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# Short note on algebraic notations: First encounter with letter variables in primary school

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*Algebraic thinking often requires generalisation processes. Generalisations of patterns are one frequently used design in early algebra settings in research and practise. Symbolical letter variables are not common in primary school and actually not needed by the children to articulate their pattern generalisations. Nevertheless, from a research perspective it is interesting to explore children's spontaneous interpretations of letter notations. The reactions of 3<sup>rd</sup>- and 4<sup>th</sup>-graders, who participated in an explorative study presented here, might provide a starting point to sensibly appreciate the individual thoughts and ideas about letter variables in further studies.*

*Keywords: Early algebra, generalisation, symbolic notations, variables.*

## Introduction

Algebraic thinking in all its possible facets are described in detail in several studies (Kieran, 2018). One of the major aspects of algebraic thinking –especially in early algebra– is generalising from given cases to rules and description of the patterns found. Symbolical notations in letter variables can condense these rules in algebra lessons in secondary schools.

In some research approaches, based on the so-called Elkonin-Davydov curriculum, algebraic thinking is assumed as equal to using letter notations (Dougherty & Simon, 2014; Dougherty, 2008; Freudenthal, 1974). Others follow the way to re-invent algebra by the children (Amerom, 2002). Nowadays, almost everyone involved in early algebra research might agree on the fact that the children will not invent letter notations on their own.

Teachers need to introduce unfamiliar terms, representations, and techniques, despite the irony that in the beginning students will not understand such things as they were intended. The initial awkwardness vis-à-vis new representations should gradually dissipate, especially if teachers listen to student's interpretation and provide students with opportunities to expand and adjust their understandings. (Carraher, Schliemann, & Schwartz, 2008, p. 237)

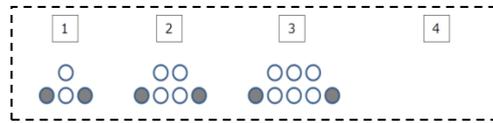
Notation in letter variables does not emerge. It also holds true that letter notation is not needed to adequately describe patterns found (Akinwunmi, 2012). However, a suitable verbal description of complex patterns is sometimes tough to find. Letter notation is therefore a useful condensation and last not least a cultural heritage or an aspect of enculturation: "Enculturation into a mathematical expectation" (Mason, Stephens & Watson, 2009, p. 14). The study described here aims to receive an impression of primary children's initial interpretations of a vis-à-vis confrontation with letter notations, and listen to these ideas. The findings might base teaching advices and further research.

## Theoretical framework

Letter notations use variables. Variables have at least three different aspects and features. They can

- represent quantities (Kieran, 2004) as an unknown (Strømskag, 2015; Freudenthal, 1983) in an equation like  $5+x=8$ ,

- express rules of numerical relationships (Kieran, 2004) like in  $a+b=b+a$ , which Strømshag (2015) identifies as parameters and Freudenthal (1983) as indeterminate,
- express generality arising from patterns (Kieran, 2004) as variables (Strømshag, 2015; Freudenthal, 1983) like  $n \rightarrow 2n+2$  (cf. Figure 1), which are the main focus in the study presented in this paper.



**Figure 1: Pattern Sequence  $n \rightarrow 2n+2$**

Working on pattern sequences, like the one given in Figure 1, allows at least four different approaches, which can be summarised in the ReCoDE (Recognise-Continue-Describe-Explain) model (Steinweg, 2014). As mentioned above early algebraic thinking deeply depends on generalisations. A typical task implicitly challenges the children to generalise already whilst recognising the regularities: “Generalizing starts when you sense an underlying pattern, even if you cannot articulate it” (Mason, Burton, & Stacey, 2010, p. 8). Nevertheless, the articulation or even notation is a substantial element in the development of algebraic abilities. Experienced learners may use variables to articulate and to communicate findings with others in describe- or explain- phases.

When students perform a pattern generalization, it basically involves mutually coordinating their perceptual and symbolic inferential abilities so that they are able to construct and justify a plausible and algebraically useful structure that could be conveyed in the form of a direct formula. (Rivera, 2010, p. 298)

Dörfler (2006) regards notation as mathematical object itself, through which mathematical learning can be initiated. In any case letter notation has to be interpreted by the learner. This, of course, is the fact for every mathematical object like digits and operations signs. In line with Steinbring (2015) one has to keep in mind, that “the elements of the mathematical world cannot be perceived directly by our senses but consist of ‘invisible structures’ and relations” (p. 292). Hence, mathematical thinking depends on working on mental objects:

I speak of the constitution of mental objects, which in my view precedes concept attainment and which can be highly effective even if it is not followed by concept attainment (...) The fact that manipulating mental objects precedes making concepts explicit seems to me more important than the division of representations into enactive, iconic, and symbolic. (Freudenthal, 1983, p. 33)

The need for individual interpretation of mathematical objects and operations by images and explanatory models is fundamental for mathematical learning (Hofe & Blum, 2016). The interpretation of letters as notation of variables is nothing else than this usual habit. Rivera (2013) identifies the guessing of a rule of a given figural pattern as abductions and therefore as a fundamental mathematical activity. Of course ‘symbol sense’ is crucial for algebra because it allows

to see algebra as a tool for understanding, expressing and communicating generalizations, for revealing structure, and for establishing connections and formulating mathematical arguments (proofs). (Arcavi, 1994, S. 24).

In the study presented here pattern sequences as “arbitrary patterns” (Strømshag, 2015, p. 475) and no figurate numbers (e.g. square numbers) are used. The given pattern in predefined coloured dots expects the learners to see a double structure or “a regularity that involves the linkage of two different structures: one spatial and the other numerical” (Radford, 2011, p. 19). In an explicit notation of the underlying rule the commonly used letter (n) corresponds with the position (ordinal) of the figure in the pattern as a whole. The cardinal amount of black and white dots needs to be set in relation to this ordinal number. The explicit formula represents in a direct approach “a functional relationship between position and numerical value of an element” (Strømshag, 2015, p. 475). As mentioned earlier the letter notation is in a way independent from generalising the pattern:

The difference between finding rules and expressing those rules in formal notation highlights a difference I see between these two aspects of mathematics. Spotting patterns and finding rules is algebraic in nature whereas how to express those rules is a matter of language and notation. (Hewitt, 2009, p. 43)

Thus, Hewitt (2012) differentiates between the necessary and the arbitrary in algebraic notations. The notation of spotted regularities also depends on (school) experiences (Radford, 2002). The didactical dilemma is, that “notation is not an afterthought, but rather an inherent part of mathematical activity” (Hewitt, 2016, p. 168).

Some pitfalls of educational approaches to letter notations are well documented. If tasks and learning-teaching-environments encourage pupils to match letters to word beginnings as abbreviation of names for objects in context situations (so-called fruit-salad-algebra (Thomas & Tall, 2001), or positions in the alphabet, these ideas hinder a deeper understanding of algebraic variables: “This ‘letter-as-object’ misconception can haunt students in their efforts to set up equations throughout their algebra careers” (Arcavi, Drijvers, & Stacey, 2017, p. 52).

A recent teaching-experiment study in two first grade classes worked on functional relations in context problems (Blanton et al. 2017). The findings indicate levels of understanding and make clear, that the letters are sometimes viewed as representing the context objects. The context situation therefore might hinder an interpretation of letters as mathematical object. Nevertheless, children in primary school of course need something to refer to while making sense of letter variables. The mediator chosen in this study is dots and dot pattern sequences.

The German situation concerning algebraic thinking is very special because algebra is not a curricular content (Steinweg, Akinwunmi, & Lenz, 2018). Patterns are mentioned in the syllabi and are therefore suitable for German primary school children, even though pattern sequences (like Fig. 1) are not frequently used. The study here tries to figure out the reactions to letter variables of German pupils with –under these conditions assumed– minor or none prior algebraic knowledge and whether the findings on children’s understanding of letters correspond to international findings:

- *What kind of spontaneous interpretations of letter notations of a pattern sequence ( $n \rightarrow kn+t$  in  $\mathbb{N}$ ) can be categorised?*
- *Does the age of the pupils or the pattern sequence given effect the interpretations?*
- *Are there any correlations between continuing the sequence, finding the 100<sup>th</sup> figure, and the ability to interpret the letter notation?*

## Methodology and Design

The study is an explorative one-shot written test design with no teaching experiments. Five mixed-age classes 3<sup>rd</sup>- and 4<sup>th</sup>-grader (8 to 10-year-olds) and in total 96 children participated in the study. The designed written test uses three different versions, each focusing on one pattern sequence ( $n \rightarrow 2n+2$ ,  $n \rightarrow 2n+1$ , and  $n \rightarrow 3+n$ ) in order to allow each child an independent reaction to the task in the classroom setting. Each test version is therefore worked on by 32 children.

Each test version includes only three tasks on one sheet of paper. The first two tasks are placed on the top half of the sheet, the third on the bottom half. The task sheet was folded in the middle, so that the children only see the first two tasks. They are permitted to unfold the sheet only after working on these two tasks.

Each child gets one of the pattern sequences and is asked to draw the next figure (like in Figure 1). The second task asks to describe how to find the 100<sup>th</sup> figure in this sequence. After unfolding the tasks sheet the third task appears. It is a confrontation with one possible algebraic letter notation for the given relation (example Figure 2). This task invites the child to think about this unfamiliar symbolic letter description and to comment on it. The reactions might reveal mental images the children use.

Adults describe the pattern like this:  $2 \cdot n + 2$   
Does this fit? What do you think?

**Figure 2: Confrontation-task with the algebraic letter notation**

The design follows the idea to embed the confrontation with the letter notation in a useful way in a purposeful task. Purposeful is meant here against the background of the definition by Ainley, Bills and Wilson (2003) alongside with utility which is understood as “knowing how, when and why that idea is useful” (p. 2). It is hoped for that the learner sees the connection of the letter symbols after working on the pattern individually. The letter notation, at best, condenses the idea of generalisation of the pattern the child already used in the 2<sup>nd</sup> task (100<sup>th</sup> figure in the pattern).

Ja!! Yes!!	Weiß ich nicht I don't know
statement	none
<p>Ich glaube <math>2 \cdot n + 2</math> heißt 2 normale <math>2n + 2 = 4</math></p>	<p style="border: 1px dashed black; padding: 5px;">I think <math>2 \cdot n + 2</math> means</p>
individual	
<p><math>+2</math> ausgemalte ●●</p>	<p style="border: 1px dashed black; padding: 5px;"><math>+2</math> painted ●●</p>
partly	
<p><math>+2</math> sind die 2 schwarzen Punkte. Das <math>n</math> ist nochmal</p>	<p style="border: 1px dashed black; padding: 5px;"><math>+2</math> are the 2 black dots. The <math>n</math> is again</p>
fully	

**Figure 3: Categories and examples of reactions of the algebraic notation  $2n+2$**

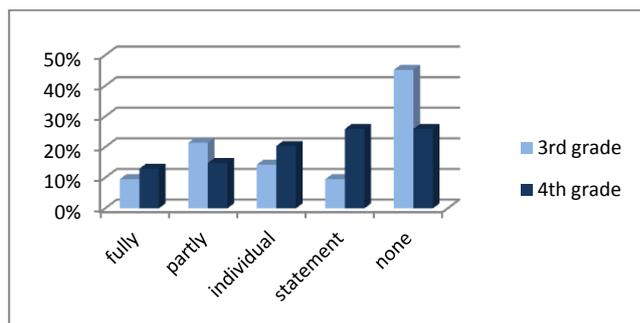
The answers to the first two tasks are analysed in a common way. The continuations are categorised in *none*, *individual*, *correct without coloration*, and *correct*. The description of the 100<sup>th</sup> figure in the pattern are categorised in *none*, *individual*, *initial approaches*, and *correct*. The reactions to the letter notation are the main focus of the study. Here a wide range of answers are found (Figure 3). The categories *none*, *individual*, *partly*, and *fully* are suitable. Additionally, the category *statement* is needed, because the task allows yes-or-no-answers, which are frequently used.

## Findings and Discussion

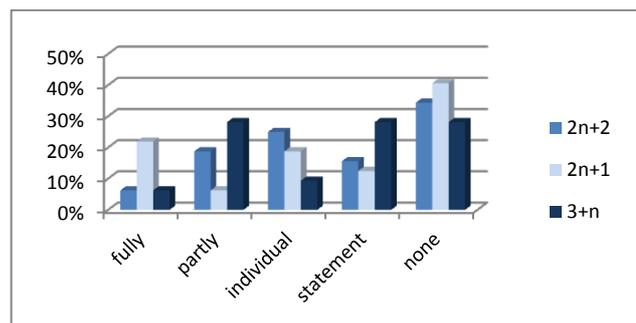
### Quantitative Results

78 % of the children are able to continue the pattern *correctly* and over one third (35 %) describes the 100<sup>th</sup> figure *correctly*. Taking into account that pattern tasks –not even on Recognise-Continue-Describe level– are not common in German primary schools, these results are quite pleasing.

The duration of schooling of the participating children differs between just started the 3<sup>rd</sup> year or rather 4<sup>th</sup> year. This might have an effect on the reactions. Although, it should be mentioned again, pattern tasks are not frequently used in German mathematics lessons. At first sight it seems like 4<sup>th</sup>-grades are getting better, because the numbers of *no answers* are dropping (Table 1). But it may only be the case that the older the children the more cheeky or assertive they are, because the number of *statements* increases inversely.

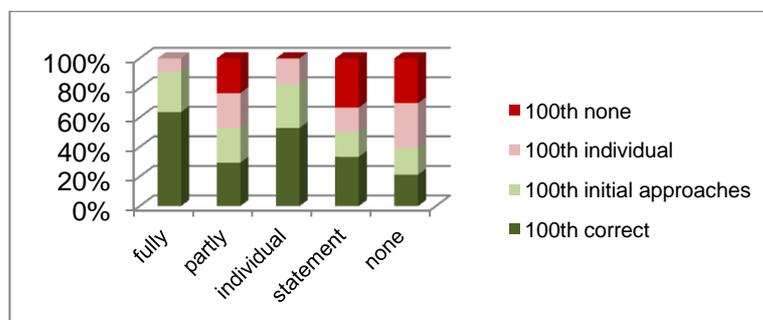


**Table 1: Impact of duration of schooling**



**Table 2: Impact of the pattern**

Furthermore, an impact of the given pattern, illustrated in Table 2, is visible. The number of *full* interpretations of the pattern  $n \rightarrow 2n+1$  is considerably higher than the others. Simultaneously the number of *no answers* is the largest as well. The  $n \rightarrow 3+n$  pattern triggers the most *partly* interpretations and the highest number of *statements*.



**Table 3: Bivariate analysis interpretation of letter notation and finding the 100<sup>th</sup> figure**

Additionally, correlations between the reactions to the first two tasks and the ability to interpret the formula are proved. It can be shown that the ability to continue the pattern *correctly* is necessary but not sufficient for a suitable interpretation. This means, every child who gives a *full* interpretation of the letter notation continued the pattern correctly, but not every child who correctly continued the pattern sequence is able to interpret the symbolical letter notation. Moreover, correlations between the experiences and reactions to the generalising task before (100<sup>th</sup> figure in the pattern) and the interpretation of the letter notation could be expected –particularly with regard to the focus on generalising in various research studies in early algebra. The bivariate analysis (Table 3) indicates that –although the number of children who are able to read the letter notation *fully* is quite small– it is striking that 9% of them have no clue how to find the 100<sup>th</sup> figure in the pattern sequence. Regarding the *partly*-readers of the letter notation, the amount of children who have not mastered a suitable 100<sup>th</sup> figure grows up to 51%. Apart from that, 52% of the children who give *individual* interpretations of the letter notation have found the 100<sup>th</sup> figure in the pattern sequence beforehand. The expected causal relation –that generalising the pattern by finding the 100<sup>th</sup> figure in the sequence directly influences the ability to interpret the letter notation– therefore cannot be confirmed in this study.

### Qualitative Findings

The reactions of the children categorised as *individual* or *partly* are of great interest to understand children’s first-encounter-interpretations of letter variables of relations in  $\mathbb{N}$  ( $n \rightarrow kn+t$ ). One striking aspect is, that none of the *partly* interpretations refer to the multiplicative part of the term ( $kn$ ) but in each case the additional part (constant element  $t$ ) is described (a typical example is given in Figure 3). A recursive idea is read into the explicit letter notation by two other children. One of these children explicitly specifies +2 as the two dots increase from pattern to pattern and does not assign the painted dots (constant element) to +2.

The *individual* reactions try to assign an individual meaning to the given letter. Some regard the variable as an unknown, which has a certain value like 2 (Figure 3). Five children strongly suggest changing the  $n$  to an  $h$ , because it means 100 in their eyes. This might be a result triggered by the second task, which asks for the 100<sup>th</sup> figure of the pattern. Two others match  $n$  to 14, because  $n$  is the 14<sup>th</sup> letter in the alphabet. One child expects  $n$  to equal nine, because nine (‘Neun’) starts with an  $N$ . These interpretations are of special interest against the background of fruit-salad-algebra (Thomas & Tall, 2001) and ‘letter–as–object’ misconception (Arcavi, Drijvers, & Stacey, 2017).

### Closing Remarks

The study presented here is limited, because research about the importance of “perceptual diversity” (Rivera, 2013, p. 5), the possibility of different colouring, and various individual interpretations of possible structures (Twohill, 2018) are not taken up in the design. The tasks only involve one predefined explicit formula at a time and recursive thinking is somewhat ignored – even though two children stick to their recursive approach anyway. This small study can only be seen as a starting point for verifying or falsifying the results. In summary, the striking findings are:

- The impact of the pattern used in the tasks is noticeable but not coherent.
- Age and duration of schooling might have an effect.

- The ability to continue patterns is necessary but not sufficient for the interpretation of letter notations (symbolic terms).
- The ability to generalise patterns (find the 100<sup>th</sup> one) seems not necessarily engender (sound) formula interpretations.

Especially the latter result requests further studies. These studies are thinkable in two ways: Larger studies to proof possible correlations or smaller study with a focus on interviews and qualitative analysis. Some interviews in different age-groups are carried out currently. The deeper inside in children's thoughts while interpreting the letter-symbols will be most interesting.

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