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An epistemological and philosophical perspective on the question of mathematical work in the Mathematical Working Space theory

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The theory of Mathematical Working Spaces (ThMWS) is a young theory that appeared in the field of didactics of mathematics a decade ago with the objective of describing and constructing the mathematical work of students in a school context. To this end, the theoretical framework aims to link and closely combine epistemological and cognitive approaches. In this article, we deepen the very notion of mathematical work by focusing first on an epistemological approach that specifies the nature of work in mathematics and then on a philosophical approach that connects work and communication in a dialectical relation. These two approaches highlight the more or less implicit ideological assumptions that guide our research in didactics.

Keywords: Mathematical work, work and communication, semiotics genesis, paradigms.

Introduction

Developed for more than a decade as a methodological and theoretical tool, the Mathematical Working Space model has recently been presented as a theory by eminent researchers in the field of mathematics education. In particular, Radford (2017) uses the theory of MWS together with Brousseau's Theory of Didactic Situations (1997) to exemplify theories in mathematics education which reflect on mathematical knowledge and know-how that the much of the mainstream research in the domain seems to neglect:

However, I think that focusing on knowledge is what educational approaches such as the French didactic theories excel at – for example, the theory of didactic situations or its young sibling theory “the mathematical working space”. (Radford, 2017, p. 503)

This focus on content is essential in the theory of MWS and is reflected in the central place given to “mathematical work”. Moreover, one of the specificities of the theory is that it was conceived to interact with a diversity of constructs and perspectives in the field. This plasticity poses the challenging question of the real nature of its relationships with other theoretical approaches, which may be grounded on very different epistemological and methodological principles. The epistemological stances from a given theoretical model shape the nature of the objects and problems being studied, but according to our view they also influence the capacity of integration of, and exchanges with, other theoretical frameworks. This capacity refers to Kidron’s *epistemological sensitivity* (2016), which is likely to foster the possible networking between different approaches.

In this contribution we would like to focus on this epistemological question which relates to a point highlighted in the call for contributions for this topics group, namely the epistemological dimension of theories for example in terms of *argumentative grammars or background philosophy*. This contribution complements the one presented at CERME10 (Kuzniak & Nechache, 2017) where we showed the interactions between methodological development and the evolution of the MWS

theory. To characterize its *epistemological sensitivity*, we propose to question the epistemological and ideological assumptions underlying our theoretical approach and stemming from the fact that we are interested in the nature of "mathematical work". We will also highlight the specificity of the theory and its originality in the field of didactics of mathematics.

The French environment of didactics of mathematics

We think it is useful to start by presenting the particular French context in which the ThMWS is embedded and developed. Despite some theological quarrels linked to the personality of the various researchers involved in its growth, it must be recognized, that the didactics of mathematics in France is based on a community of theoretical and methodological principles that contribute to establishing a real cultural homogeneity. In a fairly simple way, we can say that the didactics of mathematics in France is supported on three essential pillars associated to symbolic figureheads enjoying strong international recognition (at least at symbolic level through the award of scientific prizes and medals).

- The Theory of Didactic Situations, the foundations of which were laid by Brousseau (1997), where it is essential to link theoretical and empirical research in close connection with the classroom. Its aim is to develop an experimental didactics that goes beyond the classical didactics as described in Comenius' *Magna Didactica*, which Brousseau regularly cites in his presentations and where all guidelines and ideas have been given a priori and are not validated by field studies.
- The Anthropological Theory of Didactics, initiated by Chevallard (1992), is based on an anthropological and holistic vision of learning and teaching acts. This holistic framework, based on the study of praxeologies, aims at a theoretical autonomy of didactics which is independent of other scientific fields such as mathematics, sociology or psychology.
- The Theory of Conceptual Fields developed by Vergnaud (1991) is less well known but brings a psychological and cognitive dimension, based on Piaget's learning theory, to these two theories strongly influenced by mathematical epistemology.

As underlined above, these theories benefit from a kind of international visibility and recognition with the award of the Klein Medal to Brousseau, the Freudenthal Medal to Chevallard and the fact that Vergnaud was one of the founders of PME. The following theoretical contributions are less widely known but very present and influent in the French context:

- Duval's contribution (2006) on semiotics with the development of semiotics registers. In this approach, signs and semiosis are essential because they mediate mathematical objects that are only accessible through their different semiotic representations.
- The ergonomic approach combines Piaget's and Vygotsky's contributions and led to "French Activity Theory" as it is presented in a recent special issue of *Annales de Didactique* (2018). It is also based on a reflection on instruments and tasks and includes the developmental dimension of activity.

It is within this context that the theory of MWS (Kuzniak, Tanguay, & Elia 2016, Gómez-Chacón, Kuzniak, & Vivier 2016) has been developed by seeking to link and preserve very closely the

epistemological and cognitive viewpoints. The epistemological point of view is very focused on mathematical content with a reflection on its organization. While the cognitive viewpoint follows Duval's approach in geometry and focuses on the visible and tangible aspects of the subject's activity. The interaction of the two perspectives was achieved by placing mathematical work at the centre of the reflection. The specific study of the mathematical work is based on three dimensions and geneses: semiotic, instrumental and discursive.

A short presentation of the theoretical framework: The model with its diagrams

The purpose of the theory of Mathematical Working Space is to provide a tool for the specific study of the mathematical work in which students and teachers are effectively engaged during mathematics sessions. The abstract space thus conceived refers to a structure organised in a way that allows the mathematical activity of individuals who are facing mathematical problems.

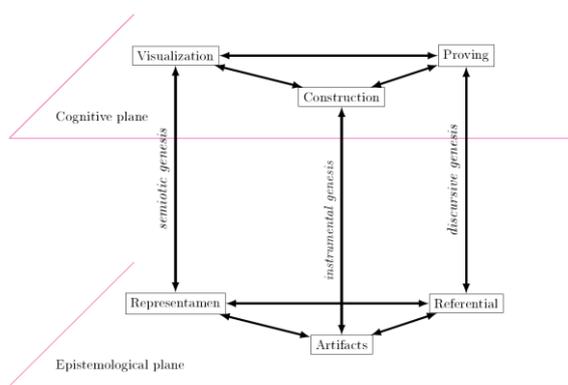


Figure 1: The MWS diagram

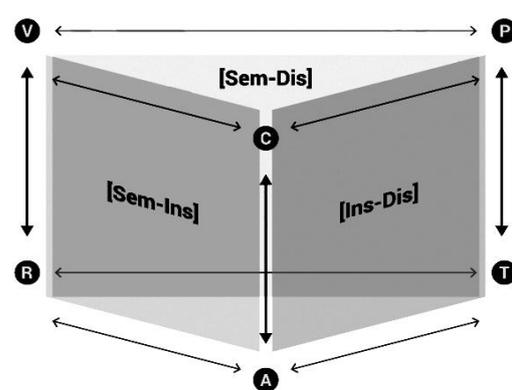


Figure 2: The three vertical planes in the MWS

To grasp the specific activity of students solving problems in mathematics, the two epistemological and cognitive facets of mathematics are present and articulated in the model into two planes: one of an epistemological nature in close relation to the mathematical content in the field being studied; the other of a cognitive nature, related to the visible action of the individual solving problems (Figure 1). Three components in interaction are characterised for the purpose of describing the work in its epistemological dimension, organised according to purely mathematical criteria: a set of concrete and tangible objects (representamen); a set of artefacts such as drawing instruments or software; a theoretical system of reference based on definitions, properties and theorems. The second level of the MWS model is centred on the subject, considered as a cognitive subject. Three cognitive components are introduced as follows: visualisation related to deciphering and interpreting signs; construction depending on the used artefacts and the associated techniques; proving conveyed through actions producing validations, and based on the theoretical frame of reference. Therefore, analysing mathematical work through the lens of MWSs allows tracking down the process of bridging the epistemological plane and the cognitive plane, in accordance with three different dimensions: An instrumental dimension (Ins) related to the experimental and empirical part of work, a discursive dimension (Dis) related to the strategic and theoretical part, and finally a semiotic dimension (Sem) that expresses the importance of signs as an intuitive source and support

for mathematical work. The shaping of work involves different but intertwined genetic developments, each identified as a genesis linked to a specific dimension of the model: semiotic, instrumental and discursive geneses.

In order to understand this complex process, the interactions that are specific to the execution of given mathematical tasks will be associated to the three vertical planes, naturally occurring in the MWS diagram: the [Sem-Dis] plane, conjoining the semiotic genesis and the discursive genesis of proof, the [Ins-Dis] plane, conjoining the instrumental genesis and the discursive genesis of proof, the [Sem-Ins] plane, conjoining the semiotic genesis and the instrumental genesis (Figure 2).

The exact definition and precise description of the nature and dynamics between these planes during the solving of a series of mathematical problems remains a central concern for a deeper understanding of mathematical work. To achieve this description, studies performed within the framework of MWS theory closely examine the design and implementation of classroom teaching situations according to a specific methodology supported on MWS diagrams (Kuzniak & Nechache, 2017).

The necessary clarification of what is “le travail mathématique” in schooling

In the following, our study focuses on the real scope and meaning of “le travail mathématique”, in particular we wish examine the theoretical construction developed in the context of the didactics of mathematics and see its solidity when the same notion is considered from other perspectives, epistemological and philosophical, that a priori do not belong to the world of education. First, we kept the expression “travail mathématique” in French, because one of the first difficulties we encounter in this study is the translation into English of the term "le travail mathématique". We have opted for "mathematical work" which is not its exact equivalent as we will see later. This difficulty in translating the meaning of the term "mathematical work" is found again in the choice of translating, after long discussions, "Espace de Travail Mathématique" into “Mathematical Working Space” to insist on "working" more than on "the workplace".

To achieve this study, we begin by following the easiest and least controversial path, that of epistemology, which makes it possible to question the term "mathematical work" by emphasizing the "mathematical" aspect in order to identify precisely the meaning of mathematics when it is considered through the idea of work. A second, steeper and more controversial path, because it may be political, will make us question ourselves on “work”, the first term of the pair, from a philosophical point of view.

An epistemological perspective on mathematical work: Thurston and Granger

Mathematical work as a recursive and progressive process

The notion of “mathematical work” must be taken as a syntactic whole which combines closely work and mathematics. In order to define the idea of work within our framework, we use the dual approach of Habermas (1969), which stresses both instrumental action and rational choice as constitutive of work.

By « work » or purposive-rational action I understand either instrumental action or rational choice or their conjunction. Instrumental action is governed by technical rules based on

empirical knowledge (...). The conduct of rational choice is governed by strategies based on analytic knowledge. (...) Purposive-rational action realizes defined goals under given conditions. (Habermas, 1969, p. 92-93).

But what exactly is this specific action that allows mathematical work to be identified? According to Thurston (1994), the difficulty in defining mathematics stems from the essentially recursive nature of the mathematical activity. So, he characterizes mathematics as the field that contains as a central core the study of numbers and space geometry, and then he defines this field in extension as what is being produced by the work of mathematicians, “those humans who advance human understanding of mathematics” (Thurston, 1994, p. 162). In this view, mathematics ‘scientificity’ comes from the critical and repeated revision of the research and results generated by mathematics and by those who produce it along processes always open to reassessment and questioning.

These processes are necessarily in the long run, and we may thus evaluate that having access to full mathematical work is not possible without being engaged in such a progressive and recursive process. This idea of mathematical work in constant evolution and construction, even in the school context, is a central point of our vision on mathematics learning and teaching.

The mathematical work and its style

Defining mathematics from the point of view of mathematicians' activity implies to look at the results of this work in order to better understand the nature and contents of mathematics. To achieve this, it will be necessary to study the resulting work, fruit of the work elaborated by mathematicians. This work is a formalisation of abstract concepts that requires a codification of the discourse. Granger (1963) calls style the particular way of presenting rational knowledge by submitting it to codified norms that give objects a specific meaning. These standards help to set the direction of work on problem solving. They make it possible to exclude certain practices by limiting the possibilities of interpretation and therefore exploration of the reader or student. In that sense, the notion relates to the idea of paradigm introduced by Kuhn (1966). For us, a paradigm will stand for a combination of beliefs, convictions, techniques, methods and values that are shared by a scientific group. Gaining access to a mathematical paradigm will involve meeting the mathematicians' work and going through the solutions and solving processes of distinctive problems that qualify as exemplars according to Kuhn. In studies conducted within the framework of the theory of MWS, the exact definition of paradigms is important because it allows revealing and understanding the expectations of teachers and educational institutions and also to explore the real distance between this expected work and the work actually carried out by the students.

A philosophical and critical perspective on mathematical work: Work and communication

In this section, we examine the second part of the formulation "mathematical work", and deal with the question of work. To move forward in this direction, we will rely mainly on the philosopher Fuchs (2016) who published a critical and social essay on work and communication in the age of the digital economy. As we will see his approach joins and sharpens our own conceptions and research on mathematical work developed now in a rich technological and digital environment which has deeply changed the traditional way of working in mathematics especially at school.

On the difficulty to translate the word “travail” in “le travail mathématique”

Fuchs revisits the question of terminology and sense of the word “travail” in different languages by pointing out the distinction between *work* and *labour*. He stresses on the fact that whereas the etymological root of terms such as *work* or *werk* (German) is creating, acting, doing, the etymological root of words such as *labour* and *Arbeit* (German) is toil, slavery and hardship (Fuchs, 2016, p. 17). This etymological distinction is also found in French—on the one hand *ouvrier* (*work*) and on the other hand *travailler* (*labour*). But at the same time the origin of the word “travail” in French is controversial and does not refer to the same roots. The word “travail” seems to have a broader meaning than *work* and the word “*labour*” (*labour*) is rarely used in French. Thus, for example, the equivalent of Labour Day is la “*fête du travail*”.

In the context of our study, we will consider three large dimensions of work close to the French idea of “travail”: work as a long-term process with a certain permanence; the result of work which can be named a work; the goals and stakes of work which will be precisely here of mathematical nature.

On the relationship between work and communication

As we have seen above, the communication of the work of proof is part of the mathematical work, it makes visible the rationality of the arguments presented. But what is the exact nature of this relationship? Fuchs studies four modes of relationship between work and communication. The first two assume a pre-eminence of one over the other, the third proposes their independence and finally the last one proposes a dialectical link between the two. For Fuchs, Habermas, whom we mentioned above, is the main defender of the dualist position. Habermas’ theory of communicative action makes a sharp distinction between on the one hand purposive (instrumental, strategic) action (including work) that is orientated on success and on the other hand communicative action that is orientated on reaching understanding. According to Fuchs, Habermas’ vision is a little naive insofar as it assumes that communication is always part of a positive and rational dynamic.

In addition to this dualist vision of the relationship between communication and work, Fuchs emphasizes authors who favour a dialectical approach to work and communication. Curiously and unexpectedly by us, he underlines the singular contribution of Lev Vygotsky and his Activity Theory and stresses on the fact that Lev Vygotsky (1896–1934) was a Marxist psychologist from Belarus who developed a theory of activity, whose basic point is that human cognition and language are grounded in human activity. As we have pointed out above, this theory is particularly important and natural in France where Vygotsky is considered for his research in psychology which is used in combination with Piaget’s studies. Moreover, the reference to Marx is common among the main French philosophers (from Althusser to Comte-Sponville via Badiou) and is not considered as sulphurous and diabolical as it is the case in some national traditions with an anti-communist or liberal influence. Perhaps, this specific relationship between Activity Theory and Marxism helps to understand the different ways this theory is received depending on countries because of the close relation existing between Marxism and communism.

To come back to the scientific content of Vygotsky’s writings, it should be retained that Vygotsky (1978) argues that. ‘The sign acts as an instrument of psychological activity in a manner analogous to the role of a tool in labor’ (Vygotsky 1978, 82). Tools as means of work and signs as means of

communication have for Vygotsky in common that they are both forms of mediated activity. The difference is for him that the tool is externally oriented on changing nature and the sign is internally oriented: The sign is a means of internal activity aimed at mastering oneself.

Again close to our approach which insists on semiotic aspects, Fuchs also points out a much less well-known Italian author, Rossi-Landi, who favours the semiotic approach as a support for this dialectic between work and communication and introduces semiotic production processes:

Rossi-Landi's semiotic theory (...) helps to clarify the dialectic of work and communication: language-use and communication are work that produce words, sentences, interconnected sentences, arguments, speeches, essays, lectures, books, codes, artworks, literature, science, groups, civilisation and the linguistic world as totality (Fuchs, p. 191).

Thus, this reflection on the dialectics of work and communication is in line with our way of considering mathematical work as the result of different processes: a semiotic process that relates to the treatment of signs; an instrumental process that promotes the link with material or digital tools; finally, a discursive process that relates to rational proof. The singular importance we attach to the discourses of proof reveals more particularly the very nature of the work associated with mathematics where proof, particularly in its demonstration form, is crucial in the realization and coherence of mathematical work.

Conclusion

The Theory of Mathematical Working Spaces aims to describe the forms of mathematical work carried out by students in the school environment. As its name suggests, it places mathematical work at the centre of reflection on teaching and learning. In this context, the primary purpose of educational institutions and teachers is to develop an environment that enables students to solve mathematical problems in an appropriate way. The organization of MWSs into two levels, epistemological and cognitive, allows the description of the work of students confronted with a set of tasks with a mathematical purpose.

In our framework, mathematical work results from the interactions between signs, tools and properties through the interweaving of the three semiotic, instrumental and discursive geneses. We were also able to verify that that our vision of mathematical work based on circulation in the MWS diagram (Figures 1 and 2) has many justifications in the field of didactics of mathematics. The reflexive detour that we have made by relying on contemporary epistemological and philosophical approaches, has also shown us that our issue has encountered some strong questions in these fields such as the relationship between forms of work and communication. Our way of seeing resonates with certain current problems of critical philosophy confronted with the evolution of the world of work in an increasingly technological environment and subject to new rules of production and alienation.

Even if the confrontation of our theory with this political and social approach does not radically transform our way of thinking, it does contribute to better positioning our research in a broader current of thought and above all it helps to reveal the ideology that often underlies some work carried out in our research field in an unconscious or implicit way. In particular, it can explain some

misunderstandings or difficulties in receiving our research: one of the most visible being the understanding and use of the term “mathematical work” which seems doubly provocative in today's world of mathematics education. The use of this term implies for the subject acting to go beyond the only laborious and technical part of work and thus that work focused on mathematics contributes to the intellectual emancipation of students and professors that is still in the making-phase.

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