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# Quasilinear Elliptic Problem in a Two-Component Domain with $L^1$ data

Rheadel G. Fulgencio\* and Olivier Guibé†

## Abstract

In the present paper we consider the following class of quasi-linear equations:

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases}$$

The domain  $\Omega$  is composed of two components,  $\Omega_1$  and  $\Omega_2$ , with  $\Gamma$  being the interface between them. The given function  $f$  belongs to  $L^1(\Omega)$ . We will first give a definition of a renormalized solution for this class of equations. The main result of this paper is the existence of such a solution.

## 1 Introduction

In the present paper, we study the existence of a solution  $u := (u_1, u_2)$  of the following class of quasi-linear equations:

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases} \quad (\text{P})$$

Here,  $\Omega$  is our two-component domain with  $\partial\Omega$  as its boundary. The open sets  $\Omega_1$  and  $\Omega_2$  are the two disjoint components of  $\Omega$  with  $\Gamma$  as the interface between them (see Figure 1) and the vector  $\nu_i$  is the unit outward normal to  $\Omega_i$ . The matrix field  $B$  is coercive but does not satisfy any growth condition, and the data  $f$  is an  $L^1$ -function. On the boundary  $\partial\Omega$ , we have a Dirichlet boundary condition, while on the interface  $\Gamma$ , we have a continuous flux and the jump of the solution is proportional to the flux.

The existence and uniqueness of solution of (P) when  $f \in L^2(\Omega)$  was studied in [3, 10, 11]. In [10, 11] the equations are linear, that is, the matrix field  $B$  does not depend on the solution  $u$ , while in [3], the equations are quasilinear, which is also the case in this study. The above mentioned papers are all motivated by homogenization, which is also our main goal (see [9]).

Since we consider in the present paper an  $L^1$ -data, we need an appropriate notion of solution. Indeed, for the elliptic equation

$$-\operatorname{div}(A(x, u)\nabla u) = f$$

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with Dirichlet boundary condition if the matrix  $A$  is bounded, a solution in the sense of distribution exists (see [5]) but it is not unique in general (see the counterexamples in [19, 20]). If the matrix field is not bounded, then we cannot expect to have a solution in the sense of distribution since there is no reason to have  $A(x, u) \in L^1_{loc}$ . In the present paper, we use the notion of renormalized solution, which is first discussed in [8] by R.J. DiPerna and P.L. Lions for first order equations. This notion was then further developed by F. Murat in [17], by P.L. Lions and F. Murat in [15] for elliptic equations with Dirichlet boundary conditions and  $L^1$  data, and by G. Dal Maso et al. in [6] for elliptic equations with general measure data. There is a wide range of literature for elliptic equations with Dirichlet boundary condition and  $L^1$  data, among them are [4, 5, 6, 7, 15, 17]. Considering elliptic equations with Neumann or Fourier boundary conditions and  $L^1$  data, which are connected to our problem, gives in general additional difficulties due the lack of Poincaré inequality or the low regularity of the solution (definition of the trace for e.g.). In the case of one-component domain (and  $L^1$  data), using the framework of entropy solution existence results are given in [1, 2, 18], convection-diffusion equations with mixed Neumann and Robin boundary conditions are studied in [12] by a duality method, and in [14] the authors prove existence and uniqueness results using the notion of renormalized solution for equations with Robin boundary conditions.

The main originality of the present paper is the jump of the solution which produces in the formulation a term in the interface  $\Gamma$ . Recalling that the regularity of the renormalized solution is given through the truncate, the first difficulty is to give a sense on the interface for functions  $(u_1, u_2)$  whose truncates belong to  $H^1$ . Following the ideas of [1, 14] (but in the case of one-component domain), we define an appropriate notion of trace (see Proposition 2.2). The second difficulty is the regularity of  $\gamma_1(u_1) - \gamma_2(u_2)$  (where  $\gamma_1$  is the trace function for  $H^1(\Omega_1)$ -functions and  $\gamma_2$  is the trace function for  $H^1(\Omega_2)$ -functions), since we have to deal with terms on the boundary like  $(\gamma_1(u_1) - \gamma_2(u_2))\chi_{\{|\gamma_1(u_1)| < k\}}$  (where  $\chi_A$  is the characteristic function of any set  $A$ ) in the renormalized formulation. To have  $(\gamma_1(u_1) - \gamma_2(u_2))\chi_{\{|\gamma_1(u_1)| < k\}}$  belonging to  $L^1(\Gamma)$  is then equivalent to have  $\gamma_2(u_2)\chi_{\{|\gamma_1(u_1)| < k\}} \in L^1(\Gamma)$ , which is unusual and is in some sense a coupled regularity on the boundary. It is worth noting that it is not a direct consequence of  $T_k(u_1) \in H^1(\Omega_1)$  and  $T_k(u_2) \in H^1(\Omega_2)$ . Using the structure of the equation, we give an extra regularity (see (2.11b) in Definition 2.4) which allows one to complete our notion of renormalized solution for problem (P). We are then able to give a definition of renormalized solution for problem (P) for which we prove the existence (see Theorem 3.1).

This paper is organized as follows. The next section discusses the assumptions on our problem and some definitions including the definition of a renormalized solution of (P). Section 3 is devoted to the proof of the existence of a renormalized solution for (P).

## 2 Assumptions and Definitions

In this section, we present the assumptions and definitions necessary for our problem. We begin by introducing the two-component domain  $\Omega$ . The domain  $\Omega$  is a connected bounded open set in  $\mathbb{R}^N$  with its boundary  $\partial\Omega$ . We can write

$\Omega$  as the disjoint union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , where  $\Omega_2$  is an open set such that  $\overline{\Omega_2} \subset \Omega$  with a Lipschitz boundary  $\Gamma$  and  $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ . We denote by  $\nu_i$  the unit outward normal to  $\Omega_i$ .

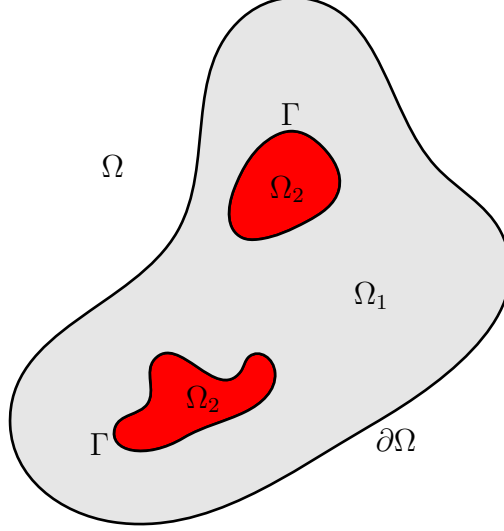


Figure 1: The two-component domain  $\Omega$

If we have a function  $u$  defined on  $\Omega \setminus \Gamma$ , then we denote  $u_i = u|_{\Omega_i}$  the restriction of  $u$  in  $\Omega_i$ . Furthermore, we have the following assumptions:

(A1) The data  $f$  belongs to  $L^1(\Omega)$ .

(A2) The function  $h$  satisfies

$$h \in L^\infty(\Gamma) \quad \text{and} \quad 0 < h_0 < h(y) \text{ a.e. on } \Gamma, \quad (2.1)$$

for some  $h_0 \in \mathbb{R}^+$ .

(A3) The matrix field  $B$  is a Carathéodory function, that is,

- (a) the map  $t \mapsto B(x, t)$  is continuous for a.e.  $x \in \Omega$ ;
- (b) the map  $x \mapsto B(x, t)$  is measurable for a.e.  $t \in \mathbb{R}$ ,

and it has the following properties:

(A3.1)  $B(x, t)\xi \cdot \xi \geq \alpha|\xi|^2$ , for some  $\alpha > 0$ , for a.e.  $x \in \Omega$ ,  $\forall t \in \mathbb{R}$ ,  $\forall \xi \in \mathbb{R}^N$ ;

(A3.2) for any  $k > 0$ ,  $B(x, t) \in L^\infty(\Omega \times (-k, k))^{N \times N}$ .

The space for this class of equations is not a usual  $L^p$ -space or a Sobolev space due to the jump on the interface. We need the normed space  $V$  defined as follows. Let  $V_1$  be the space defined by

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

Define  $V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\}$ , equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2. \quad (2.2)$$

Identifying  $\nabla v := \widetilde{\nabla}v_1 + \widetilde{\nabla}v_2$  we have that  $\|v\|_V^2 = \|\nabla v\|_{L^2(\Omega \setminus \Gamma)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2$ .

**Proposition 2.1** ([16]). *The norm given in (2.2) is equivalent to the norm of  $V_1 \times H^1(\Omega_2)$ , that is, there exist two positive constants  $c_1, c_2$  such that*

$$c_1 \|v\|_V \leq \|v\|_{V_1 \times H^1(\Omega_2)} \leq c_2 \|v\|_V, \quad \forall v \in V.$$

We now define the function  $T_k$ , which is known as the truncation function at height  $\pm k$ . The function  $T_k : \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq -k, \\ t, & \text{if } -k \leq t \leq k, \\ k, & \text{if } t \geq k. \end{cases} \quad (2.3)$$

This function will be crucial in the definition of a renormalized solution of (P).

In the case of  $L^1$ -data, we cannot expect to have the solution  $u$  belonging to  $V$ . In general, in the framework of renormalized solution, the regularity of the solution is given through the regularity of the truncate. So it is necessary in our case to define the gradient and the trace of the solution  $u$ . For the gradient, we follow the definition given in [4]. For the trace, we have to precise the trace of  $u_1$  on  $\Gamma$  and the one of  $u_2$  on  $\Gamma$ . With respect to [1, 14], we have the additional difficulty for  $u_2$  since we do not have the Poincaré inequality.

**Proposition 2.2.** *Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  be a measurable function such that  $T_k(u) \in V$  for every  $k > 0$ . For  $i = 1, 2$ ,*

1. *there exists a unique measurable function  $v_i : \Omega_i \rightarrow \mathbb{R}^N$  such that for all  $k > 0$ ,*

$$\nabla T_k(u_i) = v_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i, \quad (2.4)$$

*where  $\chi_{\{|u_i| < k\}}$  denotes the characteristic function of  $\{x \in \Omega_i : |u_i(x)| < k\}$ . We define  $v_i$  as the gradient of  $u_i$  and write  $v_i = \nabla u_i$ .*

2. *if*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty, \quad (2.5)$$

*then there exists a unique measurable function  $w_i : \Gamma \rightarrow \mathbb{R}$ , for  $i = 1, 2$ , such that for all  $k > 0$ ,*

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma, \quad (2.6)$$

*where  $\gamma_i : H^1(\Omega_i) \rightarrow L^2(\Gamma)$  is the trace operator. We define the function  $w_i$  as the trace of  $u_i$  on  $\Gamma$  and set*

$$\gamma_i(u_i) = w_i, \quad i = 1, 2.$$

*Proof.*

1. This is proved in [4] (see Lemma 2.1).

2. The case  $i = 1$ , or more generally the truncates have a zero trace on a part of the boundary (which allows one to use Poincaré-kind inequality) is presented in [14]. We just have to prove the result for  $i = 2$ .

The uniqueness is in the almost everywhere sense. Note that if we find a function that satisfies (2.6), then the uniqueness of  $w_2$  is assured by the monotonicity of  $T_k$  and the fact that  $w_2$  is finite a.e. in  $\Gamma$ .

By Proposition 2.1, we know that

$$\|T_k(u_2)\|_{H^1(\Omega_2)} \leq c_1 \|T_k(u)\|_V,$$

for some positive constant  $c_1$ , independent of  $k$ . It follows from (2.5) that

$$\|T_k(u_2)\|_{H^1(\Omega_2)}^2 \leq Mk, \quad (2.7)$$

with  $M \in \mathbb{R}^+$  independent of  $k$ . Due to the regularity of  $\Gamma$ ,  $\gamma_2(T_n(u_2))$  is well-defined and

$$\begin{aligned} k^2 \text{meas}_\Gamma\{|\gamma_2(T_k(u_2))| \geq k\} &= \int_{\Gamma \cap \{|T_k(u_2)| \geq k\}} (\gamma_2(T_k(u_2)))^2 d\sigma \\ &\leq \|\gamma_2(T_k(u_2))\|_{L^2(\Gamma)}^2. \end{aligned}$$

Hence, by Trace Theorem and (2.7), we have

$$\begin{aligned} k^2 \text{meas}_\Gamma\{|\gamma_2(T_k(u_2))| \geq k\} &\leq \|\gamma_2(T_k(u_2))\|_{L^2(\Gamma)}^2 \\ &\leq \|T_k(u_2)\|_{L^2(\Omega_2)}^2 + \|\nabla T_k(u_2)\|_{L^2(\Omega_2)}^2 \\ &\leq Mk. \end{aligned}$$

As a result,

$$\text{meas}_\Gamma\{|\gamma_2(T_k(u_2))| \geq k\} \longrightarrow 0 \text{ as } k \longrightarrow 0. \quad (2.8)$$

Define  $\Gamma_n = \{x \in \Gamma : |\gamma_2(T_n(u_2))| < n\}$  for  $n \in \mathbb{N}$ . From (2.8), it follows that

$$\Gamma = \bigcup_{n \geq 1} \Gamma_n \cup A, \quad (2.9)$$

where  $A$  is a subset of  $\Gamma$  with zero measure.

Note that for  $k < n$ , we have  $T_k(T_n(u_2)) = T_k(u_2)$ . Fix  $k > 0$ . Then for every  $n \in \mathbb{N}$  such that  $n > k$ , we have the following equality

$$T_k(\gamma_2(T_n(u_2))) = \gamma_2(T_k(T_n(u_2))) = \gamma_2(T_k(u_2)) \quad \text{a.e. on } \Gamma,$$

and then

$$\gamma_2(T_k(u_2)) = \gamma_2(T_n(u_2)) \quad \text{a.e. on } \Gamma_k. \quad (2.10)$$

Since for every  $n_1 \leq n$ , we have  $\Gamma_{n_1} \subseteq \Gamma_n$ , in view of (2.9) and (2.10), we can define  $w_2$  in the following way:

$$w_2 = \gamma_2(T_n(u_2)) \quad \text{on } \Gamma_n$$

and noting that  $\Gamma = \bigcup_{n \geq 1} \Gamma_n$  (up to measure zero set), we have for any  $k > 0$

$$\gamma_2(T_k(u_2)) = T_k(w_2) \quad \text{a.e. on } \Gamma.$$

This concludes the proof.

□

**Remark 2.3.** In the following, we give an example of a measurable function  $u$  where  $T_k(u) \in V$  but  $u_2$  is not defined on a part of the interface. We consider  $\Omega = (-1, 2)$  with  $\Omega_1 = (-1, 0) \cup (1, 2)$  and  $\Omega_2 = (0, 1)$  (so  $\Gamma = \{0, 1\}$ ), and  $u = (u_1, u_2)$  is defined as

$$u(x) = \begin{cases} u_1(x) = (x+1)(x-2) & \text{if } x \in \Omega_1 \\ u_2(x) = x^{-2} & \text{if } x \in \Omega_2. \end{cases}$$

We have for some positive constants  $C_1, C_2$ ,

$$\|\nabla T_k(u_1)\|_{L^2(\Omega_1)}^2 = \int_{|u_1| < k} (2x-1)^2 dx \leq \int_{\Omega_1} (2x-1)^2 dx \leq C_1,$$

and

$$\|\nabla T_k(u_2)\|_{L^2(\Omega_2)}^2 = \int_{k^{1/2}}^1 (-2x^{-3})^2 dx = 4 \left[ -\frac{x^7}{7} \right]_{x=k^{1/2}}^1 = \frac{4}{7}(k^{7/2} - 1).$$

Thus, we can see that

$$\frac{k^{7/2}}{C} \leq \|T_k(u)\|_V^2 \leq Ck^{7/2},$$

for some  $C > 0$  but clearly  $u_2$  does not have a trace on  $\{0\} \subset \Gamma$ .

We are now in a position to give the definition of renormalized solution.

**Definition 2.4.** Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  be a measurable function. Then  $u$  is a renormalized solution of (P) if

$$T_k(u) \in V, \quad \forall k > 0; \quad (2.11a)$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \quad (2.11b)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u dx = 0; \quad (2.12a)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) d\sigma = 0; \quad (2.12b)$$

and for any  $S \in C^1(\mathbb{R})$  (or equivalently for any  $S \in W^{1,\infty}(\mathbb{R})$ ) with compact support,  $u$  satisfies

$$\begin{aligned} & \int_{\Omega_1} S(u_1) B(x, u_1) \nabla u_1 \cdot \nabla v_1 dx + \int_{\Omega_1} S'(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 v_1 dx \\ & + \int_{\Omega_2} S(u_2) B(x, u_2) \nabla u_2 \cdot \nabla v_2 dx + \int_{\Omega_2} S'(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 v_2 dx \quad (2.13) \\ & + \int_{\Gamma} h(x)(u_1 - u_2)(v_1 S(u_1) - v_2 S(u_2)) d\sigma = \int_{\Omega} f v S(u) dx, \end{aligned}$$

for all  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ .

**Remark 2.5.** Conditions (2.11a) (the regularity of the truncate) and (2.12a) (the decay of the "truncated energy") are standard in the framework of renormalized solutions. As mentioned in Introduction, the main originality of the present paper is the presence of the trace.

In view of Proposition 2.2,  $\gamma(u_1)$  and  $\gamma(u_2)$  are well-defined. Condition (2.11b) is an extra regularity of  $(u_1 - u_2)(T_k(u_1) - T_k(u_2))$ . Indeed,  $(u_1 - u_2)(T_k(u_1) - T_k(u_2))$  cannot be written as

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2))\chi_{\{|u_1|<n\}}\chi_{\{|u_2|<n\}},$$

for any  $n \in \mathbb{R}^+$ , so that having  $(u_1 - u_2)(T_k(u_1) - T_k(u_2))$  belonging to  $L^1(\Gamma)$  is not a consequence of (2.11a).

Conditions (2.11a) and (2.11b) allow one to give a sense of all the terms in (2.13). Let  $S \in C^1(\mathbb{R})$  with compact support. Then for all  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ ,  $i = 1, 2$ , we have if  $\text{supp}(h) \subset [-k, k]$

$$\begin{aligned} S(u_i)B(x, u_i)\nabla u_i \cdot \nabla v_i &= S(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla v_i \in L^1(\Omega_i), \\ S'(u_i)B(x, u_i)\nabla u_i \cdot \nabla u_i v_i &= S'(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla T_k(u_i) v_i \in L^1(\Omega_i), \\ fvS(u) &\in L^1(\Omega). \end{aligned}$$

For the boundary term, let us define  $S_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$S_n(s) = \begin{cases} 0, & \text{if } s \leq -2n \\ \frac{s}{n} + 2, & \text{if } -2n \leq s \leq -n \\ 1, & \text{if } -n \leq s \leq n \\ -\frac{s}{n} + 2, & \text{if } n \leq s \leq 2n \\ 0, & \text{if } s \geq 2n, \end{cases} \quad (2.14)$$

then since  $S$  has a compact support, for some large enough  $n$ , we have

$$\begin{aligned} h(u_1 - u_2)v_1S(u_1) &= hv_1(u_1 - u_2)(S(u_1) - S(u_2))S_n(u_1) \\ &\quad + hv_1(u_1 - u_2)S(u_2)S_n(u_1). \end{aligned}$$

Since both  $S$  and  $S_n$  have compact support, we have that  $hv_1(u_1 - u_2)S(u_2)S_n(u_1)$  is bounded and is therefore in  $L^1(\Gamma)$ . Moreover, since

$$S(u_1) - S(u_2) = S(T_{2n}(u_1)) - S(T_{2n}(u_2))$$

and  $S$  is Lipschitz, we have

$$\begin{aligned} |hv_1(u_1 - u_2)(S(u_1) - S(u_2))S_n(u_1)| &\leq \|hv_1\|_{L^\infty(\Gamma)} \|S'\|_{L^\infty(\mathbb{R})} \\ &\quad \times |u_1 - u_2| |T_{2n}(u_1) - T_{2n}(u_2)|, \end{aligned}$$

a.e. in  $\Gamma$ . Thus, in view of (2.11b),  $h(u_1 - u_2)v_1S(u_1) \in L^1(\Gamma)$ . Similarly,  $h(u_1 - u_2)v_2S(u_2) \in L^1(\Gamma)$ .

It is worth noting that condition (2.11b) is equivalent to have

$$u_2\chi_{\{|u_1|<k\}} \in L^1(\Gamma) \quad \text{and} \quad u_1\chi_{\{|u_2|<k\}} \in L^1(\Gamma), \quad (2.15)$$

for any  $k > 0$ . Indeed,

$$\begin{aligned} u_2\chi_{\{|u_1|<k\}} &= (u_2 - u_1)\chi_{\{|u_1|<k\}}(S_n(u_1) - S_n(u_2)) + u_2S_n(u_2)\chi_{\{|u_1|<k\}} \\ &\quad + u_1S_n(u_1)\chi_{\{|u_1|<k\}} - u_1S_n(u_2)\chi_{\{|u_1|<k\}}, \end{aligned}$$



and by condition (2.11b), the first term on the right-hand side belongs to  $L^1(\Gamma)$  while the next 3 terms are bounded and thus also belong to  $L^1(\Gamma)$ .

Finally, let us comment that conditions (2.12a) and (2.12b) which are crucial to obtain uniqueness results (see other paper [13]) and also to recover that formally, for any  $k > 0$ ,  $T_k(u)$  is an admissible function in (P), that is,

$$\begin{aligned} \int_{\Omega_1} B(x, u_1) \nabla u_1 \nabla T_k(u_1) dx + \int_{\Omega_2} B(x, u_2) \nabla u_2 \nabla T_k(u_2) dx \\ + \int_{\Gamma} h(x)(u_1 - u_2)(T_k(u_1) - T_k(u_2)) d\sigma = \int_{\Omega} f T_k(u_1) dx. \end{aligned}$$

To prove this, fix  $k > 0$ . Using  $S_n(u)T_k(u)$  which is an admissible test function for any  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \int_{\Omega_1} S_n(u_1) B(x, u_1) \nabla u_1 \cdot \nabla T_k(u_1) dx \\ + \int_{\Omega_1} S'_n(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 T_k(u_1) dx \\ + \int_{\Omega_2} S_n(u_2) B(x, u_2) \nabla u_2 \cdot \nabla T_k(u_2) dx \\ + \int_{\Omega_2} S'_n(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 T_k(u_2) dx \\ + \int_{\Gamma} h(x)(u_1 - u_2)(S_n(u_1)T_k(u_1) - S_n(u_2)T_k(u_2)) d\sigma \\ = \int_{\Omega} f T_k(u) S_n(u) dx. \end{aligned} \tag{2.16}$$

Condition (2.12a) allows one to pass to the limit of the second and fourth integral in (2.16) while condition (2.12b) is useful for passing to the limit of the integral on the boundary in (2.16).

**Remark 2.6.** As observed in the previous remark, we have the condition (2.11b) for the integral on the interface to make sense. We can avoid this problem by using the Boccardo-Gallouët estimates presented in [5]. However, these estimates are heavily dependent on the Sobolev constants. With the final aim of doing the homogenization process, we try as much as possible to refrain from using these estimates (see Remark 3.2).

### 3 Existence Results

In this section, we present the proof for the existence of a renormalized solution of (P).

**Theorem 3.1.** *Suppose the assumptions (A1)-(A3) hold. Then there exists a renormalized solution to (P) in the sense of Definition 2.4.*

*Proof.* The proof is divided into 4 steps. In Step 1, we consider an approximate problem (see  $(P_\varepsilon)$ ) in which  $B$  is approximated and  $f^\varepsilon$  is an  $L^2$ -data. Using Schauder's fixed point theorem, the existence of at least a variational solution of  $(P_\varepsilon)$  can be shown. Step 2 is devoted to prove some a priori estimates and then

extracting a subsequence. In Step 3, we prove that conditions (2.11a), (2.11b), (2.12a) and (2.12b) are satisfied. Finally, in Step 4, we pass to the limit and we show that the constructed function is a renormalized solution.

From this point until the end of the proof, we let  $i \in \{1, 2\}$ .

**Step 1:** *Introducing the approximate problem and showing the existence of solution of the approximate problem*

Let  $\varepsilon > 0$ . Suppose  $\{f^\varepsilon\} \subset L^2(\Omega)$  such that

$$f^\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega)$$

as  $\varepsilon \rightarrow 0$ . Define  $B_\varepsilon(x, t) = B(x, T_{1/\varepsilon}(t))$ . We now consider the following approximate problem

$$\begin{cases} -\operatorname{div}(B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{in } \Omega_1, \\ -\operatorname{div}(B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{in } \Omega_2, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma, \end{cases} \quad (\text{P}_\varepsilon)$$

The variational formulation of problem  $(\text{P}_\varepsilon)$  is the following

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that } \forall \varphi \in V \\ \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon \cdot \nabla \varphi_1 \, dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon \cdot \nabla \varphi_2 \, dx \\ \quad + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1 - \varphi_2) \, d\sigma = \int_{\Omega} f^\varepsilon \varphi \, dx. \end{cases} \quad (3.1)$$

Using Proposition 2.1 and Schauder's Fixed Point Theorem, the proof of the existence of solution for (3.1) is quite standard (see e.g. [3]).

**Step 2:** *Extracting subsequences and examining convergences*

Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a solution to the approximate problem  $(\text{P}_\varepsilon)$ . By Stampacchia's theorem, for  $k > 0$ ,  $T_k(u^\varepsilon) \in V$  since  $u^\varepsilon \in V$ . Using  $T_k(u^\varepsilon)$  as a test function in the variational formulation (3.1), we have

$$\begin{aligned} & \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon \nabla T_k(u_1^\varepsilon) \, dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon \nabla T_k(u_2^\varepsilon) \, dx \\ & \quad + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \, d\sigma = \int_{\Omega} f^\varepsilon T_k(u^\varepsilon) \, dx. \end{aligned} \quad (3.2)$$

By the definition of  $T_k$ , the coercivity of  $B$ , and the assumption on  $h$ , we have

$$\begin{aligned} & \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon \nabla T_k(u_1^\varepsilon) \, dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon \nabla T_k(u_2^\varepsilon) \, dx \\ & \quad + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \, d\sigma \\ & \geq \alpha \|\nabla T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 + \alpha \|\nabla T_k(u_2^\varepsilon)\|_{L^2(\Omega_2)}^2 + h_0 \|T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)\|_{L^2(\Gamma)}^2 \\ & \geq C_1 \|T_k(u^\varepsilon)\|_V^2, \end{aligned}$$

for some positive constant  $C_1$ . On the other hand, by Hölder inequality,

$$\begin{aligned} \left| \int_{\Omega} f^{\varepsilon} T_k(u^{\varepsilon}) dx \right| &= \left| \int_{\Omega_1} f^{\varepsilon} T_k(u_1^{\varepsilon}) dx + \int_{\Omega_2} f^{\varepsilon} T_k(u_2^{\varepsilon}) dx \right| \\ &\leq \|f^{\varepsilon}\|_{L^1(\Omega)} k \\ &\leq Mk, \end{aligned}$$

for some positive constant  $M$ , which is independent of  $\varepsilon$ .

Thus,

$$\|T_k(u^{\varepsilon})\|_V^2 \leq \frac{Mk}{C_1}, \quad (3.3)$$

that is, the sequence  $\{T_k(u^{\varepsilon})\}$  is bounded in  $V$  for every  $k > 0$ .

By the Rellich theorem, the inclusions  $V \hookrightarrow L^2(\Omega_1) \times L^2(\Omega_2)$  and  $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$  are compact. Consequently, since  $\{T_k(u^{\varepsilon})\}$  is bounded in  $V$  for every  $k > 0$  (countable), by the diagonal process, we can extract a subsequence of  $\{T_k(u^{\varepsilon})\}$  such that for any  $k > 0$  ( $k$  being a rational number), there is a  $v_k \in V$  such that for

$$\begin{cases} T_k(u_i^{\varepsilon'}) \longrightarrow v_{k,i} & \text{strongly in } L^2(\Omega_i), \text{ a.e. in } \Omega_i, \\ T_k(u_i^{\varepsilon'}) \rightharpoonup v_{k,i} & \text{weakly in } V, \\ \gamma_i(T_k(u_i^{\varepsilon'})) \longrightarrow \gamma_i(v_{k,i}) & \text{strongly in } L^2(\Gamma), \text{ a.e. on } \Gamma. \end{cases} \quad (3.4)$$

Now, we show that  $\{u_i^{\varepsilon'}\}$  and  $\{\gamma_i(u_i^{\varepsilon'})\}$  are Cauchy sequences in measure. For  $u_i^{\varepsilon'}$ , we follow the arguments developed in [4]. For  $\gamma_i(u_i^{\varepsilon'})$ , we have additional difficulties which are overcome by using Proposition 2.1. Note that we have

$$\begin{aligned} \|T_k(u_i^{\varepsilon'})\|_{L^2(\Omega_i)}^2 &= \int_{\{|u_i^{\varepsilon'}| \geq k\}} |T_k(u_i^{\varepsilon'})|^2 dx + \int_{\{|u_i^{\varepsilon'}| < k\}} |T_k(u_i^{\varepsilon'})|^2 dx \\ &= \int_{\{|u_i^{\varepsilon'}| \geq k\}} k^2 dx + \int_{\{|u_i^{\varepsilon'}| < k\}} |u_i^{\varepsilon'}|^2 dx. \end{aligned}$$

It follows by Poincaré inequality, Proposition 2.1, and (3.3), we have

$$\begin{aligned} k^2 \text{meas}\{|u^{\varepsilon'}| \geq k\} &= \int_{\{|u_1^{\varepsilon'}| \geq k\}} k^2 dx + \int_{\{|u_2^{\varepsilon'}| \geq k\}} k^2 dx \\ &\leq \|T_k(u_1^{\varepsilon'})\|_{L^2(\Omega_1)}^2 + \|T_k(u_2^{\varepsilon'})\|_{L^2(\Omega_2)}^2 \\ &\leq C_3 \|\nabla T_k(u_1^{\varepsilon'})\|_{L^2(\Omega_1)}^2 + \|T_k(u_2^{\varepsilon'})\|_{H^1(\Omega_2)}^2 \\ &\leq C_4 \|T_k(u^{\varepsilon'})\|_V^2 \\ &\leq \frac{C_4 Mk}{C_1}. \end{aligned}$$

for some  $C_3, C_4 \in \mathbb{R}^+$ . Thus, we can find a positive constant  $C$  independent of  $\varepsilon$  such that

$$\text{meas}\{|u_i^{\varepsilon'}| \geq k\} \leq \frac{C}{k} \quad (3.5)$$

For  $\gamma_1(u_1^{\varepsilon'})$ , observe that by Poincaré inequality and (3.3),

$$\begin{aligned}
k^2 \text{meas}_\Gamma\{|\gamma_1(u_1^{\varepsilon'})| \geq k\} &= \int_{\{|\gamma_1(u_1^{\varepsilon'})| \geq k\}} k^2 d\sigma \\
&= \int_{\{|\gamma_1(u_1^{\varepsilon'})| \geq k\}} \gamma_1(T_k(u_1^{\varepsilon'}))^2 d\sigma \\
&\leq \|\gamma_1(T_k(u_1^{\varepsilon'}))\|_{L^2(\Gamma)}^2 \\
&\leq C_5 \|\nabla T_k(u_1^{\varepsilon'})\|_{L^2(\Omega_2)}^2 \\
&\leq C_6 k.
\end{aligned}$$

Consequently,

$$\text{meas}_\Gamma\{|\gamma_1(u_1^{\varepsilon'})| \geq k\} \leq \frac{C_6}{k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.6)$$

For  $\gamma_2(u_2^{\varepsilon'})$ , by the Trace Theorem, Proposition 2.1, and (3.3), we have

$$\begin{aligned}
k^2 \text{meas}_\Gamma\{|\gamma_2(u_2^{\varepsilon'})| \geq k\} &= \int_{\{|\gamma_2(u_2^{\varepsilon'})| \geq k\}} k^2 d\sigma \\
&= \int_{\{|\gamma_2(u_2^{\varepsilon'})| \geq k\}} \gamma_2(T_k(u_2^{\varepsilon'}))^2 d\sigma \\
&\leq \|\gamma_2(T_k(u_2^{\varepsilon'}))\|_{L^2(\Gamma)}^2 \\
&\leq C_7 \|T_k(u_2^{\varepsilon'})\|_{H^1(\Omega_2)}^2 \\
&\leq C_8 k.
\end{aligned}$$

It follows that

$$\text{meas}_\Gamma\{|\gamma_2(u_2^{\varepsilon'})| \geq k\} \leq \frac{C_8}{k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.7)$$

By (3.6) and (3.7), for every  $\eta > 0$ , there exists  $k_0$  such that for every  $k \geq k_0$ ,

$$\text{meas}_\Gamma\{x \in \Gamma; |\gamma_i(u_i^{\varepsilon'})| \geq k\} < \eta. \quad (3.8)$$

Let  $\omega, \eta > 0$ . By (3.5) and (3.8), we can find  $k$  large enough such that

$$\text{meas}\{|u_i^{\varepsilon'}| \geq k\} \leq \frac{\eta}{3} \quad (3.9)$$

$$\text{meas}_\Gamma\{x \in \Gamma; |\gamma_i(u_i^{\varepsilon'})| \geq k\} \leq \frac{\eta}{3}, \quad (3.10)$$

for every  $\varepsilon' > 0$ . Note that from (3.4), we can deduce that the sequences  $\{T_k(u_i^{\varepsilon'})\}$ ,  $\{\gamma_i(T_k(u_i^{\varepsilon'}))\}$  are Cauchy in measure. Hence, there exists  $\varepsilon_0 > 0$  such that

$$\text{meas}\{|T_k(u_i^{\varepsilon'}) - T_k(u_i^{\varepsilon''})| \geq \omega\} < \frac{\eta}{3} \quad (3.11)$$

$$\text{meas}_\Gamma\{|\gamma_i(T_k(u_i^{\varepsilon'})) - \gamma_i(T_k(u_i^{\varepsilon''}))| \geq \omega\} < \frac{\eta}{3}, \quad (3.12)$$

for every  $0 < \varepsilon', \varepsilon'' < \varepsilon_0$ .

Observe that

$$\{|u_i^{\varepsilon'} - u_i^{\varepsilon''}| \geq \omega\} \subset \{|u_i^{\varepsilon'}| \geq k\} \cup \{|u_i^{\varepsilon''}| \geq k\} \cup \{|T_k(u_i^{\varepsilon'}) - T_k(u_i^{\varepsilon''})| \geq \omega\}$$

and thus,

$$\begin{aligned} \text{meas}\{|u_i^{\varepsilon'} - u_i^{\varepsilon''}| \geq \omega\} &\leq \text{meas}\{|u_i^{\varepsilon'}| \geq k\} + \text{meas}\{|u_i^{\varepsilon''}| \geq k\} \\ &\quad + \text{meas}\{|T_k(u_i^{\varepsilon'}) - T_k(u_i^{\varepsilon''})| \geq \omega\}. \end{aligned}$$

It follows from (3.9) and (3.11) that

$$\text{meas}\{|u_i^{\varepsilon'} - u_i^{\varepsilon''}| \geq \omega\} < \eta,$$

that is,  $\{u_i^{\varepsilon'}\}$  is indeed Cauchy in measure. Using the inequalities (3.10) and (3.12), and similar arguments, it can be shown that  $\{\gamma_i(u_i^{\varepsilon'})\}$  is Cauchy in measure.

Consequently, there is a subsequence of  $\{u_i^{\varepsilon'}\}$  that is convergent a.e. to some measurable function  $u_i : \Omega_i \rightarrow \overline{\mathbb{R}}$ , that is

$$u_i^{\varepsilon'} \rightarrow u_i \quad \text{a.e. in } \Omega_i. \quad (3.13)$$

It follows from (3.5) that  $u_i$  is finite a.e. in  $\Omega_i$ . This  $u := (u_1, u_2)$  is our candidate for a renormalized solution for problem (P).

We now prove that  $u$  satisfies the conditions (2.11). Indeed, by the continuity of  $T_k$ , we have

$$T_k(u_i^{\varepsilon'}) \rightarrow T_k(u) = v_k \in V \quad \text{a.e. in } \Omega \setminus \Gamma. \quad (3.14)$$

Moreover, we can deduce that  $\{\gamma_i(u_i^{\varepsilon'})\}$  is convergent a.e. on  $\Gamma$  up to a subsequence. Hence there exists  $\omega_i : \Gamma \rightarrow \overline{\mathbb{R}}$  such that

$$\gamma_i(u_i^{\varepsilon'}) \rightarrow \omega_i \quad \text{a.e. on } \Gamma, \quad (3.15)$$

with  $\omega_i$  finite a.e. on  $\Gamma$  by (3.8). We now identify  $w_i$  and  $\gamma_i(u_i)$ . Using (3.3) and (3.4), we obtain

$$\frac{1}{k} \|T_k(u)\|_V^2 \leq \frac{M}{C_1},$$

for any  $k > 0$ .

By Proposition 2.2,  $\gamma_i(u_i)$  (the trace in the truncate sense) is well defined. From (3.4), (3.14) and (3.15), we obtain that for any  $k > 0$ ,

$$T_k(\omega_i) = \gamma_i(v_{k,i}) = \gamma_i(T_k(u_i)) = T_k(\gamma_i(u_i)) \quad \text{a.e. on } \Gamma.$$

Then we have  $\omega_i = \gamma_i(u_i)$  a.e. on  $\Gamma$ . By Fatou's Lemma,  $T_k$  being non-decreasing, we have for all  $k > 0$ ,

$$\int_{\Gamma} (u_1 - u_2)(T_k(u_1) - T_k(u_2)) \, d\sigma \leq \liminf_{\varepsilon' \rightarrow 0} \int_{\Gamma} (u_1^{\varepsilon'} - u_2^{\varepsilon'})(T_k(u_1^{\varepsilon'}) - T_k(u_2^{\varepsilon'})) \, d\sigma \leq kM,$$

which is (2.11b).

From this point, we just denote our sequence by  $\varepsilon$ . Rewriting all the results we got in terms of  $\varepsilon$ , we have the following: for all  $k > 0$ ,

$$\begin{cases} u_i^\varepsilon \longrightarrow u_i & \text{a.e. in } \Omega \\ T_k(u_i^\varepsilon) \longrightarrow T_k(u_i) & \text{strongly in } L^2(\Omega_i), \text{ a.e. in } \Omega_i, \\ \gamma_i(u_i^\varepsilon) \longrightarrow \gamma_i(u_i) & \text{a.e. on } \Gamma, \\ \gamma_i(T_k(u_i^\varepsilon)) \longrightarrow \gamma_i(T_k(u_i)) & \text{strongly in } L^2(\Gamma), \text{ a.e. in } \Gamma. \end{cases} \quad (3.16)$$

In addition, we have

$$\nabla T_k(u_i^\varepsilon) \rightharpoonup \nabla T_k(u_i) \quad \text{weakly in } (L^2(\Omega_i))^N. \quad (3.17)$$

**Step 3:** *Showing conditions (2.12) of Definition 2.4.*

From the continuity of  $B$  and (3.16), we have that for any fixed  $n > 0$ ,

$$B(x, T_n(u^\varepsilon)) \longrightarrow B(x, T_n(u)) \quad \text{a.e. in } \Omega \text{ and in } L^\infty(\Omega \setminus \Gamma) \text{ weak-}^*. \quad (3.18)$$

Due to (A3.1) and the lower semi-continuity of the weak convergence,

$$\begin{aligned} \frac{1}{n} \int_{\{|u|<n\}} B(x, u) \nabla u \cdot \nabla u \, dx &= \frac{1}{n} \int_{\Omega \setminus \Gamma} B(x, T_n(u)) \nabla T_n(u) \cdot \nabla T_n(u) \, dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) \, dx, \end{aligned}$$

and by Fatou's Lemma,

$$\frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u)) \, d\sigma \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma.$$

Since

$$\int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) \, dx \quad \text{and} \quad \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma$$

are nonnegative, it is sufficient to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left( \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) \, dx \right. \\ \left. + \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma \right) = 0. \end{aligned} \quad (3.19)$$

We use  $\frac{1}{n} T_n(u^\varepsilon)$  as a test function in (3.1) to arrive at

$$\begin{aligned} \frac{1}{n} \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \nabla T_n(u_1^\varepsilon) \, dx + \frac{1}{n} \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \nabla T_n(u_2^\varepsilon) \, dx \\ + \frac{1}{n} \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma = \frac{1}{n} \int_{\Omega} f^\varepsilon T_n(u^\varepsilon) \, dx. \end{aligned}$$

Consequently, for  $\varepsilon$  small enough, we have

$$\begin{aligned} \frac{1}{n} \int_{\Omega_1} B(x, T_n(u_1^\varepsilon)) \nabla u_1^\varepsilon \nabla T_n(u_1^\varepsilon) \, dx + \frac{1}{n} \int_{\Omega_2} B(x, T_n(u_2^\varepsilon)) \nabla u_2^\varepsilon \nabla T_n(u_2^\varepsilon) \, dx \\ + \frac{1}{n} \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma = \frac{1}{n} \int_{\Omega} f^\varepsilon T_n(u^\varepsilon) \, dx. \end{aligned}$$

Furthermore, since  $T_n(u^\varepsilon)$  converges to  $T_n(u)$  in  $L^\infty$  weak-\* and  $f^\varepsilon$  converges to  $f$  in  $L^1(\Omega)$ , we have

$$\frac{1}{n} \int_{\Omega} f^\varepsilon T_n(u^\varepsilon) dx \longrightarrow \frac{1}{n} \int_{\Omega} f T_n(u) dx \quad \text{as } \varepsilon \longrightarrow 0.$$

It follows that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left( \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) dx \right. \\ \left. + \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon) (T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) = \frac{1}{n} \int_{\Omega} f T_n(u) dx. \end{aligned}$$

Observe that since  $u$  is finite a.e.,

$$\frac{1}{n} T_n(u) \longrightarrow 0 \quad \text{a.e. in } \Omega \setminus \Gamma.$$

In addition, for any  $n > 0$ ,  $|T_n(u)| \leq n$  a.e. and thus,

$$\left| \frac{1}{n} f T_n(u) \right| \leq |f| \in L^1(\Omega).$$

By the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} f T_n(u) dx = 0,$$

which gives (3.19).

**Step 4.** Show that  $u$  satisfies (2.13) of Definition 2.4.

Let  $S \in C^1(\mathbb{R})$  with compact support and let  $k > 0$  such that

$$\text{supp } S \subset [-k, k]. \quad (3.20)$$

We need to show that for any  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ ,  $u$  satisfies (2.13).

We use the function  $S_n$  defined in (2.14). Note that

$$S_n(u_i^\varepsilon) = S_n(T_{2n}(u_i^\varepsilon)) \in H^1(\Omega_i) \cap L^\infty(\Omega_i)$$

and thus, for any  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ ,

$$\psi = v S(u) S_n(u^\varepsilon) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2)).$$

Using  $\psi$  as a test function in (3.1), we have

$$I_{11} + I_{12} + I_{21} + I_{22} + I_{31} + I_{32} + I_4 = I_5, \quad (3.21)$$

where

$$\begin{aligned} I_{1i} &= \int_{\Omega_i} B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla v_i S(u_i) S_n(u_i^\varepsilon) dx \\ I_{2i} &= \int_{\Omega_i} B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla u_i S'(u_i) S_n(u_i^\varepsilon) dx \\ I_{3i} &= \int_{\Omega_i} B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon S(u_i) S_n'(u_i^\varepsilon) dx \\ I_4 &= \int_{\Gamma} h(x) (u_1^\varepsilon - u_2^\varepsilon) (v_1 S(u_1) S_n(u_1^\varepsilon) - v_2 S(u_2) S_n(u_2^\varepsilon)) d\sigma \\ I_5 &= \int_{\Omega} f v S(u) S_n(u^\varepsilon) dx. \end{aligned}$$

We look at the behavior of each integral. In particular, we will pass to the limit as  $\varepsilon \rightarrow 0$  and then as  $n \rightarrow \infty$ .

Note that for  $n \geq k$ , we have

$$S_n(s)S(s) = S(s) \quad \text{and} \quad S_n(s)S'(s) = S'(s), \quad \text{for a.e. } s \in \mathbb{R}. \quad (3.22)$$

We first look at  $I_{1i}$ . Observe that if  $\varepsilon$  is small enough, we have

$$B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla v_i S(u_i) S_n(u_i^\varepsilon) = B(x, T_{1/\varepsilon}(u_i^\varepsilon)) \nabla T_{2n}(u_i^\varepsilon) S(u_i) S_n(u_i^\varepsilon).$$

Choosing  $\varepsilon$  small enough, we have

$$B(x, T_{1/\varepsilon}(u_i^\varepsilon)) S_n(u_i^\varepsilon) = B(x, T_{2n}(u_i^\varepsilon)) S_n(u_i^\varepsilon) \rightarrow S_n(u_i) B(x, T_n(u_i)),$$

a.e. in  $\Omega_i$ . Moreover, by the assumptions on  $B$ , we have

$$|B(x, T_{1/\varepsilon}(u_i^\varepsilon)) S_n(u_i^\varepsilon)| \leq \sup_{\Omega_i \times [-2n, 2n]} |B(x, t)|.$$

It follows from the Lebesgue Dominated Convergence Theorem that

$$B(x, T_{1/\varepsilon}(u_i^\varepsilon)) S_n(u_i^\varepsilon) = B(x, T_{2n}(u_i^\varepsilon)) S_n(u_i^\varepsilon) \rightarrow S_n(u_i) B(x, T_{2n}(u_i)).$$

a.e. in  $\Omega_i$  and in  $L^\infty(\Omega_i)$  weak-\*. This and (3.17) imply as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} I_{1i} &\rightarrow \int_{\Omega_i} B(x, T_{2n}(u_i)) \nabla T_{2n}(u_i) \nabla v_i S(u_i) S_n(u_i) dx \\ &= \int_{\Omega_i} B(x, u_i) \nabla u_i \nabla v_i S(u_i) S_n(u_i) dx. \end{aligned}$$

By (3.22) we have,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{1i} = \int_{\Omega_i} B(x, u_i) \nabla u_i \nabla v_i S(u_i) dx. \quad (3.23)$$

We now observe the behavior of  $I_{2i}$ . For small enough  $\varepsilon$ , we have

$$B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i v_i S'(u_i) S_n(u_i^\varepsilon) = B(x, T_{2n}(u_i^\varepsilon)) \nabla T_{2n}(u_i^\varepsilon) \nabla u_i v_i S'(u_i) S_n(u_i^\varepsilon),$$

a.e. in  $\Omega_i$ . Since  $\nabla u_i v_i S'(u_i) = \nabla T_{2n}(u_i) v S'(u_i) \in (L^2(\Omega_i))^N$  and by (3.17), we obtain as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} I_{2i} &\rightarrow \int_{\Omega \setminus \Gamma} B(x, T_{2n}(u)) \nabla T_{2n}(u) \nabla u v S'(u) S_n(u) dx \\ &= \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \nabla u v S'(u) S_n(u) dx. \end{aligned}$$

By (3.22),

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{2i} = \int_{\Omega \setminus \Gamma} B(x, u) \nabla u \cdot \nabla u v S'(u) dx. \quad (3.24)$$

For the behavior of  $I_{3i}$ , we observe that

$$|S'_n(s)| \leq \frac{1}{n}, \quad \text{for } |s| \leq 2n.$$



Consequently,

$$|I_{3i}| \leq \frac{\|v_i\|_{L^\infty(\Omega_i)} \|S\|_{L^\infty(\mathbb{R})}}{n} \int_{\{|u_i^\varepsilon| < 2n\}} B(x, u_i^\varepsilon) \nabla T_{2n}(u_i^\varepsilon) \nabla T_{2n}(u_i^\varepsilon) dx.$$

By (3.19), we have

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{3i} = 0. \quad (3.25)$$

For  $I_4$ , we note that

$$h(x)(u_1^\varepsilon - u_2^\varepsilon)v_i S(u_i) S_n(u_i^\varepsilon) = h(x)(u_1^\varepsilon - u_2^\varepsilon)v_i S(u_i) S_n(u_i^\varepsilon) S_{2n}(u_i^\varepsilon).$$

Then we can write  $I_4$  as

$$I_4 = I_{41} + I_{42} + I_{43} - I_{44},$$

where

$$\begin{aligned} I_{41} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_1 S(u_1) S_{2n}(u_1^\varepsilon) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) d\sigma \\ I_{42} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_1 S(u_1) S_n(u_2^\varepsilon) S_{2n}(u_1^\varepsilon) d\sigma \\ I_{43} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_2 S(u_2) S_{2n}(u_2^\varepsilon) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) d\sigma \\ I_{44} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_2 S(u_2) S_n(u_1^\varepsilon) S_{2n}(u_2^\varepsilon) d\sigma. \end{aligned}$$

Observe that  $S_n$  is Lipschitz and  $S_n(u_i^\varepsilon) = S_n(T_{2n}(u_i^\varepsilon))$ . This gives

$$\begin{aligned} |S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)| &= |S_n(T_{2n}(u_1^\varepsilon)) - S_n(T_{2n}(u_2^\varepsilon))| \\ &= \frac{1}{n} |T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)|. \end{aligned}$$

Consequently,

$$|I_{41}| \leq \frac{\|h\|_{L^\infty(\Gamma)} \|v_1\|_{L^\infty(\Gamma)} \|S\|_{L^\infty(\Gamma)} \|S_n\|_{L^\infty(\Gamma)}}{n} \int_{\Gamma} |u_1^\varepsilon - u_2^\varepsilon| |T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)| d\sigma,$$

and then by (3.19) we get

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{41} = 0. \quad (3.26)$$

By similar arguments, it can be shown that

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{43} = 0. \quad (3.27)$$

For  $I_{42}$ , we observe that

$$|h(u_1^\varepsilon - u_2^\varepsilon)v_1 S(u_1) S_{2n}(u_1) S_n(u_2^\varepsilon)| \leq M,$$

where the constant  $M$  depends only on the  $L^\infty$ -norms of  $h, S, S_n$  and  $S_{2n}$ , and  $n$ . Also,

$$h(x)(u_1^\varepsilon - u_2^\varepsilon)v_1 S(u_1) S_{2n}(u_1^\varepsilon) S_n(u_2^\varepsilon) \longrightarrow h(x)(u_1 - u_2)v_1 S(u_1) S_{2n}(u_1) S_n(u_2),$$

a.e. on  $\Gamma$  as  $\varepsilon \rightarrow 0$ . By Lebesgue Dominated Convergence Theorem, as  $\varepsilon \rightarrow 0$ ,

$$I_{42} \rightarrow \int_{\Gamma} h(x)(u_1 - u_2)v_1 S(u_1)S_{2n}(u_1)S_n(u_2) d\sigma$$

and similarly,

$$I_{44} \rightarrow \int_{\Gamma} h(x)(u_1 - u_2)v_2 S(u_2)S_{2n}(u_2)S_n(u_1) d\sigma.$$

For large enough  $n$ , for  $j = 1, 2$ ,  $i \neq j$ , we have  $S(u_i)S_{2n}(u_i)S_n(u_j) = S(u_i)$ . In view of (2.15) in Remark 2.5,  $(u_1 - u_2)S(u_1) \in L^1(\Gamma)$  so that by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{42} = \int_{\Gamma} h(u_1 - u_2)v_1 S(u_1) d\sigma \quad (3.28)$$

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{44} = \int_{\Gamma} h(u_1 - u_2)v_2 S(u_2) d\sigma. \quad (3.29)$$

Combining (3.26), (3.28), (3.27) and (3.29), we conclude that

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_4 = \int_{\Gamma} h(u_1 - u_2)(v_1 S(u_1) - v_2 S(u_2)) d\sigma. \quad (3.30)$$

Finally for  $I_5$ , observing that  $S_n(u^\varepsilon)$  weakly converges to  $S_n(u)$  in  $L^\infty(\Omega \setminus \Gamma)$  weak-\* and a.e. in  $\Omega \setminus \Gamma$ ,  $f^\varepsilon$  converges strongly to  $f$  in  $L^1(\Omega \setminus \Gamma)$ , we have

$$I_5 = \int_{\Omega \setminus \Gamma} f^\varepsilon v S(u) S_n(u^\varepsilon) dx \rightarrow \int_{\Omega \setminus \Gamma} f v S(u) S_n(u) dx = \int_{\Omega} f v S(u) S_n(u) dx.$$

From (3.22), we have

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_5 = \int_{\Omega} f v S(u) dx. \quad (3.31)$$

Passing through the limit of (3.21) and using (3.23), (3.24), (3.25), (3.30), and (3.31), we have the desired conclusion.

This concludes the proof for the existence of a renormalized solution.  $\square$

**Remark 3.2.** As explained in Introduction, it is possible to use the Boccardo-Gallouët estimates, that is, to show that  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  is bounded in  $W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ , and that leads to  $u = (u_1, u_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ , for all  $q < \frac{N}{N-1}$ . Such a result may simplify the proof since it implies that  $\gamma_1(u_1^\varepsilon), \gamma_2(u_2^\varepsilon) \in W^{1-\frac{1}{q},q}(\Gamma)$  and in particular,  $\gamma_1(u_1^\varepsilon), \gamma_2(u_2^\varepsilon)$  are bounded in  $L^{1+\eta}(\Gamma)$ , for some small enough  $\eta$ . It follows that we can give another definition of renormalized solution including  $u_1 - u_2 \in W^{1-\frac{1}{q},q}(\Gamma)$  instead of (2.11b) and then (2.12b) is not necessary since it is a direct consequence of the regularity  $u_1 - u_2 \in W^{1-\frac{1}{q},q}(\Gamma)$ .

However, dealing with homogenization does not allow to use the Boccardo-Gallouët estimates since they are strongly related to the Sobolev constant which may blow up in a varying domain. Moreover, our techniques allow to consider more general equations (with a nonlinear boundary terms) for which the Boccardo-Gallouët estimates are not useful.

**Remark 3.3** (Stability). *By adapting the proof of Theorem 3.1, it is possible to derive a stability result. More precisely, let us consider  $u^\varepsilon$ , a renormalized solution of*

$$\begin{cases} -\operatorname{div}(B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{in } \Omega_1, \\ -\operatorname{div}(B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{in } \Omega_2, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma, \end{cases} \quad (3.32)$$

where

1.  $f^\varepsilon \in L^1(\Omega)$ ;
2.  $B_\varepsilon(x, t)$  is a Carathéodory matrix verifying
  - (a)  $B(x, t)\xi \cdot \xi \geq \alpha|\xi|^2$ , a.e.  $x \in \Omega$ , for all  $t \in \mathbb{R}$ , for any  $\xi \in \mathbb{R}^N$ , and
  - (b) for any  $k > 0$ ,  $B(x, t) \in L^\infty(\Omega \times (-k, k))^{N \times N}$ ;
3.  $h^\varepsilon \in L^\infty(\Gamma)$  with  $0 < h_0 < h_\varepsilon(y)$  a.e. on  $\Gamma$  and  $h^\varepsilon(y) < M$  (uniform), for some  $M > 0$ .

If  $f \in L^1(\Omega)$ ,  $B : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$  is a Carathéodory function, and  $h : \Gamma \rightarrow \mathbb{R}$  with  $h \geq 0$  are such that

$$f^\varepsilon \rightarrow f \quad \text{strongly in } L^1(\Omega);$$

$$\begin{cases} B_\varepsilon(x, r_\varepsilon) \rightarrow B(x, r) \\ \text{for every sequence } r_\varepsilon \in \mathbb{R} \text{ such that} \\ r_\varepsilon \rightarrow r \text{ a.e. on } \mathbb{R}; \end{cases}$$

$$h^\varepsilon \rightarrow h \quad \text{a.e. in } \Gamma,$$

then  $u^\varepsilon$  converges to  $u$  a.e. where  $u$  is a renormalized solution of

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2 \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases}$$

The main point is to obtain the a priori estimates of Step 2. In view of (2.16) in Remark 2.5,  $T_k(u^\varepsilon)$  is an "admissible" test function, so that

$$\begin{aligned} & \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon \cdot \nabla T_k(u_1^\varepsilon) dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon \cdot \nabla T_k(u_2^\varepsilon) dx \\ & + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma = \int_{\Omega} f^\varepsilon T_k(u^\varepsilon) dx, \end{aligned}$$

which gives all the necessary estimates. Then we can extract subsequences so that (3.16) hold true. In view of conditions on  $f^\varepsilon$ ,  $B_\varepsilon$ , and  $h^\varepsilon$ , we can perform Step 3 of the proof of Theorem 3.1.

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