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Dealing with quantitative difference: A study with second graders

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The main goal of this paper is to understand students' reasoning when solving a task that involves quantitative difference. A qualitative methodology was used within the modality of teaching experiment. The data collection was done through the participant observation supported by video and audio recording of the work developed by two pairs of second graders, as well as the records of whole class discussions. To analyse data, we organized them into two categories: additive comparisons and complex additive relationships. The results show that these students were able to deal with quantitative difference, establishing relationships between quantities, even in a situation where initial numbers were unknown.

Keywords: Quantitative difference, quantitative additive reasoning, algebraic thinking.

Introduction

This paper reports part of the research developed by a project focused on flexible calculation and quantitative reasoning developed by teachers of the Higher Education Schools of Lisbon, Setúbal and Portalegre in Portugal. The main goal of the project is to build knowledge about the development of quantitative reasoning and calculation flexibility of students from 6 to 12 years old. The paper aims to understand second graders' reasoning when exploring and discussing a task that involves quantitative difference (Thompson, 1993). Specifically, the study aims to contribute to the knowledge of how early years students are capable of modelling situations focusing on their structures, abstracting from computation.

Theoretical framework and empirical studies

Nowadays, algebra is viewed as a generalizing activity (Mason, 2008) with instruments that represent the generality of mathematical relationships, patterns, and rules. According to Kaput (2008), there are two essential aspects of algebraic thinking: (a) generalization and the expression of generalizations in increasingly systematic, conventional symbol systems, and (b) syntactically guided action on symbols within organized systems of symbols. The aspect (b) is developed later than aspect (a). These two aspects are embodied in three strands of algebra:

1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics. (Kaput, 2008, p. 11)

The first strand is the primary route into algebra. In the early years, it is important to approach arithmetic focusing on number and operations properties or relationships, thus building generalizations from them. It is also fundamental to focus on mathematical processes (not on the final product), in order to put relationships and transformations as objects of study (Cusi & Malara,

2007). Such an approach is conducive to the development of students' algebraic thinking. There is another approach within this first strand that is based on quantitative reasoning. This kind of reasoning is not based on the arithmetic of numbers but rather on quantities.

Quantitative reasoning is defined as an analysis of a situation into a network of quantities and quantitative relationships (Thompson, 1993). This kind of reasoning is related with the algebraic reasoning insofar as the focus is on the description and modelling of situations. It involves the relationships between quantities. As argued by Smith and Thompson (2008), situations involving complex additive relationships, that is to say, situations with more than three related quantities, could be fundamental for the development of students' quantitative reasoning. And this development prepares students for algebra in multiple ways. "If we want students to learn and use algebra as a sensible tool for expressing their thinking and solving problems, then work with complex problems must come first" (Smith & Thompson, 2008, p. 113). Quantitative reasoning has two roles: (i) to provide the content for algebraic expressions; and (ii) to support a flexible and general reasoning not necessarily relied on symbolic expressions (Smith & Thompson, 2008). Therefore, quantitative reasoning is foundational for both arithmetic and algebra, providing meaning and content for numerical and algebraic expressions.

There is a conceptual difference between quantity and number. According to Thompson (1993), "a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (p. 165). For instance, we can reason about the lengths of two objects and the amount by which one is longer than the other without having the numerical values of their measures. Furthermore, numerical difference is not synonymous with quantitative difference. Numerical difference is the result of subtracting, while quantitative difference is the excess found when one compares two quantities additively (Smith & Thompson, 2008; Thompson, 1993). For example, 9 is a numerical difference when we subtract 6 from 15 (doing a computation), and it is a quantitative difference when we compare the quantities 15 and 6, assuming it is the amount by which 15 exceeds of 6 or 6 falls short of 15. Besides this, we can reason about 9 as a quantitative difference without having absolute numbers, conceiving 9 as a relative change related to variables.

Several studies have reported results related with additive quantitative reasoning (e.g. Schliemann, 2015; Thompson, 1993). Schliemann (2015) referred to a study conducted through interviews with second graders and showed that they were able to understand that the equality of two quantities relative to concrete sets of objects, with known or unknown number of elements, remains after performing identical transformations in the two compared quantities. She also reported the results of longitudinal studies from third to fifth graders in USA about algebraic reasoning and function representations. In these studies, the verbal representation of functions emerged from the consideration of relationships between physical quantities and situations of daily life, and involved additive comparison. She concluded that the students, from the age of 8 years old, showed understanding of quantitative relationships which was fostered by the discussion on those functional relations. The students used informal strategies for solving early algebra problems. Examples of such problems are: to represent additive comparison in a situation of having physically two chocolate boxes with equal quantities but unknown absolute number and 3 more chocolates above one of the boxes; to represent additive operations as movements along a line marked with $N-3$, $N-2$,

N-1, N, N+1, N+2, N+3, N+4,... to solve problems where initial value was unknown. The evaluation of the intervention impact, at the end of 5th grade, demonstrated that the performance of the intervention group in the written test was significantly better in the items related to the intervention and similar to that of the control group in the other items. Another study, driven by a teaching experiment that involved 5th-grade students (Thompson, 1993), showed students' trouble with the distinction between the quantitative operation of comparing two quantities additively and the arithmetical operation of subtraction: they conceived a quantitative difference as an invariant numerical relationship. Thus, they assumed relative change as an absolute amount and needed to know absolute values before they could make comparisons. These studies allow us to compare the respective results with those of the study of our project. On the one hand, comparing results with second and third graders in similar early algebra problems but in more concrete conditions, and on the other hand, comparing results with fifth graders, looking at the difficulties or facilities in the performances of students of different grades in similar algebra problems. Hence, this confrontation allows a deeper understanding of the effective quantitative reasoning abilities of second graders, through the extension to progressively more abstract situations not based on arithmetic computation.

Methodology

The project follows a qualitative approach framed in an interpretative paradigm (Bogdan & Biklen, 1994). It is focused on the educational processes and the meanings of the study's participants. It adopts the modality of teaching experiment conceived with the purpose to develop students' flexibility for calculation and quantitative reasoning. The Project team defined a sequence of nine tasks involving additive structure and discussed it with the classroom teacher.

The task “*More? Or Less?*” was the fifth task in the sequence and was created by the Project team. It has a context of games with marbles and it was presented after another task that had the same context. It is composed by three parts. The first and the second parts are identical but the first one, besides involving different numbers, presents, in the middle, empty and white squares, while the second one presents black squares. The second part aimed to focus on quantitative difference corresponding to the excess found when gains and losses are compared. Figure 1 presents one example of each part.

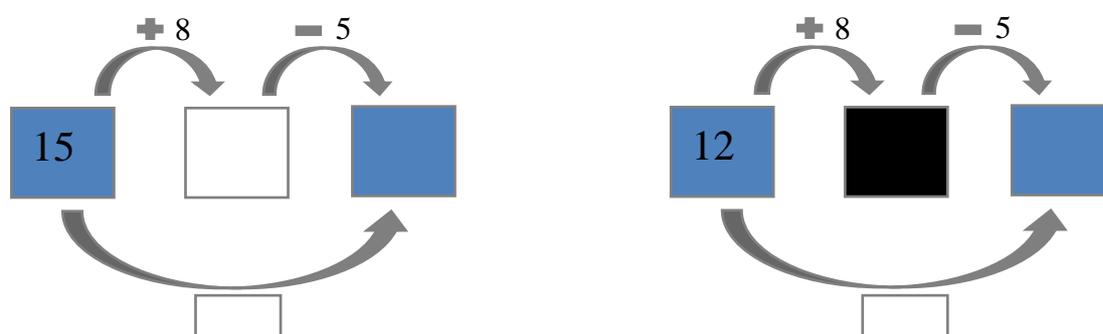


Figure 1: The first and second parts of the task

The 3rd part aimed to focus on the balance between gains and losses and it does not present the absolute initial number of marbles (see Figure 2). At the end, students were asked to circle the face at the end of two days (Monday and Tuesday) and at the end of three days (Monday, Tuesday, and

Wednesday). The students were requested to circle a smiley or a sad face depending on their sense of whether there were more gains or more losses at the end of the day(s). Besides this, the students were expected to find the quantity representing how many marbles have been gained or lost at the end of the day(s), that is, the final quantitative difference. It should be noted, that they had not worked with notation of negative numbers before this task.

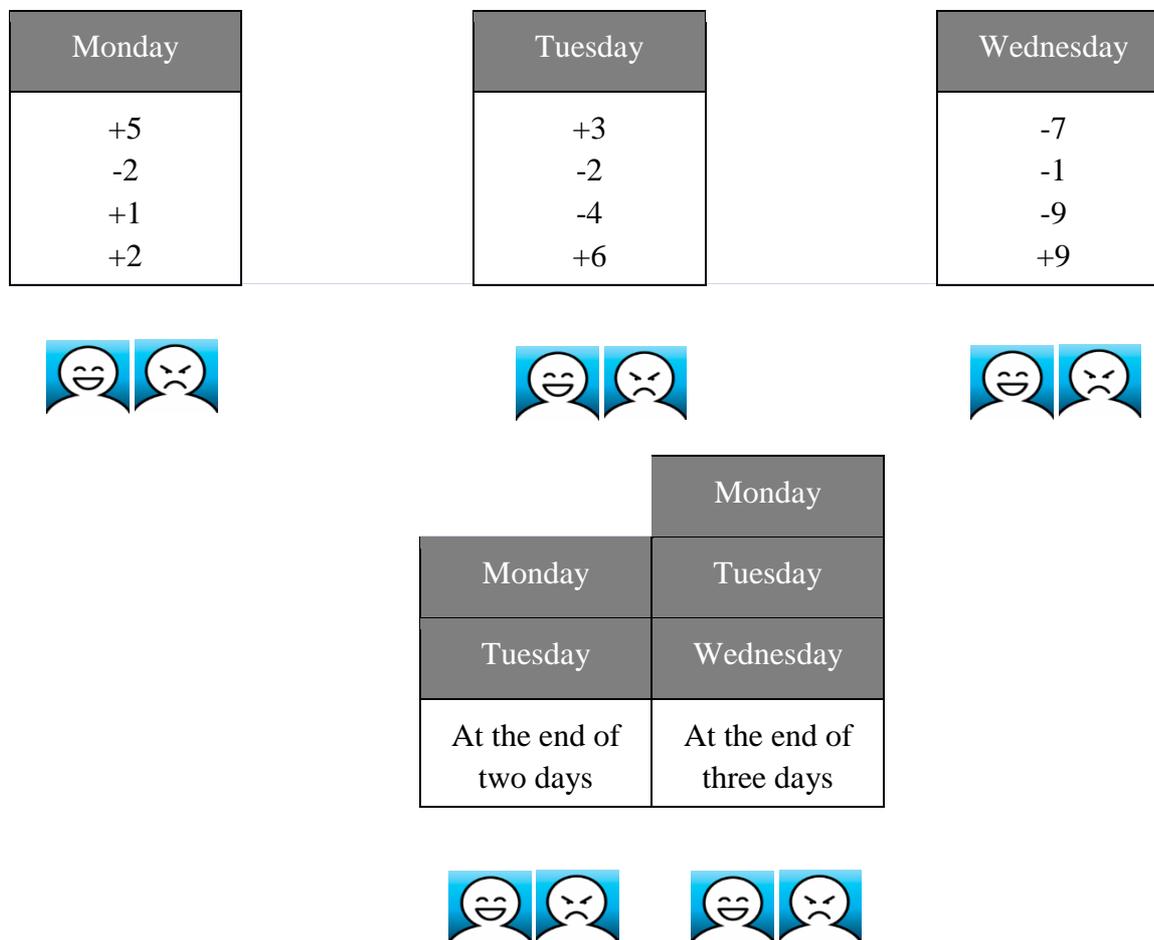


Figure 2: The third part of the task

The data reported in the current article were collected during two lessons of one hour and a half each that took place in the same class and they were conducted in the same way, within an exploratory teaching approach. The names of the students have been changed to ensure confidentiality. The data collection was done through the participant observation of the authors of this article, complemented with field notes and videotaping of the lessons. The students' productions were also collected. In the part of the exploration of the task, the videotaping focused on two pairs of students, Luís and Lúcia, and Paulo and João, selected because they habitually verbalized their reasoning among themselves.

The analysis focused on students' productions, field notes, video recordings and their transcriptions. To analyse data we defined categories built from the theoretical framework of Thompson (1993): (i) additive comparison; and (ii) complex additive relationship. The first one is illustrated for example by comparing 8 gains and 5 losses and determining 3 as the quantitative difference (there are 3 more gains than losses) or by comparing 6 gains and 10 losses and determining 4 as the quantitative

difference (there are 4 more losses than gains). The second corresponds, for example, to the balance between several gains (+3, +4) and several losses (-8, -6) conceived as quantities without the necessity of having the absolute number of marbles in situations with more than three related quantities: at the end, the quantitative difference is -7.

Results

During the first lesson, the pair Luís and Lúcia explored the first and second parts of the task in a consecutive manner, prior to the discussion of the first part. Luís asked: "Why is it here black?". Although informed that they should write nothing there, Luis wrote the various results within the black squares, despite Lucia's reaction ("We cannot!"), thus addressing the second part in the same way as the first one. To do the calculations, they used compensation strategies, adding first 10. For example, for calculating $12+8$, they did: $12+10=22$; $22-2=20$. Also the pair Paulo and João performed all the calculations, as in the first part.

In the whole class discussion, the teacher asked different pairs to go to the blackboard to present their solutions for the first part of the task.

Nádia: Here is 15. Then we added another 8 that gives 23. Minus 5 gives 18.

Teacher: Yes? ... So how did you find the bottom one?

Nádia: Because here (*pointing to 15*) to reach the 18 is 3 more.

Teacher: So the first and the last were enough; the jumps above do not matter?

Maria hesitates.

Teacher: I want to know if the jumps above also help.

Paulo: Yes, they help.

Teacher: Why?

Paulo: (*goes to the blackboard*) Because if from 8 you take out 5, you get only 3 from here (*points to the bottom square*). And knowing that only 3 were missing, here from the first to reach the last we only needed 3 more. For the last.

Teacher: So did they gain or lose?

Guilherme: They gained because they got 3 more.

The students were able to focus on the balance between gains and losses of marbles, rather than on numerical calculations with the help of the teacher's questioning. The classroom discourse was centered on quantitative difference ("3 more") instead of absolute number of marbles (15 and 18).

During the second lesson, the teacher gave the second part of the task. Since Luís and Lúcia had already done it in the previous lesson, the teacher asked them to talk to each other about how they had done it. When the teacher approached the pair, she asked:

Teacher: Do you need this middle square?

Luís and Lúcia: No.

Teacher: Tell me another way to reach this without saying the number in the middle. If it's painted of black, maybe it would not be necessary.

Luís: 8 minus 5 gives 3 (*pointing to the arrows*).

Teacher: Gains or losses?

Luís: Gains. 3 more.

Teacher: And here? (*pointing to the situation below*)

Luís: 15 + 9...

Teacher: But is the middle square necessary?

Luís: Plus 9, minus 10.

Teacher: Okay. (*the teacher moves away*)

Luís erased the numbers written in the black squares, indicating that he followed the teacher's guidance that these numbers would not be necessary. Luís seemed to be able to focus on quantitative difference between gains and losses in both games, as encouraged by the teacher. However, in the other situation, he began verbalizing the numerical computations and just after the teacher's intervention, he focused on quantitative difference resulting from *additive comparison* ("Plus 9, minus 10").

With regard to the other pair, Paulo showed the ability to focus on the balance between gains and losses. He decided to invent two other situations, asking João to determine the jumps. When the researcher got closer, Paulo explained the "-6+4" situation: "It's 6 minus 4. It's going to give a negative number". Probably Paulo used a vocabulary from his home context since the teacher had not used this expression in classroom in the previous lessons.

At the moment of whole class discussion, the teacher asked:

Teacher: Why do you think the square is painted of black?

Student: It's to guess the number and keep it in our head.

Teacher: Does it make sense?

Gil: To not write.

Maria: To help us think.

Paulo: The square is painted because the number under that square should not interest.

Teacher: So how does it solve?

Luís: It's +8-5 that gives +3.

Teacher: Gained or lost, Gil?

Gil: Gained +3.

Teacher: Without looking to the jumps, how can we see if he gained?

Students: We see in the numbers, from 12 to 15.

With the help of the teacher's guidance, the students were able to relate the balance between gains and losses with the quantitative difference between 15 and 12 ("It's $+8-5$ that gives $+3$ "; "Gained $+3$ "; "We see in the numbers, from 12 to 15"). Then the teacher introduced the third part of the task:

Teacher: What if there were no numbers neither at the beginning nor at the end?

Luís: It's just to find out if you gain or lose and how much.

Teacher: This new proposal has only the gains and losses in different games. Let's see what happens at the end of each day.

Paulo seemed to deal easily with quantitative difference whether it is an excess or a deficit. He did this part very quickly, in two minutes. When the researcher got closer, Paulo explained: "these annul each other" (for example, $-2, +2$), having calculated only the remaining ones. His solution presents positive and negative numbers notation (with the signals $+$ and $-$) and it can be seen in Figure 3.

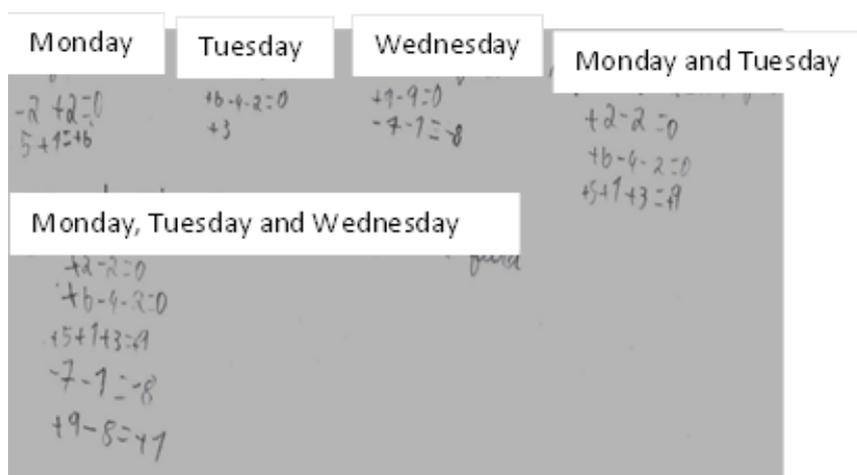


Figure 3: Paulo's resolution of the third part of the task

The pair Luís and Lúcia approached the third part using a different strategy. They added the gains, added the losses, and then did the additive comparison. In the Wednesday's situation, the students surrounded the sad face, understanding that at the end of the day, the player lost 8 marbles. They registered " $-17; +9$ " and " -8 " below the sad face.

In the whole class discussion, the most presented strategy was the one used by Luís and Lúcia: to add the gains and the losses, and then determine the quantitative difference. For example, for Tuesday, the pair Marta and Monica quickly recorded on the blackboard " $+9-6=+3$ " and drew a 'smile'. The students concluded: "The difference between 9 and 6 is 3 to gain".

Teacher: Is there another explanation?

Paulo: Yes, the $+6, -4$ and -2 annul themselves. (*The teacher records on blackboard: $+6-4-2=0$*)

Paulo: And only $+3$ left over.

This idea of 'annulling' was used by other students in situations that followed and many of them used it correctly.

Conclusion

The students were able to focus on difference as a quantitative structure, looking at gains and losses. Thus, on the contrary to what happened with fifth-grade students in the study of Thompson (1993), these second graders were able to conceive a quantitative difference independently of numerical information about quantities (Smith & Thompson, 2008); they did not need to know the initial number of marbles, in the third part of the task, to be able to think about gains and losses. These students were able to focus on quantities, building relationships from them as stated by Cusi and Malara (2007) and as happened in the studies reported by Schliemann (2015). Hence, they conceived difference as a relative change independently of actual values, considering them as variables within a relational dimension.

This study stresses the fundamental importance of modeling the situations, revealing their structure. Both structural and relational dimensions are key elements of algebraic thinking. The results highlight the easy way in which students manipulated the quantitative differences attributing a contextual meaning, and also suggest the potential of a work focused on quantitative relationships to enhance early algebraic thinking. They open new lines for research in order to understand the multiple factors that may influence students' learning, namely the role of task design, in particular the black square and the final part with unknown initial numbers, as well as the role of teacher's questioning, in particular the funneling questioning, in the development of students' quantitative reasoning.

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