambda(n)}(x_1)| \geq n_1 \right\} \circ \dots \circ \left\{ |C_{\Lambda(n)}(x_t)| \geq n_t \right\}.$$

Note that, for all $i \in \{1, \dots, t\}$, the event $\{|C_{\Lambda(n)}(x_i)| \geq n_i\}$ is an increasing event, thus by the BK inequality,

$$\begin{aligned} \mathbb{P}_p \left(|\mathcal{M}_n| > An^a \right) &\leq \sum_{\substack{0 \leq n_1, \dots, n_t \leq n^d \\ n_1 + \dots + n_t > An^a}} \prod_{i=1}^t \mathbb{P}_p \left(|C_{\Lambda(n)}(x_i)| \geq n_i \right) \\ &\leq \sum_{\substack{0 \leq n_1, \dots, n_t \leq n^d \\ n_1 + \dots + n_t > An^a}} \prod_{i=1}^t \mathbb{P}_p \left(|C(0)| \geq n_i \right). \end{aligned}$$

Furthermore, according to theorem 6.75 in [Gri99], for $p < p_c$, there exists a constant $\lambda(p) > 0$ such that, for all $n \geq 1$,

$$\mathbb{P}_p \left(|C(0)| \geq n \right) \leq e^{-n\lambda(p)},$$

which is also true if $n = 0$. It follows that

$$\begin{aligned} \mathbb{P}_p \left(|\mathcal{M}_n| > An^a \right) &\leq \sum_{\substack{0 \leq n_1, \dots, n_t \leq n^d \\ n_1 + \dots + n_t > An^a}} \prod_{i=1}^t \exp \left(-\lambda(p)n_i \right) \\ &\leq \sum_{0 \leq n_1, \dots, n_t \leq n^d} \exp \left(-\lambda(p)An^a \right) \\ &\leq (n^d + 1)^t \exp \left(-\lambda(p)An^a \right) \\ &= \exp \left(|\partial\Lambda(n)| \ln(n^d + 1) - \lambda(p)An^a \right). \end{aligned}$$

To conclude, note that

$$|\partial\Lambda(n)| \ln(n^d + 1) = O \left((\ln n)n^{d-1} \right) = o(n^a).$$

This completes the proof of the lemma. \square

5.2 Exponential decay in the supercritical phase

We now state a similar exponential decay property in the supercritical regime.

Lemma 10. *For all $a < d$, for $p > p_c$ and $A > 0$, we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p \left(|\mathcal{M}_n| < An^a \right) < 0.$$

Proof. Let $p > p_c$ and $A > 0$. As in the proof of lemma 6, we show that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln \mathbb{P}_p \left(|\mathcal{M}_n| \leq \frac{\theta(p)n^d}{2} \right) < 0.$$

For $d \geq 3$, the result follows from theorem 1.2 of [Pis96], which proves it for p larger than \widehat{p}_c , which was proved to be equal to p_c in [GM90]. In dimension $d = 2$, the claim follows from theorem 6.1 in [ACC90]. \square

5.3 Lower bound on the partition function

We now establish a lower bound on the normalization constant Z_n .

Lemma 11. *For any real a such that $d - 1 < a < d$, we have*

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^{a(d-1)/d}} > -\infty.$$

Heuristics of the proof: We wish to apply the same technique as in the proof for the case of the largest cluster (section 4.3), by constructing a decreasing coupling between the distributions $\mathbb{P}_{\varphi_n(t)}$ for t varying from 0 (all edges open) to n^d (almost all edges closed). We monitor the evolution of the variable $|\mathcal{M}_n|$ until an instant $t = T'$ when $|\mathcal{M}_n|$ is of order T' . Then we find a set of edges $H \subset \mathbb{E}_n$ whose closure would lead to $|\mathcal{M}_n| = T' + 2$ at the instant $T' + 2$.

The hurdle is that, in order to find such a set H which is not too big (and thus whose closure is likely enough), we need a control on the size of the clusters which are connected to the boundary of the box at the instant T' . To obtain such a control, a natural idea is to monitor first the evolution of the size of the clusters connected to the boundary, to wait for an instant T when these clusters have become small enough, and then to define the instant T' in a way which ensures that it is later than T . However, unlike the size of the largest cluster C_{max} , which can be at most divided by a factor 2 when closing an edge, the size of the largest cluster connected to the boundary can fall drastically with the closure of one edge. To avoid this, we choose to monitor the size of the largest cluster on the torus, that is to say in the square box but with periodic boundary conditions. This variable has the advantage of being at most halved at each edge closure.

Proof. Sketch of the proof: We first define a decreasing coupling of configurations $(\omega(t, s))_{t, s}$ but on the edges of the torus. We then consider the first instant (T, S) when any of the clusters on the torus contains at most $2T + 3$ vertices. In what follows, we will reason conditionally on the fact that, at this instant, the largest cluster on the torus touches the boundary of the box. We will show that, at this instant, we have $|\mathcal{M}_n(\omega)| \geq T + 2$. Next we will construct a second instant $(T', S') \geq (T, S)$ and a set of edges H such that, if the only edges of \mathcal{M}_n which are closed between (T', S') and $(T' + 2, 0)$ are the edges of H , then we have $|\mathcal{M}_n(\omega(T' + 2))| = T' + 2$. We will call this scenario the ‘‘happy event’’, and our aim is to obtain a lower bound on its probability. To this end, we will show that, with sufficiently high probability, we have $T' = O(n^a)$, which implies that, from the instant (T, S) onward, any of the clusters on the torus contains $O(n^a)$ vertices. This control will allow us to show that it is possible to find H small enough to ensure that the happy event is likely enough.

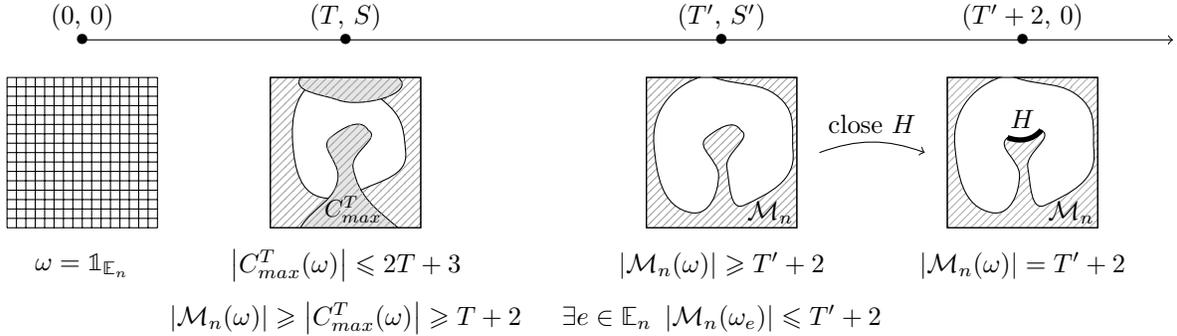


Figure 8: Illustration for the sketch of the proof of lemma 11.

Construction of the coupling and definition of T : Take $n \geq 2$. We use the same notations and definitions as in the proof of lemma 8, but we now consider configurations on the torus. To define the torus, write $p : \mathbb{Z}^d \rightarrow \Lambda(n)$ for the projection application, with is such that $p(x) - x \in n\mathbb{Z}^d$ for every $x \in \mathbb{Z}^d$. The torus is the graph whose

vertex set is $\Lambda(n)$, and whose edge set is

$$\mathbb{E}_n^T = p(\mathbb{E}^d) = \left\{ \{p(x), p(y)\} : \{x, y\} \in \mathbb{E}^d \right\},$$

which amounts to adding edges between corresponding vertices on opposite faces of the box. We then write $\mathbb{E}_n^T = \{e_1, \dots, e_r\}$ with $r = |\mathbb{E}_n^T|$, and we consider a collection of i.i.d. random variables

$$(X_{t,e})_{t \in \{0, \dots, n^d - 1\}, e \in \mathbb{E}_n^T}$$

all distributed with a Bernoulli law of parameter $\exp(-1/n^a)$. We set, for $t_0 \in \{0, \dots, n^d\}$,

$$\omega(t_0) : e \in \mathbb{E}_n^T \mapsto \min_{0 \leq t < t_0} X_{t,e},$$

and for $t \in \{0, \dots, n^d - 1\}$ and $s_0 \in \{0, \dots, r\}$, we define

$$\omega(t, s_0) : e_s \in \mathbb{E}_n^T \mapsto \begin{cases} \omega(t+1)(e_s) & \text{if } s \leq s_0, \\ \omega(t)(e_s) & \text{otherwise.} \end{cases}$$

For a configuration $\omega : \mathbb{E}_n^T \rightarrow \{0, 1\}$ and $v \in \Lambda(n)$, we denote $C_{\Lambda(n)}^T(v, \omega) \subset \Lambda(n)$ the cluster of v in the configuration ω on the torus, that is to say the connected component of the vertex v in the graph

$$\left(\Lambda(n), \{e \in \mathbb{E}_n^T : \omega(e) = 1\} \right).$$

For any $\omega : \mathbb{E}_n^T \rightarrow \{0, 1\}$, write $C_{max}^T(\omega)$ for the largest cluster on the torus in the configuration ω . In case of equality between several clusters, we choose one with an arbitrary order on the subsets of $\Lambda(n)$. We consider the pair of random variables

$$(T, S) = \min \left\{ (t, s) \in \{0, \dots, n^d - 2\} \times \{0, \dots, r\} : |C_{max}^T(\omega(t, s))| \leq 2t + 3 \right\},$$

which is well-defined because $|C_{max}^T(\omega(n^d - 2, 0))| \leq n^d$. Let us show that, at this instant (T, S) , we have

$$|C_{max}^T(\omega(T, S))| \geq T + 2. \tag{25}$$

We distinguish several cases :

- If $S \geq 1$ then, (T, S) being minimal, we have $|C_{max}^T(\omega(T, S-1))| \geq 2T + 4$. To obtain (25), note that closing a single edge can at most divide $|C_{max}^T|$ by a factor two.
- If $T \neq 0$ and $S = 0$ then, by minimality of (T, S) , we have that

$$|C_{max}^T(\omega(T-1, r))| \geq 2(T-1) + 4 = 2T + 2 \geq T + 2,$$

which implies inequality (25), because the configurations $\omega(T-1, r)$ and $\omega(T, 0)$ are identical.

- The case $(T, S) = (0, 0)$ never occurs because we have $|C_{max}^T(\omega(0, 0))| = n^d > 3$.

We have thus shown that (25) holds.

Definition of the reference vertex : We now consider the vertex $V = \min C_{max}^T(\omega(T, S))$, by taking the minimum with an arbitrary order on $\Lambda(n)$. Given that

$$\sum_{v \in \Lambda(n)} \mathbb{P}(V = v) = 1,$$

we can find a vertex $v_0 \in \Lambda(n)$ such that

$$\mathbb{P}(V = v_0) \geq \frac{1}{|\Lambda(n)|} = \frac{1}{n^d}.$$

In what follows, we will reason conditionally on this event $\{V = v_0\}$. Until now, everything took place on the torus, which is translation-invariant. If $v_0 \notin \partial\Lambda(n)$, we can apply a translation on the torus so as to have $v_0 \in \partial\Lambda(n)$, modifying at the same time the orders considered on \mathbb{E}_n^T , on $\Lambda(n)$ and on the subsets of $\Lambda(n)$. Therefore, without loss of generality, we can assume that $v_0 \in \partial\Lambda(n)$. Hence, if $V = v_0$ then we have $C_{max}^T(\omega(T, S)) \subset \mathcal{M}_n(\omega(T, S))$, whence

$$V = v_0 \quad \Rightarrow \quad |\mathcal{M}_n(\omega(T, S))| \geq |C_{max}^T(\omega(T, S))| \geq T + 2, \quad (26)$$

following (25).

Construction of the second instant T' : We now consider

$$(T', S') = \min \left\{ (t, s) \geq (T, S) \quad : \quad \exists e \in \mathbb{E}_n \quad |\mathcal{M}_n(\omega(t, s)_e)| \leq t + 2 \right\}.$$

The fact that $T \leq n^d - 2$ and $|\mathcal{M}_n(\omega(n^d - 2, r))| \leq n^d$ ensures that (T', S') is well-defined and that $T' \leq n^d - 2$. Let us show, by distinguishing several cases, that

$$V = v_0 \quad \Rightarrow \quad |\mathcal{M}_n(\omega(T', S'))| \geq T' + 2. \quad (27)$$

- If $(T', S') = (T, S)$, then the claim follows from (26).
- If $(T', S') > (T, S)$ and $S' = 0$, then the minimality of (T', S') implies that

$$T' - 1 + 2 < |\mathcal{M}_n(\omega(T' - 1, r))| = |\mathcal{M}_n(\omega(T', S'))|.$$

- Else if $(T', S') > (T, S)$ and $S' \neq 0$, then by minimality of (T', S') , we know that for all $e \in \mathbb{E}_n$,

$$|\mathcal{M}_n(\omega(T', S' - 1)_e)| > T' + 2,$$

which entails in particular that $|\mathcal{M}_n(\omega(T', S'))| > T' + 2$, because the configuration $\omega(T', S')$ is obtained from the configuration $\omega(T', S' - 1)$ by closing at most one edge.

We conclude that (27) holds in all cases.

Construction of the happy event: We now wish to define a set of edges H that we want to be closed between the configuration $\omega(T', S')$ and the configuration $\omega(T' + 2, 0)$ in order to have

$$|\mathcal{M}_n(\omega(T' + 2))| = T' + 2.$$

We are only interested in situations where $V = v_0$, thus we set arbitrarily $H = \emptyset$ if the event $\{V = v_0\}$ does not occur. We now assume that $V = v_0$. By definition of (T', S') , we may consider an edge $e \in \mathbb{E}_n$ such that

$$|\mathcal{M}_n(\omega(T', S')_e)| \leq T' + 2. \quad (28)$$

We choose this edge e minimal (for the order e_1, \dots, e_r we have considered on \mathbb{E}_n^T) among the edges satisfying (28), which ensures that e only depends on T' , $\omega(T', S')$ and V . We then construct the set H by distinguishing two cases depending on whether this inequality (28) is strict or not.

- In case of equality in (28), we take $H = \{e\}$.
- Assume that equation (28) is a strict inequality. It follows from (27) that

$$|\mathcal{M}_n(\omega(T', S')_e)| < |\mathcal{M}_n(\omega(T', S'))|,$$

which means that closing the edge e changes the number of vertices connected to the boundary of the box. Consequently, one end of the edge e , say v , must be disconnected from $\partial\Lambda(n)$ when closing e in the configuration $\omega(T', S')$. Write (C_v, E_v) for the graph of the open cluster of v in the configuration $\omega(T', S')_e$. We have, using (27),

$$|\mathcal{M}_n(\omega(T', S')_e)| = |\mathcal{M}_n(\omega(T', S'))| - |C_v| \geq T' + 2 - |C_v|.$$

Combining this with the (strict) inequality (28) yields

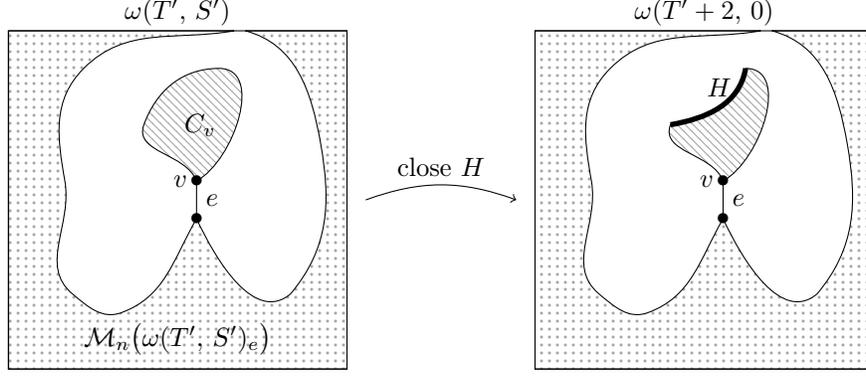


Figure 9: If (28) is a strict inequality, then closing the edge e in the configuration $\omega(T', S')$ changes the number $|\mathcal{M}_n|$ of vertices connected to the boundary of the box. This means that one end of the edge e , say v , happens to be disconnected from the boundary when e is closed. We then choose a subset H of the edges of the cluster C_v which is disconnected by the closure of e , such that closing all the edges of H and no other edges of $\mathbb{E}[\mathcal{M}_n]$ between (T', S') and $(T' + 2, 0)$ implies $|\mathcal{M}_n(\omega(T' + 2))| = T' + 2$.

$$1 \leq T' + 2 - |\mathcal{M}_n(\omega(T', S')_e)| \leq |C_v|.$$

Applying lemma 1 to the graph (C_v, E_v) and the vertex v , we can choose a set $H \subset E_v$ satisfying

$$|H| \leq K |C_v|^{\frac{d-1}{d}} \quad (29)$$

and such that the cluster of v in the graph $(C_v, E_v \setminus H)$ contains exactly $T' + 2 - |\mathcal{M}_n(\omega(T', S')_e)|$ vertices. We then have

$$|\mathcal{M}_n(\omega(T', S')_H)| = T' + 2.$$

The edge e (and thus the vertex v) depends only on T' , $\omega(T', S')$ and V , thus we can choose such a set H which also depends only on T' , $\omega(T', S')$ and V . Besides, we have the following control over $|C_v|$:

$$|C_v| = |C_{\Lambda(n)}(v, \omega(T', S')_e)| \leq |C_{\Lambda(n)}(v, \omega(T', S'))| \leq |C_{max}^T(\omega(T', S'))|. \quad (30)$$

Note now that $|C_{max}^T(\omega)|$ is a decreasing function of ω and that, by definition, $(T', S') \geq (T, S)$, whence

$$|C_{max}^T(\omega(T', S'))| \leq |C_{max}^T(\omega(T, S))| \leq 2T + 3. \quad (31)$$

Combining (30) and (31), we get $|C_v| \leq 2T + 3$, and therefore the upper bound (29) becomes

$$|H| \leq K(2T + 3)^{\frac{d-1}{d}}. \quad (32)$$

To sum up these two cases, we have defined a (random) set of edges $H \subset \mathbb{E}_n$ whose size is controlled by (32) and which satisfies

$$|\mathcal{M}_n(\omega(T', S')_H)| = T' + 2.$$

Therefore, conditionally on $\{V = v_0\}$, if the edges belonging to H and no other edges of $\mathbb{E}[\mathcal{M}_n(\omega(T', S'))]$ are closed between the configurations $\omega(T', S')$ and $\omega(T' + 2, 0)$, then we have

$$|\mathcal{M}_n(\omega(T' + 2))| = T' + 2. \quad (33)$$

This leads us to consider the event

$$\mathcal{E} = \left\{ \begin{array}{ll} \forall s > S' & e_s \in \mathbb{E}[\mathcal{M}_n(\omega(T', S'))] \Rightarrow X_{T', e_s} = 1 \\ \forall e \in H & X_{T'+1, e} = 0 \\ \forall e \in \mathbb{E}[\mathcal{M}_n(\omega(T', S'))] \setminus H & X_{T'+1, e} = 1 \end{array} \right\}$$

which, if it occurs and if $V = v_0$, implies (33). Also, our expression (7) becomes

$$Z_n = \mathbb{P}\left(\exists t \in \{0, \dots, n^d\} \mid \mathcal{M}_n(\omega(t)) = t\right) \geq \mathbb{P}\left(\mathcal{E} \cap \{V = v_0\}\right). \quad (34)$$

Conditional probability of the happy event: As in the proof of lemma 8, we consider (t_0, t'_0, s'_0) and a configuration $\omega_0 : \mathbb{F}_n^T \rightarrow \{0, 1\}$ such that

$$\mathbb{P}(\mathcal{C}_{t_0, t'_0, s'_0, \omega_0}) > 0 \quad \text{where} \quad \mathcal{C}_{t_0, t'_0, s'_0, \omega_0} = \left\{ (T, T', S') = (t_0, t'_0, s'_0) \right\} \cap \left\{ \omega(T', S') = \omega_0 \right\} \cap \left\{ V = v_0 \right\}.$$

By definition of T and T' , we have

$$|\mathcal{M}_n(\omega_0)| \leq (T' + 2) + (2T + 3) \leq 3T' + 5.$$

The event $\mathcal{C}_{t_0, t'_0, s'_0, \omega_0}$ only depends on the variables X_{t, e_s} with $(t, s) \leq (t'_0, s'_0)$ and, conditionally on this event, the event \mathcal{E} only depends on the variables X_{t, e_s} for $(t'_0, s'_0) < (t, s) < (t'_0 + 2, 0)$. What's more, the set H only depends on T' , $\omega(T', S')$ and V , which allows us to write $H = H(T', \omega(T', S'), V)$. Therefore, we have

$$\begin{aligned} \mathbb{P}(\mathcal{E} \mid \mathcal{C}_{t_0, t'_0, s'_0, \omega_0}) &= \prod_{\substack{s > s'_0 \\ e_s \in \mathbb{E}[\mathcal{M}_n(\omega_0)]}} \mathbb{P}(X_{t'_0, e_s} = 1) \times \prod_{e \in H(t'_0, \omega_0, v_0)} \mathbb{P}(X_{t'_0+1, e} = 0) \\ &\quad \times \prod_{e \in \mathbb{E}[\mathcal{M}_n(\omega_0)] \setminus H(t'_0, \omega_0, v_0)} \mathbb{P}(X_{t'_0+1, e} = 1) \\ &\geq \left(e^{-1/n^a}\right)^{2|\mathbb{E}[\mathcal{M}_n(\omega_0)]} \left(1 - e^{-1/n^a}\right)^{|H(t'_0, \omega_0, v_0)|} \\ &\geq \exp\left(-\frac{2d|\mathcal{M}_n(\omega_0)|}{n^a}\right) \left(\frac{1}{2n^a}\right)^{|H(t'_0, \omega_0, v_0)|} \\ &\geq \exp\left(-\frac{2d(3t'_0 + 5)}{n^a} - 2a(\ln n)|H(t'_0, \omega_0, v_0)|\right). \end{aligned}$$

Using the upper bound (32) on $|H|$ leads to

$$\mathbb{P}\left(\mathcal{E} \mid T, T', S', \omega(T', S'), V\right) \geq \mathbb{1}_{V=v_0} \exp\left(-\frac{2d(3T' + 5)}{n^a} - 2Ka(\ln n)(2T + 3)^{\frac{d-1}{d}}\right).$$

Taking the conditional expectation with respect to (T', V) , we obtain

$$\mathbb{P}(\mathcal{E} \mid T', V) \geq \mathbb{1}_{V=v_0} \exp\left(-\frac{2d(3T' + 5)}{n^a} - 2Ka(\ln n)(2T' + 3)^{\frac{d-1}{d}}\right). \quad (35)$$

Upper bound on T' : It follows from lemma 9 that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_{p_c/2} \left[|\mathcal{M}_n| \geq n^a \left(-\ln \left(\frac{p_c}{2} \right) \right) \right] < 0.$$

Therefore, we have

$$\mathbb{P}_{p_c/2} \left[|\mathcal{M}_n| \geq n^a \left(-\ln \left(\frac{p_c}{2} \right) \right) \right] = o\left(\frac{1}{n^d}\right),$$

and thus, if we take τ_n^+ defined as in (17) then, for n large enough,

$$\mathbb{P}\left(|\mathcal{M}_n(\omega(\tau_n^+))| \geq \tau_n^+\right) \leq \mathbb{P}_{p_c/2} \left[|\mathcal{M}_n| \geq n^a \left(-\ln \left(\frac{p_c}{2} \right) \right) \right] \leq \frac{1}{2n^d}.$$

We then have, using the fact that $\mathbb{P}(V = v_0) \geq 1/n^d$,

$$\mathbb{P}\left(V = v_0 \text{ and } |\mathcal{M}_n(\omega(\tau_n^+))| < \tau_n^+\right) \geq \frac{1}{n^d} - \frac{1}{2n^d} = \frac{1}{2n^d}.$$

Yet, if $V = v_0$ and $|\mathcal{M}_n(\omega(\tau_n^+))| < \tau_n^+$, then inequality (27) entails that $T' < \tau_n^+$. From this we can deduce that, for n large enough,

$$\mathbb{P}\left(V = v_0 \text{ and } T' < \tau_n^+\right) \geq \frac{1}{2n^d}.$$

Therefore, we can find $\kappa \geq 2$ such that, for n large enough,

$$\mathbb{P}\left(V = v_0 \text{ and } T' \leq \kappa n^a\right) \geq \frac{1}{2n^d}. \quad (36)$$

Conclusion: Combining (35) and (36) yields

$$\begin{aligned} \mathbb{P}\left(\mathcal{E} \cap \{V = v_0\}\right) &\geq \mathbb{P}\left(V = v_0 \text{ and } T' \leq \kappa n^a\right) \mathbb{P}\left(\mathcal{E} \mid V = v_0 \text{ and } T' \leq \kappa n^a\right) \\ &\geq \frac{1}{2n^d} \exp\left(-\frac{2d(3\kappa n^a + 5)}{n^a} - 2Ka(\ln n)(2\kappa n^a + 3)^{\frac{d-1}{d}}\right) \\ &\geq \frac{1}{2n^d} \exp\left(-6d\kappa - \frac{10d}{n^a} - 8K\kappa a(\ln n)n^{a(d-1)/d}\right). \end{aligned}$$

Given (34), we obtain

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^{a(d-1)/d}} \geq \liminf_{n \rightarrow \infty} \frac{\ln \mathbb{P}(\{V = v_0\} \cap \mathcal{E})}{(\ln n)n^{a(d-1)/d}} \geq -8K\kappa a > -\infty,$$

which is the required lower bound. \square

5.4 Proof of the convergence result

We now explain how the second case of theorem 1 follows from the above lemmas.

Proof of theorem 1, case $F_n = |\mathcal{M}_n|$. Let $\varepsilon > 0$ and $a \in (d-1, d)$. Exactly as in section 4.4, the upper bounds

$$\limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mu_n(p_n < p_c - \varepsilon) < 0. \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln \mu_n(p_n > p_c + \varepsilon) < 0$$

follow from our lower bound on Z_n (lemma 11) and from the results of exponential decay in the subcritical (lemma 9) and supercritical phases (lemma 10). To obtain the lower bound, we go back to our computation (3) to write

$$\mu_n(p_n > p_c + \varepsilon) = \frac{1}{Z_n} \sum_{t=0}^{t_n^+ - 1} \mathbb{P}_{\varphi_n(t)}(|\mathcal{M}_n| = t) \geq \mathbb{P}_{\varphi_n(t_n^+ - 1)}(|\mathcal{M}_n| = t_n^+ - 1), \quad (37)$$

where t_n^+ is still given by (2). We now implement a simplified surgery procedure to force $|\mathcal{M}_n| = t_n^+ - 1$ starting from a configuration ω such that $|\mathcal{M}_n(\omega)| > t_n^+ - 1$. According to lemma 10, we have

$$\mathbb{P}_{\varphi_n(t_n^+ - 1)}(|\mathcal{M}_n| > t_n^+ - 1) \geq \mathbb{P}_{p_c + \varepsilon}(|\mathcal{M}_n| > t_n^+ - 1) \xrightarrow{n \rightarrow \infty} 1. \quad (38)$$

Let $\omega \in \{0, 1\}^{\mathbb{E}_n}$ be a configuration such that $|\mathcal{M}_n(\omega)| > t_n^+ - 1$. Consider the set E of the edges of \mathbb{E}_n which have exactly one endpoint in $\partial\Lambda(n)$, which is such that

$$|\mathcal{M}_n(\omega_E)| = |\partial\Lambda(n)| \leq 2dn^{d-1} < t_n^+ - 1 \quad (39)$$

for n large enough, because $a > d - 1$. We write $E = \{e_1, \dots, e_{|E|}\}$ with $|E| \leq 2dn^{d-1}$, and we let

$$B = \max \left\{ b \in \{1, \dots, |E|\} : |\mathcal{M}_n(\omega_{\{e_1, \dots, e_b\}})| \geq t_n^+ - 1 \right\}.$$

It follows from (39) that $B < |E|$, whence by maximality of B ,

$$|\mathcal{M}_n(\omega_{\{e_1, \dots, e_{B+1}\}})| < t_n^+ - 1.$$

Therefore, if we write $e_{B+1} = \{x, y\}$ with $x \in \partial\Lambda(n)$ and $y \notin \partial\Lambda(n)$, and if we consider the cluster which is disconnected from the boundary when closing this edge e_{B+1} , namely :

$$C_y = C_{\Lambda(n)}(y, \omega_{\{e_1, \dots, e_{B+1}\}}),$$

we have

$$|\mathcal{M}_n(\omega_{\{e_1, \dots, e_{B+1}\}})| = |\mathcal{M}_n(\omega_{\{e_1, \dots, e_B\}})| - |C_y|,$$

so that $m = (t_n^+ - 1) - |\mathcal{M}_n(\omega_{\{e_1, \dots, e_{B+1}\}})|$ satisfies $1 \leq m \leq |C_y|$. Hence, lemma 1 provides us with $H_1 \subset \mathbb{E}_n$, of size $|H_1| \leq |C_y|^{(d-1)/d} \leq n^{d-1}$, such that

$$|C_{\Lambda(n)}(y, \omega_{\{e_1, \dots, e_{B+1}\} \cup H_1})| = m.$$

Writing $H = H(\omega) = \{e_1, \dots, e_B\} \cup H_1$, we then have $|H| \leq (2d + 1)n^{d-1}$ and

$$|\mathcal{M}_n(\omega_H)| = |\mathcal{M}_n(\omega_{\{e_1, \dots, e_{B+1}\}})| + m = t_n^+ - 1.$$

Using lemma 6.3 in [CP00], we can deduce that

$$\mathbb{P}_{\varphi_n(t_n^+ - 1)} \left(|\mathcal{M}_n| = t_n^+ - 1 \right) \geq \left(\frac{1}{C_n} \right)^{(2d+1)n^{d-1}} \times \mathbb{P}_{\varphi_n(t_n^+ - 1)} \left(|\mathcal{M}_n| > t_n^+ - 1 \right),$$

where

$$C_n = \left(1 \vee \frac{\varphi_n(t_n^+ - 1)}{1 - \varphi_n(t_n^+ - 1)} \right) |E_n| = O(n^d).$$

Plugging this into (37) and using (38) then yields

$$\liminf_{n \rightarrow \infty} \frac{1}{(\ln n)n^{d-1}} \ln \mu_n(|p_n - p_c| > \varepsilon) \geq \liminf_{n \rightarrow \infty} \frac{1}{(\ln n)n^{d-1}} \ln \mu_n(p_n > p_c + \varepsilon) \geq -(2d + 1)d > -\infty,$$

as announced in theorem 1. □

6 The distribution of the cluster sizes

The goal of this section is to prove the remaining part of theorem 1, namely the case of $F_n = |B_n^b|$, along with theorem 2. To this end, we fix $0 < b < a < d$.

6.1 Exponential decay in the subcritical phase

We now prove the following exponential decay in the subcritical regime :

Lemma 12. *For every $p < p_c$ and any $A > 0$, we have*

$$-\infty < \liminf_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_p \left(|B_n^b| > An^a \right) \leq \limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mathbb{P}_p \left(|B_n^b| > An^a \right) < 0.$$

Proof. Let $p < p_c$ and $A > 0$. Writing $N_n = 1 + \lfloor An^{a-b} \rfloor$ and using the BK inequality as in the proof of lemma 9, we get

$$\begin{aligned} \mathbb{P}_p\left(|B_n^b| > An^a\right) &\leq \sum_{k=1}^{N_n} \sum_{x_1, \dots, x_k \in \Lambda(n)} \sum_{\substack{n^b \leq n_1, \dots, n_k \leq n^d \\ n_1 + \dots + n_k > An^a}} \mathbb{P}_p\left(\{|C_{\Lambda(n)}(x_1)| \geq n_1\} \circ \dots \circ \{|C_{\Lambda(n)}(x_k)| \geq n_k\}\right) \\ &\leq \sum_{k=1}^{N_n} \sum_{x_1, \dots, x_k \in \Lambda(n)} \sum_{\substack{n^b \leq n_1, \dots, n_k \leq n^d \\ n_1 + \dots + n_k > An^a}} \prod_{i=1}^k e^{-\lambda(p)n_i} \\ &\leq N_n (n^d)^{2N_n} e^{-\lambda(p)An^a}. \end{aligned}$$

Therefore, we obtain

$$\frac{1}{n^a} \ln \mathbb{P}_p\left(|B_n^b| > An^a\right) \leq \frac{\ln N_n}{n^a} + \frac{2N_n d \ln n}{n^a} - \lambda(p)A \xrightarrow{n \rightarrow \infty} -\lambda(p)A < 0,$$

which proves the upper bound. The lower bound follows from the lower bound given by lemma 5, since $An^a \geq n^b$ for n large enough. \square

6.2 Exponential decay in the supercritical phase

We now deal with the deviations in the regime $p > p_c$. We wish to thank an anonymous referee for having improved our proof, leading to a better (and in fact optimal) exponent.

Lemma 13. *We have the upper bound*

$$\forall p > p_c \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n^{d-b/d}} \ln \mathbb{P}_p\left(|B_n^b| < An^a\right) < 0.$$

Proof. Let $p > p_c$ and $A > 0$. To prove this result, we partition the box $\Lambda(n)$ into hypercubic boxes of side

$$N_n = \left\lceil \left(\frac{8n^b}{\theta(p)}\right)^{1/d} \right\rceil.$$

Therefore, we let

$$M_n = \min \left\{ m \in \mathbb{N} : \Lambda(n) \subset \bigcup_{j \in \Lambda(m)} (N_n j + \Lambda(N_n)) \right\},$$

so that we have a partition

$$\Lambda(n) = \bigsqcup_{j \in \Lambda(M_n)} \left[(N_n j + \Lambda(N_n)) \cap \Lambda(n) \right].$$

By definition of B_n^b , we have

$$\begin{aligned} |B_n^b| &= \left| \left\{ x \in \Lambda(n) : |C_{\Lambda(n)}(x)| \geq n^b \right\} \right| \\ &\geq n^b \left| \left\{ j \in \Lambda(M_n) : \left| C_{\max} \left[(N_n j + \Lambda(N_n)) \cap \Lambda(n) \right] \right| \geq n^b \right\} \right|. \end{aligned}$$

Now note that

$$\frac{n^b |\Lambda(M_n)|}{2} = \frac{n^b M_n^d}{2} \stackrel{n \rightarrow \infty}{\sim} \frac{n^b}{2} \left(\frac{n}{N_n}\right)^d \stackrel{n \rightarrow \infty}{\sim} \frac{n^{b+d}\theta(p)}{16n^b} = \frac{\theta(p)n^d}{16}.$$

Given that $An^a = o(n^d)$, this implies that, for n large enough,

$$An^a < \frac{n^b |\Lambda(M_n)|}{2}.$$

Therefore, we have the following implication :

$$\begin{aligned} |B_n^b| < An^a &\Rightarrow \left| \left\{ j \in \Lambda(M_n) : |C_{max}[(N_n j + \Lambda(N_n)) \cap \Lambda(n)]| \geq n^b \right\} \right| < \frac{|\Lambda(M_n)|}{2} \\ &\Rightarrow \left| \left\{ j \in \Lambda(M_n) : |C_{max}[(N_n j + \Lambda(N_n)) \cap \Lambda(n)]| < n^b \right\} \right| \geq \frac{|\Lambda(M_n)|}{2}. \end{aligned}$$

The problem now is that the boxes on the boundaries might be truncated, while the inside boxes are full, that is to say

$$\forall j \in \Lambda(M_n - 2) \quad N_n j + \Lambda(N_n) \subset \Lambda(n).$$

But the number of boxes on the boundaries is

$$|\Lambda(M_n) \setminus \Lambda(M_n - 2)| = o(|\Lambda(M_n)|),$$

so that we have, for n large enough,

$$|B_n^b| < An^a \quad \Rightarrow \quad \left| \left\{ j \in \Lambda(M_n - 2) : |C_{max}(N_n j + \Lambda(N_n))| < n^b \right\} \right| \geq \frac{|\Lambda(M_n)|}{4}.$$

Using now the independence of the sizes of the largest cluster inside disjoint boxes and the fact that the number of choices of at least $|\Lambda(M_n)|/4$ boxes is at most $2^{|\Lambda(M_n)|}$, we get

$$\mathbb{P}_p(|B_n^b| < An^a) \leq 2^{M_n^d} \mathbb{P}_p(|C_{max}(\Lambda(N_n))| < n^b)^{M_n^d/4},$$

which implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-b/d}} \ln \mathbb{P}_p(|B_n^b| < An^a) \leq \limsup_{n \rightarrow \infty} \left[\frac{M_n^d \ln 2}{n^{d-b/d}} + \frac{M_n^d}{4n^{d-b/d}} \ln \mathbb{P}_p(|C_{max}(\Lambda(N_n))| < n^b) \right].$$

Now note that

$$\frac{M_n^d}{n^{d-b/d}} \underset{n \rightarrow \infty}{\sim} \frac{(n/N_n)^d}{n^{d-b/d}} \underset{n \rightarrow \infty}{\sim} \frac{n^{b/d}}{N_n} \frac{1}{N_n^{d-1}} \underset{n \rightarrow \infty}{\sim} \left(\frac{\theta(p)}{8} \right)^{1/d} \frac{1}{N_n^{d-1}}.$$

Therefore, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-b/d}} \ln \mathbb{P}_p(|B_n^b| < An^a) \leq \frac{1}{4} \left(\frac{\theta(p)}{8} \right)^{1/d} \limsup_{n \rightarrow \infty} \frac{1}{N_n^{d-1}} \ln \mathbb{P}_p(|C_{max}(\Lambda(N_n))| < n^b).$$

The result then follows from lemma 7, noting that $n^b \leq \theta(p)N_n^d/8$. \square

6.3 Lower bound on the partition function

It remains to prove a lower bound on the normalization constant Z_n . Adapting the technique of lemmas 8 and 11, we can easily obtain a bound with an exponent b . This is done in lemma 14, and the proof is much simpler than in the previous sections because, instead of performing a surgery step, we only “freeze” the edges of B_n^b during a certain number of steps. But this bound may not be sufficient to outweigh the bound in the supercritical phase, since it may be the case that $b > d - b/d$. To solve this problem, we show an other lower bound in lemma 15 with a different exponent, using a more geometrical technique.

Lemma 14. *For every a and b such that $0 < b < a < d$, we have*

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{n^b} > -\infty.$$

Proof. We use the same monotone coupling $(\omega(t, s))_{0 \leq t \leq n^d, 0 \leq s \leq r}$ as in the proof of lemma 8. Following a strategy similar to that of lemma 8, we define

$$(T, S) = \min \left\{ (t, s) \in \{0, \dots, n^d - 2\} \times \{0, \dots, r\} : |B_n^b(\omega(t, s))| \leq t + 1 + 2n^b \right\}. \quad (40)$$

When closing one single edge, $|B_n^b|$ cannot decrease by more than $2n^b$ (in the worst case, the edge cuts a cluster of $2 \lceil n^b \rceil - 2$ vertices in two equal parts). Therefore, we always have $|B_n^b(\omega(T, S))| \geq T + 1$. Thus, if we consider the instant $T' = |B_n^b(\omega(T, S))|$, we have $T + 1 \leq T' \leq T + 1 + 2n^b$. In view of this, our strategy is to force all the edges of $B_n^b(\omega(T, S))$ to remain open until the configuration $\omega(T', 0)$. This idea is much simpler than the strategy of the previous sections, because we do not perform any surgery step. Considering the event

$$\mathcal{E} = \left\{ \begin{array}{l} \forall s > S \quad e_s \in \mathbb{E}[B_n^b(\omega(T, S))] \Rightarrow X_{T, e_s} = 1 \\ \forall t \in \{T + 1, \dots, T' - 1\} \quad \forall e \in \mathbb{E}[B_n^b(\omega(T, S))] \quad X_{t, e} = 1 \end{array} \right\},$$

equation (7) becomes

$$Z_n \geq \mathbb{P} \left(|B_n^b(\omega(T', 0))| = |B_n^b(\omega(T, S))| = T' \right) \geq \mathbb{P}(\mathcal{E}). \quad (41)$$

A lower bound on the probability of \mathcal{E} is easily obtained by writing

$$\begin{aligned} \mathbb{P}(\mathcal{E} \mid (T, S, \omega(T, S))) &= \prod_{\substack{s > S \\ e_s \in \mathbb{E}[B_n^b(\omega(T, S))]} } \mathbb{P}(X_{T, e_s} = 1) \prod_{t=T+1}^{T'-1} \prod_{e \in \mathbb{E}[B_n^b(\omega(T, S))]} \mathbb{P}(X_{t, e} = 1) \\ &\geq \left(e^{-1/n^a} \right)^{(T'-T)|\mathbb{E}[B_n^b(\omega(T, S))]|} \geq \left(e^{-1/n^a} \right)^{(2n^b+1)d(T+1+2n^b)}. \end{aligned} \quad (42)$$

We then show an upper bound on T , using the same technique as in the proof of lemma 8. With τ_n^+ defined as in (17), we can write

$$\mathbb{P}(T \leq \tau_n^+) \geq \mathbb{P} \left(|B_n^b(\omega(\tau_n^+))| \leq \tau_n^+ + 1 + 2n^b \right) \geq \mathbb{P}_{p_c/2} \left(|B_n^b| \leq \tau_n^+ + 1 + 2n^b \right) \xrightarrow{n \rightarrow \infty} 1,$$

thanks to lemma 12. Combining this with (41) and (42) then leads to

$$\frac{\ln Z_n}{n^b} \geq \frac{\ln \mathbb{P}(T \leq \tau_n^+)}{n^b} + \frac{\ln \mathbb{P}(\mathcal{E} \mid T \leq \tau_n^+)}{n^b} \geq o(1) - \frac{(2n^b + 1)d(\tau_n^+ + 1 + 2n^b)}{n^{a+b}} \xrightarrow{n \rightarrow \infty} -2d \left(-\ln \left(\frac{p_c}{2} \right) \right),$$

using that $\tau_n^+ \sim (-\ln(p_c/2))n^a$. \square

We now state the other lower bound we obtain using a more geometrical technique:

Lemma 15. *For every a and b such that $0 < b < a < d$, we have*

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^c} > -\infty \quad \text{where} \quad c = \left(1 - \frac{a}{d} + \frac{b}{d} \right) \vee \left(a - \frac{a}{d} \right).$$

Proof outline: We use again the same coupling $(\omega(t, s))_{t, s}$ as in the previous proof, and the same instant (T, S) . We then want to close edges to reach a fixed point, but the problem is that it is not always possible to do so by only closing edges. Imagine for example that, in the configuration $\omega(T, S)$, no cluster contains more than $\lceil n^b \rceil$ vertices, meaning that B_n^b is only made of clusters containing exactly $\lceil n^b \rceil$ vertices. Then, in this very unfavourable situation, closing edges either does not affect $|B_n^b|$ or it diminishes $|B_n^b|$ by $\lceil n^b \rceil$, thus we cannot finely tune B_n^b only by closing edges.

To circumvent this problem, we will change what happened before the instant (T, S) so as to ensure that, at this instant, we have at our disposal a cluster containing at least $2n^b$ vertices. This will enable us to use our surgery procedure on this cluster, to reach the exact desired value for $|B_n^b|$.

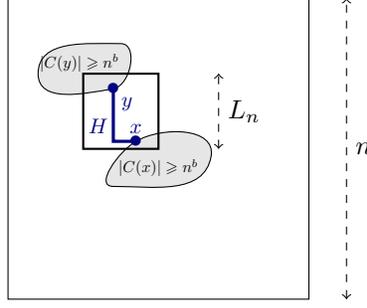


Figure 10: If there is no cluster with size $\geq 2n^b$, we reconstitute one by joining two clusters of size at least n^b , with a path using less than dL_n edges.

But, to do so, we need to intervene on the past of the instant (T, S) , which will make notations more complicated. Namely, we will define a second coupling of configurations $(\omega'(t, s))_{t,s}$ which is a copy of the first coupling, except that the closure times of a certain number of edges are drawn again, allowing us to close or to open these edges at different times.

Before diving into the proof, we detail our surgery procedure in the following two lemmas. The first one is a lower bound on the number of edges to reopen to create a cluster of size at least $2n^b$.

Lemma 16. *Let $b \in (0, d)$. For every $n \geq 1$ and for any configuration $\omega \in \{0, 1\}^{\mathbb{E}_n}$ such that $|B_n^b(\omega)| \geq 2^{d+1}n^b$, there exists a set of edges $H \subset \mathbb{E}_n$, such that*

$$|C_{max}(\omega^H)| \geq 2n^b \quad \text{and} \quad |H| \leq 4d \frac{n^{1+b/d}}{|B_n^b(\omega)|^{1/d}}.$$

Proof. Let $b \in (0, d)$ and $\omega \in \{0, 1\}^{\mathbb{E}_n}$ such that $|B_n^b(\omega)| \geq 2^{d+1}n^b$. If $|C_{max}(\omega)| \geq 2n^b$, then we choose $H = \emptyset$. Let us now assume that $|C_{max}(\omega)| < 2n^b$. Then all the clusters in $B_n^b(\omega)$ contain between n^b and $2n^b$ vertices. Therefore, there are at least $|B_n^b(\omega)|/(2n^b)$ such clusters in the configuration ω . Now, if we divide the box $\Lambda(n)$ into hypercubic boxes of side

$$L_n = \left\lceil 2n \left(\frac{2n^b}{|B_n^b(\omega)|} \right)^{1/d} \right\rceil,$$

with boxes which may be smaller along the boundaries of $\Lambda(n)$, the number of boxes is

$$\left\lceil \frac{n}{L_n} \right\rceil^d \leq \left\lceil \frac{1}{2} \left(\frac{|B_n^b(\omega)|}{2n^b} \right)^{1/d} \right\rceil^d < \left\lceil \frac{1}{2} \left(\frac{|B_n^b(\omega)|}{2n^b} \right)^{1/d} + 1 \right\rceil^d \leq \frac{|B_n^b(\omega)|}{2n^b}.$$

Hence, there are strictly more boxes than the number of clusters with size at least n^b . Therefore, by the pigeonhole principle, at least one of these boxes must intersect two such clusters, which means that we can find $x, y \in \Lambda(n)$ such that

$$|C(x, \omega)| \geq n^b, \quad |C(y, \omega)| \geq n^b, \quad x \xleftrightarrow{\omega} y \quad \text{and} \quad \|x - y\|_\infty \leq L_n.$$

We then have $\|x - y\|_1 \leq dL_n$, implying that there exists a path $H \subset \mathbb{E}_n$ of at most dL_n edges which connects x and y . Opening the edges of this path in ω creates a connection between two different clusters of size at least n^b , whence $|C_{max}(\omega^H)| \geq 2n^b$. What's more, we have

$$|H| \leq dL_n \leq d \left(\frac{2^{1+1/d} n^{1+b/d}}{|B_n^b(\omega)|^{1/d}} + 1 \right) \leq (2^{1+1/d} + 1)d \frac{n^{1+b/d}}{|B_n^b(\omega)|^{1/d}} \leq 4d \frac{n^{1+b/d}}{|B_n^b(\omega)|^{1/d}},$$

which completes the proof of this lemma. \square

The second geometrical lemma will tell us how many edges we need to close to adjust the size of B_n^b :

Lemma 17. *Let $b \in (0, d)$. There exists $K_1 = K_1(d) > 0$ such that, for $n \geq 1$, for any configuration $\omega \in \{0, 1\}^{\mathbb{E}_n}$ and any $s \in \mathbb{N}$, if*

$$12n^b \leq s \leq |B_n^b(\omega)| \leq s + 6n^b \quad \text{and} \quad |C_{max}(\omega)| \geq 2n^b, \quad (43)$$

then there exists $H \subset \mathbb{E}_n$ such that

$$|B_n^b(\omega_H)| = s \quad \text{and} \quad |H| \leq K_1(s + n^b)^{\frac{d-1}{d}}.$$

Sketch of the proof: If we have at our disposal a big enough cluster, then we may reach $|B_n^b| = s$ by only closing edges of this cluster. In this case, using the geometrical results of section 3, the idea is to cut $C_{max}(\omega)$ into one large piece of size $m \geq n^b$ and remaining pieces all of size $< n^b$ (see the left part of figure 11). Adjusting the cutting so that $m = |C_{max}(\omega)| - |B_n^b(\omega)| + s$ yields the desired result.

But, for the above technique to work, we need this m to be greater than n^b , so that the large piece of size m still belongs to B_n^b after the cutting. This is the case if $|C_{max}(\omega)| \geq 7n^b$, because then $m \geq 7n^b - 6n^b = n^b$.

In the case when $|C_{max}(\omega)| < 7n^b$, we proceed differently. In this case, we first cut other intermediate clusters, to reach a situation where $s \leq |B_n^b| < s + n^b$ (see the right part of figure 11). Then, we can use $C_{max}(\omega)$ to reach exactly $|B_n^b| = s$, by disconnecting $|B_n^b| - s$ vertices from $C_{max}(\omega)$. Because we have assumed that $|C_{max}(\omega)| \geq 2n^b$ and $|B_n^b| - s < n^b$, we can ensure that the resulting cluster still contains at least n^b vertices, and thus still belongs to B_n^b .

Proof. Let us now detail the strategy presented above. Let $b \in (0, d)$, $\omega \in \{0, 1\}^{\mathbb{E}_n}$ and $s \in \mathbb{N}$ such that (43) holds. We distinguish between two cases :

• **First case:** Assume that $|C_{max}(\omega)| \geq 7n^b$. Letting $m = |C_{max}(\omega)| - |B_n^b(\omega)| + s$, we can use lemma 1 to find $H_1 \subset \mathbb{E}_n$ with

$$|H_1| \leq K |C_{max}(\omega)|^{(d-1)/d} \leq K(s + 6n^b)^{(d-1)/d}$$

and such that closing the edges of H_1 divides $C_{max}(\omega)$ into one connected component of size exactly m , and one or several other pieces, whose total size is $|C_{max}(\omega)| - m = |B_n^b(\omega)| - s \leq 6n^b$. Using the butcher's lemma (lemma 2), this remaining part can be cut into pieces smaller than $3n^b$. Using again the butcher's lemma on the connected subpieces which contain strictly more than $(3/2)n^b$ vertices (there are at most 3 such subpieces), we can cut them into pieces smaller than $(3/2)n^b$. Repeating the operation on the eventual pieces containing strictly more than $(3/4)n^b$ vertices (there are at most 7 such subpieces), we can cut them into pieces smaller than $(3/4)n^b$. Thus, using at most 11 times the butcher's lemma, we obtain $H_2 \subset \mathbb{E}_n$ such that $|H_2| \leq 11 \times 4^{d+1} d^2 (6n^b)^{(d-1)/d}$ and such that in the configuration $\omega_{H_1 \cup H_2}$, the vertices of $C_{max}(\omega)$ are separated into one cluster of size exactly m and the remaining clusters which are all smaller than $(3/4)n^b < n^b$ (see the left part of figure 11). Therefore, writing $H = H_1 \cup H_2$ and using that

$$m = |C_{max}(\omega)| - |B_n^b(\omega)| + s \geq 7n^b - 6n^b = n^b,$$

we obtain

$$|B_n^b(\omega_H)| = |B_n^b(\omega)| - |C_{max}(\omega)| + m = s$$

and

$$|H| \leq K(s + 6n^b)^{(d-1)/d} + 11 \times 4^{d+1} d^2 (6n^b)^{(d-1)/d} \leq 6(K + 11 \times 4^{d+1} d^2)(s + n^b)^{(d-1)/d}.$$

• **Second case:** Now assume that $2n^b \leq |C_{max}(\omega)| < 7n^b$. The first step in this case is to find $H_1 \subset \mathbb{E}_n$ such that

$$s \leq |B_n^b(\omega_{H_1})| < s + n^b. \quad (44)$$

If we already have $|B_n^b(\omega)| < s + n^b$, then we take $H_1 = \emptyset$. Assume now that $|B_n^b(\omega)| \geq s + n^b$. As explained in the sketch of the proof, the idea to obtain (44) is to cut one or several intermediate clusters. By intermediate, we mean clusters containing at least n^b vertices but which are not C_{max} . Each cutting of an intermediate cluster will have to either yield directly (44) or to decrease $|B_n^b|$ by at least $\lceil n^b \rceil - 1$, maintaining $|B_n^b| \geq s$. Because we

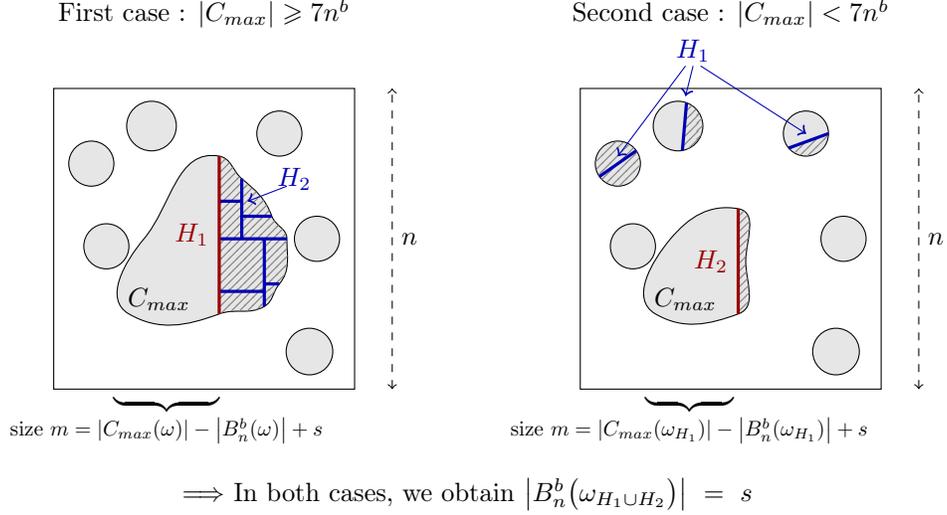


Figure 11: First case: when $|C_{max}| \geq 7n^b$ (picture on the left), we cut a piece of C_{max} with the desired size m (by closing H_1) and we divide the remaining part of C_{max} into pieces smaller than n^b (by closing H_2). Second case: when $|C_{max}| < 7n^b$, we cut some intermediate clusters (by closing H_1) and we adjust by cutting a piece of C_{max} with the desired size m (by closing H_2). In both figures, the hatched region is $B_n^b(\omega) \setminus B_n^b(\omega_{H_1 \cup H_2})$, i.e., the vertices whose cluster is no longer larger than n^b after the cutting procedure.

start from $s + n^b \leq |B_n^b(\omega)| \leq s + 6n^b$, at most 6 such cuttings are necessary to eventually obtain (44). Let us now detail the cutting we perform on these intermediate clusters. We look for $H_0 \subset \mathbb{E}_n$ such that

$$s \leq |B_n^b(\omega_{H_0})| \leq \left(|B_n^b(\omega)| - (\lceil n^b \rceil - 1) \right) \vee (s + n^b - 1). \quad (45)$$

Notice that

$$|B_n^b(\omega)| \geq s \geq 12n^b > |C_{max}(\omega)|,$$

which allows us to choose $x \in B_n^b(\omega) \setminus C_{max}(\omega)$, meaning that the cluster $C_x = C(x, \omega)$ is larger than n^b but it is not the largest cluster (it is what we call an intermediate cluster). We now distinguish between several subcases depending on the size of this cluster C_x :

- If $|C_x| \geq 2 \lceil n^b \rceil - 2$, then lemma 1 provides us with $H_0 \subset \mathbb{E}_n$ such that

$$|C(x, \omega_{H_0})| = |C_x| - (\lceil n^b \rceil - 1).$$

We then have

$$|B_n^b(\omega_{H_0})| = |B_n^b(\omega)| - (\lceil n^b \rceil - 1).$$

- If $\lceil n^b \rceil \leq |C_x| < 2 \lceil n^b \rceil - 2$ and $|B_n^b(\omega)| - |C_x| \geq s$, then we choose a set of edges $H_0 \subset \mathbb{E}_n$ such that $|C(x, \omega_{H_0})| = \lceil n^b \rceil - 1$. In this case, we get

$$|B_n^b(\omega_{H_0})| = |B_n^b(\omega)| - |C_x| \leq |B_n^b(\omega)| - \lceil n^b \rceil.$$

- Otherwise, if $\lceil n^b \rceil \leq |C_x| < 2 \lceil n^b \rceil - 2$ and $|B_n^b(\omega)| - |C_x| < s$, then we choose H_0 such that $|C(x, \omega_{H_0})| = \lceil n^b \rceil$, which entails that

$$|B_n^b(\omega_{H_0})| = |B_n^b(\omega)| - (|C_x| - \lceil n^b \rceil) < s + n^b.$$

In all three cases, H_0 satisfies (45) and lemma 1 ensures that H_0 can be chosen with

$$|H_0| \leq K |C_x|^{(d-1)/d} \leq K |C_{max}(\omega)|^{(d-1)/d} \leq 7Kn^{b(d-1)/d}.$$

After closing this set of edges H_0 , we still have $|B_n^b| \geq s$ and, either $|B_n^b| < s + n^b$ or $|B_n^b|$ has decreased by at least $n^b - 1$. Therefore, we can repeat this operation, and after at most 6 steps, we obtain $s \leq |B_n^b| < s + n^b$. Thus, we end up with $H_1 \subset \mathbb{E}_n$ satisfying (44) and such that $|H_1| \leq 42Kn^{b(d-1)/d}$. As we have not touched $C_{max}(\omega)$ during this procedure, we still have

$$2n^b \leq |C_{max}(\omega_{H_1})| < 7n^b.$$

Letting now

$$m = |C_{max}(\omega_{H_1})| - |B_n^b(\omega_{H_1})| + s,$$

it follows from (44) that

$$n^b \leq m \leq |C_{max}(\omega_{H_1})|.$$

Hence, using again lemma 1, we can find $H_2 \subset \mathbb{E}_n$ with $|H_3| \leq K(7n^b)^{(d-1)/d}$ and such that closing the edges of H_2 divides the cluster $C_{max}(\omega_{H_1})$ into one connected component of size exactly m and one or several other connected components, whose total size is

$$|C_{max}(\omega_{H_1})| - m = |B_n^b(\omega_{H_1})| - s < n^b.$$

Therefore, writing $H = H_1 \cup H_2$, we obtain

$$|B_n^b(\omega_H)| = |B_n^b(\omega_{H_1})| - |C_{max}(\omega_{H_1})| + m = s,$$

with

$$|H| \leq 42Kn^{b(d-1)/d} + 7Kn^{b(d-1)/d} = 49Kn^{b(d-1)/d}.$$

In both cases, we obtain the claimed result, with $K_1 = (6K + 66 \times 4^{d+1}d^2) \vee (49K)$. \square

We are now in a position to prove our second lower bound on Z_n .

Proof of lemma 15. As explained above, we define two couplings, in order to be able not only to close edges, but also to reopen edges.

Definition of the two couplings: The first coupling is defined as in the previous proofs: we write $\mathbb{E}_n = \{e_1, \dots, e_r\}$ with $r = |\mathbb{E}_n|$, and we consider i.i.d. random variables

$$(X_{t,e})_{t \in \{0, \dots, n^d - 1\}, e \in \mathbb{E}_n}$$

with Bernoulli law of parameter $\exp(-1/n^a)$. For $t_0 \in \{0, \dots, n^d\}$, we define

$$\omega(t_0) : e \in \mathbb{E}_n \mapsto \min_{0 \leq t < t_0} X_{t,e}.$$

In addition to this, we draw a uniform random $M \in \{0, \dots, n^d\}$, uniform independent edges $\varepsilon_1, \dots, \varepsilon_M \in \mathbb{E}_n$ and i.i.d. random variables $(X'_{t,e})_{t \leq n^d - 1, e \in \mathbb{E}_n}$ again with Bernoulli law of parameter $\exp(-1/n^a)$. The second coupling of configurations is then defined by

$$\forall t_0 \in \{0, \dots, n^d\} \quad \omega'(t_0) : e \in \mathbb{E}_n \mapsto \begin{cases} \min_{0 \leq t < t_0} X'_{t,e} & \text{if } e \in \{\varepsilon_1, \dots, \varepsilon_M\}, \\ \omega(t_0) & \text{otherwise,} \end{cases}$$

with again intermediate configurations defined for all $t \in \{0, \dots, n^d - 1\}$ and $s_0 \in \{0, \dots, r\}$ by

$$\omega'(t, s_0) : e_s \in \mathbb{E}_n \mapsto \begin{cases} \omega'(t+1)(e_s) & \text{if } s \leq s_0, \\ \omega'(t)(e_s) & \text{otherwise.} \end{cases}$$

Hence, the two decreasing couplings have the same law, with $\omega(t) \stackrel{d}{=} \omega'(t) \stackrel{d}{=} \mathbb{P}_{\varphi_n(t)}$, and the second coupling differs from the first one only on the edges $\varepsilon_1, \dots, \varepsilon_M$. This set of edges is chosen at random, but we will be interested in the probability that $\{\varepsilon_1, \dots, \varepsilon_M\} = H_1 \cup H_2$, where H_1 is a set of edges which we want to leave

open longer in the second coupling, and H_2 is a set of edges which we want to close sooner in the second coupling. Thus, this double coupling will allow us to perform the surgery procedure of lemmas 16 (which involves opening edges) and 17 (which involves closing edges), starting from a given configuration in the first coupling. Note that, instead of constructing such a double coupling, we could also have used the standard estimate of, for example, lemma 6.3 in [CP00], about the price to open or close edges.

Reconstitution of a big enough cluster: As in the proof of lemma 14, we consider the instant (T, S) defined by (40), which is such that

$$T + 1 \leq |B_n^b(\omega(T, S))| \leq T + 1 + 2n^b.$$

With τ_n^+ defined as in (17) and τ_n^- given by

$$\tau_n^- = \left\lfloor n^a \left(-\ln \left(\frac{p_c + 1}{2} \right) \right) \right\rfloor,$$

we have

$$\begin{aligned} \mathbb{P}(T \notin [\tau_n^-, \tau_n^+]) &\leq \mathbb{P}\left(|B_n^b(\omega(\tau_n^-))| \leq \tau_n^- + 1 + 2n^b \quad \text{or} \quad |B_n^b(\omega(\tau_n^+))| > \tau_n^+ + 1 + 2n^b\right) \\ &\leq \mathbb{P}_{(p_c+1)/2}\left(|B_n^b| \leq \tau_n^- + 1 + 2n^b\right) + \mathbb{P}_{p_c/2}\left(|B_n^b| > \tau_n^+ + 1 + 2n^b\right) \xrightarrow{n \rightarrow \infty} 0, \end{aligned} \quad (46)$$

thanks to lemmas 12 and 13. This allows us, in the sequel, to reason conditionally on the event that $\tau_n^- \leq T \leq \tau_n^+$. Lemma 16 then provides us with $H_1 = H_1(\omega(T, S)) \subset \mathbb{E}_n$ such that the configuration $\omega(T, S)^{H_1}$, where the edges of H_1 are reopened, contains a cluster with at least $2n^b$ vertices, and such that

$$|H_1| \leq 4d \frac{n^{1+b/d}}{|B_n^b(\omega)|^{1/d}} \leq 4d \frac{n^{1+b/d}}{T^{1/d}} \leq 4d \frac{n^{1+b/d}}{(\tau_n^-)^{1/d}} \leq K' n^{1+b/d-a/d} \quad (47)$$

for a certain $K' > 0$. We can choose this H_1 minimal in the sense of inclusion, so that either $H_1 = \emptyset$ if we already had $|C_{max}(\omega(T, S))| \geq 2n^b$ or all the edges of H_1 belong to $C_{max}(\omega(T, S)^{H_1})$ which contains at most $4n^b$ vertices, so that in both cases,

$$T + 1 \leq |B_n^b(\omega(T, S)^{H_1})| \leq |B_n^b(\omega(T, S))| + 4n^b \leq T + 1 + 6n^b.$$

Surgery step: The next step is to find a set of edges $H_2 = H_2(\omega(T, S)^{H_1}) \subset \mathbb{E}_n$ such that

$$|B_n^b(\omega(T, S)^{H_2})| = T + 1. \quad (48)$$

Still assuming $\tau_n^- \leq T \leq \tau_n^+$, applying lemma 17 with $s = T + 1 \geq \tau_n^- \geq 12n^b$, we can construct $H_2 \subset \mathbb{E}_n$, which can be defined as a function of the configuration $\omega(T, S)^{H_1}$, satisfying (48) and whose cardinality is bounded by

$$|H_2| \leq K_1 (T + 1 + 6n^b)^{(d-1)/d} \leq K_1 (\tau_n^+ + 1 + 6n^b)^{(d-1)/d} \leq K'' n^{a(d-1)/d}, \quad (49)$$

for a certain $K'' > 0$, since $\tau_n^+ = O(n^a)$ and $a > b$.

The happy event: We now consider the event (where $H_1 = H_2 = \emptyset$ if $T \notin [\tau_n^-, \tau_n^+]$)

$$\mathcal{E} = \left\{ \begin{array}{l} M = |H_1| + |H_2|, \quad \{\varepsilon_1, \dots, \varepsilon_M\} = H_1 \cup H_2, \\ \forall s > S \quad e_s \in \mathbb{E}[B_n^b(\omega(T, S))] \Rightarrow X_{T, e_s} = 1, \\ \forall e \in H_1 \quad \forall t \in \{0, \dots, T\} \quad X'_{t, e} = 1, \\ \forall e \in H_2 \quad X'_{0, e} = 0 \end{array} \right\}.$$

If this event occurs and $\tau_n^- \leq T \leq \tau_n^+$, then we have

$$|B_n^b(\omega'(T + 1, 0))| = |B_n^b(\omega(T, S)^{H_2})| = T + 1,$$

whence

$$Z_n \geq \mathbb{P}\left(|B_n^b(\omega'(T+1, 0))| = T+1\right) \geq \mathbb{P}\left(\mathcal{E} \cap \{\tau_n^- \leq T \leq \tau_n^+\}\right). \quad (50)$$

As in the proof of lemma 8, we now take (t_0, s_0) and $\omega_0 \in \{0, 1\}^{\mathbb{E}^n}$ such that $\tau_n^- \leq t_0 \leq \tau_n^+$ and

$$\mathbb{P}(\mathcal{C}_{t_0, s_0, \omega_0}) > 0 \quad \text{where} \quad \mathcal{C}_{t_0, s_0, \omega_0} = \left\{ (T, S) = (t_0, s_0) \quad \text{and} \quad \omega(T, S) = \omega_0 \right\}.$$

Because H_1 and H_2 only depend on T and $\omega(T, S)$, we may consider the deterministic sets H_1 and H_2 associated with $T = t_0$ and $\omega(T, S) = \omega_0$. Then we consider the event

$$\tilde{\mathcal{E}}_{t_0, s_0, \omega_0} = \left\{ \begin{array}{l} M = |H_1| + |H_2|, \quad \{\varepsilon_1, \dots, \varepsilon_M\} = H_1 \cup H_2, \\ \forall s > s_0 \quad e_s \in \mathbb{E}[B_n^b(\omega_0)] \Rightarrow X_{t_0, e_s} = 1, \\ \forall e \in H_1 \quad \forall t \in \{0, \dots, t_0\} \quad X'_{t, e} = 1, \\ \forall e \in H_2 \quad X'_{0, e} = 0 \end{array} \right\},$$

which is independent of $\mathcal{C}_{t_0, s_0, \omega_0}$ because $\mathcal{C}_{t_0, s_0, \omega_0}$ depends only on the variables X_{t, e_s} with $(t, s) \leq (t_0, s_0)$. Therefore, we can write

$$\begin{aligned} \mathbb{P}(\mathcal{E} | \mathcal{C}_{t_0, s_0, \omega_0}) &= \mathbb{P}\left(\tilde{\mathcal{E}}_{t_0, s_0, \omega_0} | \mathcal{C}_{t_0, s_0, \omega_0}\right) = \mathbb{P}\left(\tilde{\mathcal{E}}_{t_0, s_0, \omega_0}\right) \\ &= \frac{1}{n^d + 1} \left(\frac{1}{n^d}\right)^{|H_1| + |H_2|} \prod_{\substack{s > s_0 \\ e_s \in \mathbb{E}[B_n^b(\omega_0)]}} \mathbb{P}(X_{t_0, e_s} = 1) \prod_{e \in H_1} \prod_{t=0}^{t_0} \mathbb{P}(X'_{t, e} = 1) \prod_{e \in H_2} \mathbb{P}(X'_{0, e} = 0) \\ &\geq \left(\frac{1}{n^d + 1}\right)^{1 + |H_1| + |H_2|} (e^{-1/n^a})^{|\mathbb{E}[B_n^b(\omega_0)]| + |H_1|(t_0+1)} (1 - e^{-1/n^a})^{|H_2|}. \end{aligned}$$

We now use the bounds (47) and (49) on $|H_1|$ and $|H_2|$, the upper bound

$$|\mathbb{E}[B_n^b(\omega_0)]| \leq d|B_n^b(\omega_0)| \leq d(t_0 + 1 + 2n^b) \leq d(\tau_n^+ + 1 + 2n^b)$$

and the fact that, for n large enough, $1 - e^{-1/n^a} \geq 1/(n^d + 1)$, to obtain

$$\mathbb{P}(\mathcal{E} | \mathcal{C}_{t_0, s_0, \omega_0}) \geq \left(\frac{1}{n^d + 1}\right)^{1 + K'n^{1+b/d-a/d} + 2K''n^{a(d-1)/d}} \exp\left(-\frac{d(\tau_n^+ + 1 + 2n^b) + (\tau_n^+ + 1)K'n^{1+b/d-a/d}}{n^a}\right).$$

This bound being uniform with respect to t_0, s_0 and ω_0 (as long as $\tau_n^- \leq t_0 \leq \tau_n^+$), we obtain that

$$\mathbb{P}(\mathcal{E} | \tau_n^- \leq T \leq \tau_n^+) \geq \left(\frac{1}{n^d + 1}\right)^{1 + K'n^{1+b/d-a/d} + 2K''n^{a(d-1)/d}} \exp\left(-\frac{(d + K'n^{1+b/d-a/d})(\tau_n^+ + 1) + 2dn^b}{n^a}\right).$$

Plugging this into (50) and recalling (46), we get

$$\begin{aligned} \ln Z_n &\geq \ln \mathbb{P}(\tau_n^- \leq T \leq \tau_n^+) - (1 + K'n^{1+b/d-a/d} + 2K''n^{a(d-1)/d}) \ln(n^d + 1) \\ &\quad - \frac{(d + K'n^{1+b/d-a/d})(\tau_n^+ + 1) + 2dn^b}{n^a} \\ &= o(1) + O(n^{1+b/d-a/d} \ln n) + O(n^{a(d-1)/d} \ln n) = O(n^c \ln n), \end{aligned}$$

with $c = (1 + b/d - a/d) \vee (a - a/d)$. □

6.4 Proof of the convergence result

We now obtain the third case of theorem 1, proceeding as in section 4.4.

Proof of theorem 1, case $F_n = |B_n^b|$. Let $\varepsilon > 0$ and $0 < b < a < d$. The upper bound

$$\limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \mu_n(p_n < p_c - \varepsilon) < 0 \quad (51)$$

follows from the exponential decay in the subcritical regime (lemma 12) and the lower bound on Z_n given by lemma 14, using that $b < a$. Similarly, lemma 13 together with the other lower bound on Z_n given by lemma 15 implies

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-b/d}} \ln \mu_n(p_n > p_c + \varepsilon) < 0, \quad (52)$$

using that

$$c = \left(1 - \frac{a}{d} + \frac{b}{d}\right) \vee \left(a - \frac{a}{d}\right) < 1 \vee \left(d - \frac{a}{d}\right) < d - \frac{b}{d}.$$

To obtain a lower bound on $\mu_n(p_n < p_c - \varepsilon)$, we use the same technique as in section 4.4, choosing this time a parameter $a' \in (a, 2a - b)$. Using the notations of the proof of lemma 14 and $t_n^- = \lfloor n^a (-\ln(p_c - \varepsilon)) \rfloor$, we write

$$\mu_n(p_n < p_c - \varepsilon) \geq \mathbb{P}(\mathcal{E} \cap \{T \geq t_n^-\}) \geq \mathbb{P}(\mathcal{E} \cap \{t_n^- \leq T \leq n^{a'}\}).$$

As in section 4.4, using that $\varphi_n(t_n^-) \geq p_c - \varepsilon$ and $\varphi_n(n^{a'}) \rightarrow 0 < p_c/2$, and using the exponential estimate of lemma 12, we have

$$\mathbb{P}(t_n^- \leq T \leq n^{a'}) \geq \mathbb{P}_{p_c - \varepsilon}(|B_n^b| > t_n^- + 1 + 2n^b) - \mathbb{P}_{p_c/2}(|B_n^b| > n^{a'}) \geq e^{-Cn^a} - e^{-C'n^{a'}} \geq \frac{e^{-Cn^a}}{2}.$$

Combining this with the lower bound (42) on the conditional probability $\mathbb{P}(\mathcal{E} | T)$, we obtain

$$\frac{1}{n^a} \ln \mu_n(p_n < p_c - \varepsilon) \geq O\left(\frac{1}{n^{2a-b-a'}}\right) - C - \frac{\ln 2}{n^a} \xrightarrow{n \rightarrow \infty} -C > -\infty. \quad (53)$$

We now turn to the other lower bound. With t_n^+ given by (2) we have

$$\mu_n(p_n > p_c + \varepsilon) \geq \mathbb{P}_{\varphi_n(t_n^+ - 1)}(|B_n^b| = t_n^+ - 1). \quad (54)$$

We now consider a configuration $\omega \in \{0, 1\}^{\mathbb{E}_n}$ such that $|B_n^b(\omega)| \geq t_n^+ - 1$, and we use again the surgery procedure detailed in section 6.3 to force $|B_n^b| = t_n^+ - 1$. Dividing the box $\Lambda(n)$ into hypercubic boxes of side $N_n = \lceil n^{b/d} \rceil - 1$ (and hence of volume $< n^b$), and closing all the edges on the boundaries of these boxes, we can find $H_1 \subset \mathbb{E}_n$ such that $B_n^b(\omega_{H_1}) = \emptyset$ and

$$|H_1| = O\left(\frac{n}{N_n} n^{d-1}\right) = O(n^{d-b/d}).$$

Rather than closing all this set of edges H_1 , we choose a maximal subset $H_2 \subset H_1$ such that $|B_n^b(\omega_{H_2})| \geq t_n^+ - 1$. The maximality of H_2 then ensures that

$$t_n^+ - 1 \leq |B_n^b(\omega_{H_2})| \leq t_n^+ - 1 + 2n^b,$$

since closing one edge can at most diminish $|B_n^b|$ by $2n^b$, as explained in the proof of lemma 14. Using now the geometrical lemmas 16 and 17, we can find $H_3, H_4 \subset \mathbb{E}_n$ such that

$$\left|B_n^b(\omega_{H_2 \cup H_4}^{H_3})\right| = t_n^+ - 1 \quad \text{with} \quad |H_3| = O(n^{1+b/d-a/d}) \quad \text{and} \quad |H_4| = O(n^{a(d-1)/d}).$$

Following lemma 6.3 in [CP00], we obtain

$$\begin{aligned} \mathbb{P}_{\varphi_n(t_n^+-1)}\left(|B_n^b| = t_n^+ - 1\right) &\geq \left(\frac{1}{O(n^d)}\right)^{|H_2|+|H_3|+|H_4|} \mathbb{P}_{\varphi_n(t_n^+-1)}\left(|B_n^b| \geq t_n^+ - 1\right) \\ &\geq \exp\left(-O((\ln n)n^{d-b/d})\right) \mathbb{P}_{p_c+\varepsilon}\left(|B_n^b| \geq t_n^+ - 1\right). \end{aligned}$$

Plugging this into (54) and recalling that the probability on the right-hand side tends to 1 according to lemma 13, we obtain

$$\frac{1}{(\ln n)n^{d-b/d}} \ln \mu_n(p_n > p_c + \varepsilon) > -\infty. \quad (55)$$

The last case of theorem 1 then follows from (51), (52), (53) and (55). \square

6.5 A control on the convergence speed

We now make the previous arguments more precise, in order to obtain an estimate on the convergence speed of p_n towards the critical point p_c . We assume that there exist real numbers $\beta > 0$, $\gamma > 0$ such that

$$\limsup_{\substack{p \rightarrow p_c \\ p > p_c}} \frac{\ln \theta(p)}{\ln(p - p_c)} \leq \beta \quad \text{and} \quad \liminf_{\substack{p \rightarrow p_c \\ p < p_c}} \frac{\ln \chi(p)}{\ln(p_c - p)} \geq -\gamma,$$

and we fix for all this part some real numbers a , b and c such that

$$0 < b < a < d \quad \text{and} \quad 0 < c < \min\left(\frac{b}{2\gamma}, \frac{a-b}{2\gamma}, \frac{d-a}{\beta}, \frac{d-bd-b}{2\beta}\right).$$

We also choose β' and γ' such that

$$\beta < \beta' < \frac{1-b}{c} \wedge \frac{d-a}{c} \quad \text{and} \quad \gamma < \gamma' < \frac{b}{2c} \wedge \frac{a-b}{2c}.$$

Therefore, we can find $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\theta(p_c + \varepsilon) \geq \varepsilon^{\beta'} \quad \text{and} \quad \chi(p_c - \varepsilon) \leq \frac{1}{\varepsilon^{\gamma'}}.$$

6.5.1 Subcritical phase

Lemma 18. *We have the upper bound*

$$\forall \varepsilon > 0 \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n^{a-2\gamma'c}} \ln \mathbb{P}_{p_c - \varepsilon/n^c}\left(|B_n^b| > An^a\right) < 0.$$

Proof. Take $A > 0$ and $0 < \varepsilon < p_c$. Without loss of generality, we can assume that $\varepsilon < \varepsilon_0$. We repeat the proof of lemma 12, but replacing p with $p_c - \varepsilon/n^c$. To control $\mathbb{P}_p(|C_{\Lambda(n)}(x_i)| \geq n_i)$, the upper bound (8) is no longer sufficient, because we would need to specify the dependence in n of $\lambda(p_c - \varepsilon/n^c)$. Thus, we use another inequality provided by the same theorem 6.75 in [Gri99], which states that

$$\forall p < p_c \quad \forall n > \chi(p)^2 \quad \mathbb{P}_p(|C(0)| \geq n) \leq 2 \exp\left(-\frac{n}{2\chi(p)^2}\right). \quad (56)$$

With our choice of γ' , we have that

$$\chi\left(p_c - \frac{\varepsilon}{n^c}\right)^2 \leq \frac{n^{2\gamma'c}}{\varepsilon^{2\gamma'}} = o(n^b).$$

Hence, the condition $n^b \geq \chi(p_c - \varepsilon/n^c)^2$ is satisfied for n large enough. This allows us to apply (56) to get, with $N_n = 1 + \lfloor An^{a-b} \rfloor$ as in the proof of lemma 12,

$$\begin{aligned} \mathbb{P}_{p_c - \varepsilon/n^c} \left(|B_n^b| > An^a \right) &\leq \sum_{k=1}^{N_n} \sum_{x_1, \dots, x_k \in \Lambda(n)} \sum_{\substack{n^b \leq n_1, \dots, n_k \leq n^d \\ n_1 + \dots + n_k > An^a}} \prod_{i=1}^k 2 \exp \left(-\frac{n_i}{2\chi(p_c - \varepsilon/n^c)^2} \right) \\ &\leq N_n n^{2dN_n} \exp \left(-A\varepsilon^{2\gamma'} n^{a-2\gamma'c} \right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{n^{a-2\gamma'c}} \ln \mathbb{P}_{p_c - \varepsilon/n^c} \left(|B_n^b| > An^a \right) &\leq \frac{\ln N_n}{n^{a-2\gamma'c}} + \frac{2N_n d \ln n}{n^{a-2\gamma'c}} - A\varepsilon^{2\gamma'} = O \left(\frac{\ln n}{n^{a-2\gamma'c}} \right) + O \left(\frac{1}{n^{b-2\gamma'c}} \right) - A\varepsilon^{2\gamma'} \\ &= o(1) - A\varepsilon^{2\gamma'}, \end{aligned}$$

which proves the desired upper bound. \square

6.5.2 Supercritical phase

Lemma 19. *We have the upper bound*

$$\forall \varepsilon > 0 \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n^{d-bd-2\beta'c}} \ln \mathbb{P}_{p_c + \varepsilon/n^c} \left(|B_n^b| < An^a \right) < 0.$$

This bound is much rougher than the bound we proved in lemma 13, but it presents the advantage of using only $\theta(p)$ which we have assumed to scale as $(p - p_c)^\beta$. The counterpart is that it only works with $b < 1$.

Proof. We let $N = \lfloor n^b \rfloor$, and we divide the box $\Lambda(n)$ into smaller boxes of side $3N$, leaving apart the remainder, meaning that we write

$$\Lambda(n) \supset \bigsqcup_{i=1}^{\lfloor n/(3N) \rfloor^d} B_i,$$

where the boxes $B_i = \Lambda(3N) + \tau_i$ are disjoint translates of $\Lambda(3N)$. If a vertex $x \in \Lambda(N)$ is connected to the boundary $\partial\Lambda(3N)$, then the cluster of x in the box $\Lambda(3N)$ contains at least $N \geq n^b$ vertices, whence

$$S_n \stackrel{\text{def}}{=} \sum_{i=1}^{\lfloor n/(3N) \rfloor^d} \left| \left\{ x \in (\Lambda(N) + \tau_i) : x \overset{\omega}{\longleftrightarrow} \partial B_i \text{ inside } B_i \right\} \right| \leq |B_n^b|.$$

The boxes B_i being disjoint, the variables in the above sum are pairwise independent. Besides, the expectation of this sum is

$$\mathbb{E}_{p_c + \varepsilon/n^c} [S_n] \geq \left[\frac{n}{3N} \right]^d N^{d\theta} \left(p_c + \frac{\varepsilon}{n^c} \right) \geq \left[\frac{n}{3N} \right]^d N^d \frac{\varepsilon^{\beta'}}{n^{\beta'c}} \stackrel{n \rightarrow \infty}{\sim} \frac{\varepsilon^{\beta'}}{3^d} n^{d-\beta'c}.$$

Using that $d - c\beta' > a$, we deduce that, for n large enough, we have

$$\mathbb{E}_{p_c + \varepsilon/n^c} [S_n] \geq 2An^a.$$

Therefore, applying Hoeffding's inequality (see [Hoe63]) yields that, for n large enough,

$$\begin{aligned} \mathbb{P}_{p_c + \varepsilon/n^c} \left(B_n^b < An^a \right) &\leq \mathbb{P}_{p_c + \varepsilon/n^c} \left(S_n - \mathbb{E}_{p_c + \varepsilon/n^c} [S_n] < -\frac{1}{2} \mathbb{E}_{p_c + \varepsilon/n^c} [S_n] \right) \\ &\leq \exp \left(-\frac{\mathbb{E}_{p_c + \varepsilon/n^c} [S_n]^2}{2 \lfloor n/(3N) \rfloor^d N^{2d}} \right) \leq \exp \left(-\left[\frac{n}{3N} \right]^d \frac{\varepsilon^{2\beta'}}{2n^{2\beta'c}} \right), \end{aligned}$$

which concludes the proof, using that $\lfloor n/(3N) \rfloor^d \sim n^{d-bd}/3^d$. \square

6.5.3 Conclusion

Combining the lower bound on Z_n obtained in lemma 14 and the results of lemmas 18 and 19, we get the convergence of $n^c(p_n - p_c)$ to 0, using that

$$a - 2\gamma'c > b \quad \text{and} \quad d - bd - 2\beta'c > b.$$

In fact, using also the lower bound of lemma 15, a slightly larger admissible window for c can be obtained, namely

$$0 < c < \min \left(\frac{b}{2\gamma}, \frac{d-a}{\beta}, \frac{\max[a-b, a/d + \min(a-1-b/d, 0)]}{2\gamma}, \frac{d-bd + \max[-b, a/d - \max(1+b/d, a)]}{2\beta} \right),$$

but we have preferred to present the simpler condition in the statement of the theorem, since none of them is optimal anyway.

6.6 An alternative model with cluster diameters

The variant obtained by replacing B_n^b with the function \tilde{B}_n^b defined by (1) can be dealt with using the same techniques. The main difference is that, instead of using theorem 6.75 of [Gri99], we use the theorem 5.4 therein, which states that

$$\forall p < p_c \quad \exists \psi(p) > 0 \quad \forall n \geq 1 \quad \mathbb{P}_p \left(0 \overset{\omega}{\longleftrightarrow} \partial\Lambda(n) \right) \leq e^{-n\psi(p)}.$$

Acknowledgments: We wish to thank an anonymous referee who suggested an improvement of our previous version of lemma 13.

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