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# Structures identified by second graders in a teaching experiment in a functional approach to early algebra

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*We describe structures and generalisation in four second graders (7- to 8-year-old) at the beginning and end of a classroom teaching experiment involving an early algebra-related functional context. No significant differences were found in their ability to identify structures and generalise at the beginning and at the end of the teaching experiment. All the identified structures in the questions involving specific cases and all the students generalised when they were explicitly prompted.*

*Keywords: Functional thinking, structures, generalisation, teaching experiment.*

## Background and conceptual framework

Traditionally, algebra is not introduced until secondary school in most countries, Spain is one example (Ministerio de Educación Cultura y Deporte, 2014<sup>1</sup>). That approach has induced failure in many students who react to the difficulties encountered by rejecting algebra and mathematics in general (Kieran, 2004). In early algebra the discipline is introduced in the first few years of schooling by working, among others, with the generalisation of functional relationships, known as the functional approach to early algebra. Children and adults both generalise by identifying regularity in inter-variable relationships. Notions such as generalisation and the structures in functional relationships are key elements in the study of students' functional thinking (Twohill, 2018). This paper addresses second-year primary school students' ability to identify structures and generalise in a functional approach to early algebra.

Functional thinking, as an approach to introduce algebra in the early grades, involves the relationship between two or more variables, and is based on the construction, description and representation about functions and their constituent elements (Cañadas & Molina, 2016). Patterns and regularities are of significant importance in the study of such thinking, for they interrelate the variables in a situation. Earlier studies reported that primary school students of different ages with no specific prior training were able to perform tasks involving functional thinking (Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens, 2017). The notion of structure acquires different meanings in early algebra (Molina & Cañadas, 2018). In the functional approach, it is associated with how inter-variable regularity is organised, which pupils may represent in different ways when working with specific values as well as when generalising the relationship between variables (Pinto & Cañadas, 2017a). Structures can be semantically equivalent. For example, in algebraic symbolism,  $x + 4$ ,  $x + 2 + 2$  and  $x + 1 + 1 + 1 + 1$  are equivalent expressions (equivalent semantic structures) because these expressions are the same thing through arithmetic computations, but their external structure is different.

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<sup>1</sup> Ministry of Culture, Education and Sports of Spain.

Limited studies have addressed these notions in primary education students. Pinto and Cañadas (2017a, 2017b), in research on structures and generalisation in third and fifth-year students, described differences in the amount and variety of the structures identified, both of which were greater among the younger children. Third-year students tended to work with specific cases while those in fifth-year were able to establish generalizations; most of them generalised structures and used the same structure for both specific situations and the general case and their answers were more coherent. Three of the fifth-year students generalised in tasks involving specific cases. The authors also distinguished two types of generalisation: spontaneous (when students generalised in tasks describing specific situations) and prompted (when they generalised when asked to do so in connection with the general case). Tasks or worksheets were designed assuming that generalisation would arise in a later stage, so these students evidenced prompted generalisation (Pinto & Cañadas, 2017a). Generalisation and structure are not independent notions. Identifying structure, whether through specific or general situations, is requisite to generalising. We take generalising to mean seeing general patterns in specific situations (Mason, 1996). The inductive model proposed by Cañadas and Castro (2007) was implemented here to foster this kind of abstraction. At the outset, we used specific situations (variables with specific values), aiming to prompt generalisation (of the functional relationship) through the detection of regularities. Structures and generalisation may be represented in different ways by primary school pupils: natural language, algebraic symbolism, tables or graphs (Radford, 2018). The present study explored the identification of structures and generalisation in second year primary school students at the beginning and at the end of a teaching experiment involving functional thinking.

## **Method**

In this paper, we focus on part of a broader teaching experiment (Steffe & Thompson, 2000), about functional thinking in second grades (7-8 years old). We present different problems during the five sessions of the teaching experiment and the two interviews. Three of the purposes of the teaching experiment were to: (a) explore how students relate the variables involved; b) identify structures on students' responses; (c) explore students' generalization process. This paper analyses the information gathered in two interviews: one held after the first session and the second after all the sessions were completed. The data were analysed to ascertain differences between the first and second interviews in terms of the structures identified and generalisation expressed by students.

## **Participants**

The subjects were four 7- to 8-year-old, second graders from a Spanish school. They had not been previously exposed to generalisation tasks involving linear functions. Their prior knowledge included the numbers from 0 to 399, addition with carrying and subtraction with borrowing. After the first session of the teaching experiment, in which a worksheet was administered to the entire class (24 pupils), the students were classified into two groups (intermediate and advanced). Students in advanced group generalized the functional relationship; students in intermediate group did not generalize. Four students, two from each group, were selected for the interviews. The subjects selected in each group were chosen by the classroom teacher considering that they should be participative.

## Information gathering instrument

The five classroom sessions conducted were video-recorded. In each session we posed one or two tasks in different contexts involving different functions. Each task was described on a separate worksheet. Two semi-structured individual interviews were held with the four students selected, one after the first session and the other after the last.

## Classroom sessions

A contextualized task involving a linear function was posed in each session. In all sessions, students first worked with specific cases of the function from which, after identifying the structure, they could pose new specific cases or generalise. Some of the tasks were original and others taken from the studies cited in section 1 above. The context and function addressed in each session are listed in Table 1. We first involved an additive structure (it is supposed to be easier following previous studies) and then we combined the additive and multiplicative structures. We observed that the students had more difficulties to reach the generalization in the later, so we involved only multiplicative structure. We selected contexts close to the students' experiences according to the students' teacher. The objective of each session was to explore students' generalization within different contexts.

Context	Functional relationship
<b>Session 1:</b> ball dispenser <b>Session 2:</b> amusement park 1	$y = x+3$
<b>Session 3:</b> amusement park 2	$y = 2x+1$
<b>Session 4:</b> birthday <b>Session 5:</b> train stations	$y = 2x$

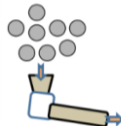
**Table 1: Characteristics of session tasks**

The dynamics were the same throughout the experiment. Three members of the research team were present in the classroom: the teacher-researcher (he was not the usual teacher), a support researcher and a researcher who video-recorded the session. Each session was divided into several parts: first we explained the task to the whole class, then we distributed the session-specific worksheets and lastly the task was discussed by the group as a whole. The students received no feedback on their replies. The worksheet used in the first session is described below as example and because the interviewees were selected on the grounds of their performance in that session.

## Session 1 worksheet

The session 1 worksheet contained five questions about a ball dispenser that when fed balls returned a certain number further to the functional relationship  $f(x) = x+3$ . We introduced the context to the students as follows: "we have a machine in which we introduce and leave balls". Four questions involving specific cases were posed as shown in Figure 1. We organized the balls in such a way that did not facilitate the structures identification through visualization.

How many balls will come out from the machine if we feed it 8 balls? \_\_\_\_ balls



**Figure 1. Specific case question**

The last question, involving generalisation, is shown in Figure 2.

Now we're going to play a game. The winner has to figure out how the machine works. How can you tell how many balls will come out?

**Figure 2. General case question**

This worksheet helped to classify the students into two groups based on their answers: intermediate and advanced. The first group included students who had identified regularity in several of the questions and the other students who were able to generalise. The classroom teacher then chose two students from each category (intermediate, S2, S1; advanced, S4, S3). Hereafter, the students are identified by the letter S and a randomised number for anonymity purposes.

### **Interviews**

We designed two semi-structured interviews, each with a different protocol. Two tasks involving additive linear functions were designed for each interview. In the first the context was the same as in session 1 ( $y = x + 3$ ). In the second interview the context was the age of two superheroes, one 4 years older than the other ( $y = x + 4$ ). The structure was the same in the two interviews: We began with specific, non-consecutive (to avoid recursive patterning in students' answers) situations and progressed inductively toward generalisation. The design of the second interview, which built on the first, also involved an addition function to ensure inter-sessional comparability of the students' replies. The guides for the two interviews were analogous. The guide for the second interview is presented in Figure 3. We note that in the last question about generalization, letters were random values that served to express examples, the question did not involve or asked for symbolic letters.

### 1. Exploration of specific cases

- Identify the functional relationship
- Apply the functional rule in different specific cases
  1. Specific cases
    - When Iron Man was 5 years old, Captain America was 9.
    - When Iron Man was 7 years old, Captain America was 11.
    - When Iron Man was 3 years old, Captain America was 7.
  2. Specific cases proposed by the student
    - Give me an age for Iron Man (\_\_\_). If he's that old, how old is Captain America?

### 2. Generalisation

- Express the generalisation
  - How would you explain to a friend what she has to do to figure out Captain America's age?
- Reason with the generalisation
  - One of your classmates said that 'When Iron Man is XX, Captain America is YY'. Do you agree?

**Figure 3. Protocol for the second interview**

### Data analysis

The data gathered during the two interviews were analysed with a system of categories based on the structures identified by the students in both specific and general cases. Students were considered to have identified a structure when they answered two or more questions referred to the same task with the same regularity or generalised a regularity.

### Results

Table 2 summarises the results for the specific and general cases posed in the first interview. The analogous information for the second interview is given in Table 3. The structures identified by the students as they worked are listed in tables 2 and 3 in the order in which they were observed and expressed using algebraic symbolism, although the students did not use that system.

Student	Specific cases	General case
S1	$y = x + 3$	$y = x + 3$
S2	$y = x + 3$ $y = x + x$	$y = x + 3$
S3	$y = x + 3$	$y = x + 3$
S4	$y = x + x$ $y = x + 3$ $y = x + x$	$y > x$

**Table 2: Structures identified in the first interview**

Further to the data in Table 2, each student identified one or two structures for the specific cases during the interview. Overall, three types of structures were observed in the particular and general cases, namely:  $x+x$ ,  $x+3$  and  $y>x$ . In the following extract of the interview, S4 identified different structures for the specific cases, as listed in the table in chronological order.

- I (Interviewer): What happened when you put 8 balls in the machine?
- S4 (Student 4): If there were 8, we add another 8, 16 balls.
- I: I'm going to write down the numbers you give me. When we put 5 balls in, how many came out?
- S4: 8
- I: And if we put 2 in, how many came out?
- S4: 4, I add 2+2

All four students generalised the functional relationship verbally (Table 2, last column). Three identified the structure as  $y = x+3$ , which they expressed verbally in different ways. S1 and S3 generalised both the specific cases (spontaneous generalisation) and the general case (prompted generalisation). S2 and S4 generalised only when they were prompted. S1 first gave examples of specific cases and then generalised by adding that the answer was found by adding 3: '1 ball we get 3, 2 balls we get 5, 3 balls we get 6, 4 balls we get 7. You always have to add 3. One million, you get 1 million and three.' That answer was classified as prompted generalisation because the student formulated it when asked the general case question (Figure 2). S4, in turn, noted that 'more balls come out than went in', generalising without quantifying: 'I know the machine spits out identical balls and what I know is that inside it has lots more balls' (prompted generalisation). The other two students' answers showed that they were thinking in terms of  $x+3$ . Three students identified the structure correctly ( $x+3$ ) in both the general and the specific cases. Table 3 lists the structures identified in the second interview.

Student	Specific cases	General case
S1	$y = x + 4$	$y = 4x$ $y = x + 4$
S2	$y = x + x$ $y = x + 4$	$y = x + 4$
S3	$y = x + 4$	$y = x + 4$
S4	$y = x + 4$ $y = x + x$	$y > x$

**Table 3: Structures identified in the second interview**

As in the first interview, students identified one or two structures for the specific cases. Three types of structures, including the ones given for the general case, were found:  $x+4$ ,  $4x$  and  $x+x$ . S2 and S4 exhibited the same type of structures as in the first interview, identifying the relationship in the specific cases to consist in doubling the value given. Generalisation was expressed verbally by all four students. The structures observed in their generalisations were:  $4x$  and  $x+4$ . The second was identified by all the interviewees. S1's initial assertion that the age of the older superhero was found by 'multiplying by four' was indicative of their difficulty in distinguishing between addition and multiplication. Student S4 generalised the relationship to be  $x+4$  saying: 'you have to add plus 4'. S2 and S3 both instantly answered that what they had to do was 'add plus 4'. S1 generalised both

the specific cases (spontaneous generalisation) and the general case (prompted generalisation). S2, S3 and S4 generalised only when prompted. The following extract exemplifies S1's spontaneous generalisation when answering questions about specific situations.

I (Interviewer): when Iron Man was 5 years old Captain America was 9, when Iron Man was 7 Captain America was 11 and when Iron Man was 3 Captain America was 7. Can you tell me how those numbers are related?

S1 (Student 1): Captain America always wins: he's 4 years older.

### **Comparison of interviews**

In the first and second interviews, S1 and S3 identified only one structure, correctly in both cases. S2 and S4 both identified more than one structure, likewise in both interviews. In the second interview S2 continued to identify  $x+x$ , although in the specific cases only, ultimately identifying the right structure as the interview proceeded. The contrary was observed in S4, who incorrectly identified the structure  $x+x$  in the second interview. Fewer differences were observed around the general case. S2 and S3 correctly identified the structure, whereas S1 initially identified a multiplicative structure in the second interview,  $4x$ , despite having correctly identified the structure in the first. In the second interview S4 correctly identified the relationship between the variables, having failed to do so in the first. All the students correctly identified at least one structure in the specific or general cases and all expressed generalisation in at least one case. In the second interview, however, only one student, S3, identified the sole same structure in the specific and general cases. Two students did so in the first interview. While seeing no clear structure in the specific cases, S4 correctly identified it in the general case. Generalisation was expressed in various ways in both interviews. In the second interview, only S1 generalised spontaneously, whilst in the first both S1 and S3 did so.

### **Discussion and conclusions**

This paper presents evidence of functional thinking in second-year primary school students. Like the fifth-year primary students who participated in the Pinto and Cañadas (2017a) study, most of the second-year pupils in our survey generalised structures, identifying the same structure in the specific and the general cases in the first and second interviews. Their answers were found to be coherent. The number and variety of the structures identified were greater in the first interview. Although the students might have identified  $2x$  instead of  $x+x$  for the specific cases in both interviews, none did. Students evidenced the  $x + x$  structure more easily than the  $2x$  because of the kind of structure (additive versus multiplicative structure). All the students expressed generalisation verbally, a finding consistent with the Cañadas and Fuentes (2015) results. All the students generalised in both the first and the second interview. That finding differed from the results reported by Pinto and Cañadas (2017b) for third-year students, only one of whom generalised. In the second interview generalisation was consistently and simply phrased as 'adding plus 4'. In our study generalisation was normally prompted. In one case, S4 in the first interview, expressed the general relationship as  $y>x$ . This student was able to see that 'y is always greater', without quantifying to what extent is greater.

Whilst differences between the intermediate and advanced groups was not explored in this study, the findings showed that students' classification in one or the other had no impact on their ability to



correctly identify structures. In other words, the advanced students did not correctly identify more structures than the others. In fact, the contrary was found: S1, an intermediate group student, was the only one to correctly identify the structure involved in the specific and general cases in both interviews. Work to enlarge on this study is ongoing to determine how the working sessions might have affected the differences between the two interviews.

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