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How students in 5th and 8th grade in Norway understand the equal sign

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Having a relational understanding of the equal sign is seen as important to students in algebra in lower secondary school. The basis for their understanding of the equal sign is laid in elementary school. This paper presents results from two studies in Norway, a quantitative survey and a qualitative task-based interview. In both studies, students in 5th and 8th grade solved an equal sign task. 1230 students participated in the survey and 8 students were interviewed. These latter students also responded to questions about the meaning of the equal sign. Results from both studies show that a majority of the students in 5th grade have an operational understanding of the equal sign, while a majority of those in the 8th grade have a relational understanding of the equal sign.

Keywords: Equal sign, operational understanding, relational understanding.

Introduction

According to Ma (1999, p. 111) the equal sign is “the soul of mathematical operations” and Falkner, Levi, & Carpenter (1999) say that to understand the meaning of the equal sign is a core element to understanding algebra. In Norway, the results from TIMSS show that a major problem in mathematics is the weak results in algebra, with a significant decline in students’ performances from 2011 to 2015 in 8th grade (Grønmo, Hole, & Onstad, 2017). To prepare students in elementary school for algebra in lower secondary school, it is important to work with their understanding of the equal sign. The aim of this paper is to examine how students in 5th and 8th grade in Norway understand the equal sign.

Operational and relational understanding of the equal sign

According to Kieran (2004), one aspect of going from arithmetical thinking to algebraic thinking is for the students to refocus the meaning of the equal sign. Prediger (2010) distinguishes between six different meanings of the equal sign, grouped into three main categories: operational meaning, relational meaning and specification. In this paper, the focus will be on the first two categories. The operational meaning is an asymmetric use of the equal sign. In elementary arithmetic, we often have a calculation to perform on the left-hand side of the equal sign that gives an answer on the right-hand side. By looking at the equal sign as “add the numbers” or “the answer”, students have an operational understanding (Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008). In the relational meaning of the equal sign, the focus is on a symmetric use of the sign. Students with a relational understanding realize that “the equal sign [is] symbolizing the sameness of the expressions or quantities represented by each side of an equation” (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012, p. 222). In arithmetic contexts, the symmetric use of the equal sign can both help to express general relations, and also numerical identities (Prediger, 2010).

Knuth, Alibali, McNeil, Weinberg, and Stephens (2005, p. 69) claim that “in algebra, students must view the equal sign as a relational symbol”. This is especially important when they learn to solve equations with operations on both sides of the equal sign. In order to understand that the transformations performed when solving an equation preserve the equivalence relation, it is essential to have a relational understanding of the equal sign. In a study of middle school students’ (6th to 8th grade) understanding of equivalence, they found that “students who have a relational view of the equal sign outperformed their peers who hold alternative views on a problem that requires use of the idea of mathematical equivalence” (Knuth et al., 2005, p. 74).

Rittle-Johnson, Matthews, Taylor, and McEldoon (2011) have made a construct map for knowledge of the equal sign as an indicator of mathematical equality, from *rigid operational* on level 1 to *comparative relational* on level 4. Students at level 1 are only successful at solving task “equations-equals-answer” and giving an operational definition of the equal sign. At level 2, *flexible operational*, students are successful at solving, evaluating and encoding atypical equation structures, for example tasks with the operations to the right of the equal sign. At level 3, *basic relational*, students can handle operations on both sides of the equal sign in equations. They can also recognize and generate a relational definition of the equal sign. At level 4, students can successfully solve and evaluate equations “by comparing the expressions on the two sides of the equal sign, including using compensatory strategies and recognizing that performing the same operations on both sides maintains equivalence” (Rittle-Johnson et al., 2011, p. 87). After using this construct map in the analysis of students’ understanding of equivalence, they comment that “describing children as having an operational or relational view of equivalence is overly simplistic” (Rittle-Johnson et al., 2011, p. 97). There may be students who are in the transition between an operational and a relational understanding of equivalence. They also noted that it was more difficult for students to give a relational definition of the equal sign than to solve equations with operations on both sides of the equal sign. In another study they also mention that the use of letters as variables instead of using blanks, may add difficulty to a task (Matthews et al., 2012).

Rittle-Johnson et al. (2011) claim that 35 years of research on elementary school students’ understanding of the equal sign shows that a majority of them have an operational understanding of the equal sign. It seems that if they get a task such as $8 + 4 = \square + 5$, they either add the numbers to the left of the equal sign or they add all numbers. According to Molina and Ambrose (2006), tasks like this one can only be solved correctly if the students have a “broad” understanding of the equal sign. In this study, I focus on how students answered a similar task. They were also asked about the meaning of the equal sign. Hopefully this study will provide further knowledge of the transition between an operational and a relational understanding of the equal sign. My research question is: *how do students in 5th and 8th grade in Norway understand the equal sign?*

Methods

The data in this paper comes from two projects, one quantitative survey study and a qualitative interview study. In 2013, 1230 students in 5th and 8th grade (age 10 and 13) from two different municipalities in Norway completed a mathematics test as part of a large classroom study (Haug, 2017). The test had 40 tasks for the students in 5th grade and 52 tasks

for those in 8th grade (the same tasks as in 5th grade and in addition 12 others). Results from 584 students in 5th grade and 646 students in 8th grade will be presented in this paper. In 2018, a task-based interview (Goldin, 1993) was conducted of 8 students from other schools in Norway, 4 from 5th grade and 4 from 8th grade. The tasks used in these interviews were all multiple-choice tasks picked from the survey study. All of them had seven response alternatives, including “don’t know”. In addition to the correct answers, the rest were so-called distractors. These interviews also form part of this paper.

In this paper I mainly focus on students’ responses to one of the tasks (Figure 1). In this task they are asked about which number is hidden behind the smiley face to make the arithmetic correct. A student with an operational understanding of the equal sign may answer 9 that matches “operation equal answer” or 16 that corresponds to “add all” (taking the subtract 5 to be negative five). A student who answers correctly (2) probably has a relational understanding of the equal sign. It may also be possible for those who answer 7 or 12 to have a relational understanding. By ignoring subtraction of 5, 7 will be a correct answer and if they do not notice that there are different signs on the left- and the right-hand sides and think there is addition on both sides of the equal sign, 12 will be correct. It is unclear how students get 0 in response to this task.

21. Hvilket tall må skjule seg bak smilefjeset for at regnestykket (ligningen) skal være riktig?

$$14 - 5 = \text{☺} + 7$$

Vet ikke

7 12 0 16 2 9

Figure 1: The English translation of the question is: Which number must hide behind the smiley to make the arithmetic (equation) correct? “Vet ikke” in English “Don’t know”

Results

The results from the survey in the equal sign task (Figure 1) show that 29.6% of the students in 5th grade and 62.2% in 8th grade gave a correct answer (2) to this task (Table 1).

| | 7 | | 12 | | 0 | | 16 | | 2 | | 9 | | Don't know | Total | |
|-----------------------|----|-----|----|-----|---|-----|-----|------|------------|-------------|-----|------|------------|-------|-----|
| | N | % | N | % | N | % | N | % | N | % | N | % | N | % | |
| 5th | 14 | 2.4 | 13 | 2.2 | 0 | 0.0 | 100 | 17.1 | 173 | 29.6 | 250 | 42.8 | 34 | 5.8 | 584 |
| 8th | 13 | 2.0 | 8 | 1.2 | 3 | 0.5 | 42 | 6.5 | 402 | 62.2 | 149 | 23.1 | 29 | 4.5 | 646 |

Table 1: Result of the equal sign task in the survey study

These students probably have a relational understanding of the equal sign. The result shows also that the answer that most students in 5th grade have chosen is 9 (42.8%). This corresponds to calculating what is on the left of the equal sign. Of the 8th grade students, 23.1% chose this answer. We also note that the answer with the second highest percentage of incorrect answers among the students is 16, which corresponds to “add all”. In total, 17.1% in

5th grade and 6.5% in 8th grade chose this incorrect answer. This means that approximately 60% of students in 5th grade and 30% of those in 8th grade in this study probably have an operational understanding of the equal sign.

There are 4.6% in 5th grade and 3.2% in 8th grade who answer either 7 or 12. Since only 3 students in 8th grade answer 0, this may indicate that there are few students just guessing an answer. If many students had simply guessed, this answer option would probably also have been chosen by more of them.

As mentioned earlier, a task-based interview was conducted where 8 students were asked to solve the equal sign task. Their names are fictional and the transcripts are translated into English by the author. First, all students were given another task: Calculate $275 - 84 =$. Three of those in 5th grade and all four in 8th grade solved this task without any difficulties. One student, Knut, had problems performing the subtraction algorithm he had chosen to use.

The second task they were given was the equal sign task (Figure 1). First, I will present the results from the students in 5th grade.

The first student, Leif, answered 16.

Leif: Ok ... 14 minus one 5 is equal ... plus ... yes ok ... 14 minus 5 that's 9 and 9 plus 7 is 16 [ticks the box with 16]

Interviewer: That sign there [points to the equal sign in the task] ... what does that mean? ... the sign before the smiley

Leif: Equals? [In Norwegian: Er lik]

Interviewer: Equals. What does that mean?

Leif: It means that it will be ... it's equal ... it should be just the same answer as it is on the other side

Leif then suggested the answer 7, before stating “this was difficult”. When the student solved this task, he showed an operational understanding of the equal sign. But when explaining the meaning of “equals”, he said that it should be similar to what is on the other side, seeing a relation between the left- and the right-hand side of the equal sign. Based on only this one task it is difficult to say whether he has an operational or relational understanding of the equal sign. Perhaps he is a student in the transition between operational and relational understanding.

Olav answered 9 in the equal sign task

Interviewer: How did you think?

Olav: Because 14 minus 5 it becomes 9

Interviewer: What does that sign mean?

Olav: Equals

Interviewer: Equals. What does that mean? Do you know ... can you say anything more than that means equals?

Olav: Like, if you take 1 plus 1 ... then you always have equals because there should be some answer

Interviewer: In order for there to be some answer, yes ... when there is 7 at the end there, what does it mean? Did you>

Olav: < It means that the answer is 9, but then you add 7 afterwards ... then it will be 16

Olav showed an operational understanding of the equal sign both when solving the task and when explaining the meaning of the equal sign. For him the answer was on the right-hand side and the operations on the left of the equal sign. Another student, Knut, showed an operational understanding of the equal sign. He “added all numbers” and answered 16 in this task. On the question of what equals means, he responded “the answer is there”.

Arne answered 12 in the equal sign task.

Interviewer: How did you think?

Arne: Em ... since 4 minus 5 is 9 ... then the answer must be something ... which is under 10 ... so then I thought ... I don't know, I just chose something

Interviewer: You just chose 12?

Arne confirms it with a “yes”. Interviewer then asks about the meaning of the equal sign.

Arne: Equals?

Interviewer: Equals. What does that mean?

Arne: For example, if there is 3 plus 3 then it's 6 ... so then you take equals and then 6

Interviewer: ... That plus 7 there [points to the task] ...

Arne: Mm ... I just thought that ... but I think I thought wrong because that ... I thought I took 2 from there [points to 14 in the task] and gave it to the five and then it became 7 and then I thought what was there then it became 12. But it is ... it's a bit wrong

This student at first tried to explain his thinking when solving the task, then he said he just picked an answer. When asked about the meaning of equals, he gave a concrete example, indicating an operational understanding of the equal sign. Afterwards, he came up with a new explanation of how he solved this task. By subtracting 2 from 14 and adding this 2 to 5 to get 7, he had 7 on both sides of the equal sign. Since he now had 12 on the left-hand side of the equal sign, he thought the answer must be 12. This would be a correct answer if there were addition on both sides of the equal sign. What Arne explained here is an attempt to make the left- and right-hand sides of the equal sign equal. This second explanation indicated that he might have a relational understanding of the equal sign.

Three of the four interviewed students in 8th grade gave a correct answer to this task. These three connected the task to solving an equation, something they know how to do. I will only present a short extract of what they said that shows this.

Lise: [write 2] Well, I thought of the smiley as an x . Then I used the move-swap rule and changed plus 7 to minus 7. Then I took 14 minus 5 minus 7

...

Kari: ... 4 minus 5 is 9. In order for it to be 9 on the other side, there must be 2 ...

Interviewer: Yes, what did you mean by 9 on the other side?

Kari: 14 minus 5, that is 9 ...and an equation ... equal on both sides

...

Nils: ... equation yes ... 14 minus 5 is equal ... yes 14 minus 5 it's now 11 ... 4 mi [interrupt] no ... 14 minus 5 no not 11, 9 is ... and then it must be 2

These three students could solve tasks with calculations on both sides of the equal sign. Two of them said that we must have the same on both sides of the equal sign. Lise solved the task by using rules she had learned about how to solve equations. She was not asked about the meaning of the equal sign. The fourth student in 8th grade, Sven, had problems solving this task. After reading the task he first asked about the “plus 7”. Afterwards, he commented that he had seen such tasks before, and then he answered 9 because “14 minus 5 equals 9”. At the end of the interview, we returned to this task and asked him more about it. First the interviewer reminded him about the answer he gave earlier.

Interviewer: But what does plus 7 means?

Sven: No, I don't know ... oh yes, I'll add 7 or something

Interviewer: If you look at the sign before the smiley [points to the equal sign], what does that sign mean?

Sven: Equals

Interviewer: Yes ... and what does equals mean?

Sven: Aa ... no, I have never thought about that. I don't know

Interviewer: No ... when you see such a sign in your book, what do you think?

Sven: That I will write the answer before

It looks as if this student has an operational understanding of the equal sign. He has not thought about the meaning of the equal sign, even though he has most likely been taught how to solve simple equations.

Discussion, conclusion and implications

In order to answer the question as to how students in 5th and 8th grade in Norway understand the equal sign, I have presented results from a quantitative survey. These show that approximately 60% of the students in 5th grade and 30% in 8th grade give an answer to an equal sign task which corresponds to “operational equal answer” or “add all”. These students most probably have an operational understanding of the equal sign, according to Rittle-Johnson et al. (2011). 29.6% of the students in 5th grade and 62.2% in 8th grade respond correctly to this task. These students have a “broad” understanding of the equal sign, most

likely a relational understanding (Molina & Ambrose, 2006). These results are in line with what 35 years of research have shown (Rittle-Johnson et al., 2011). A higher percentage of students in 8th grade than in 5th grade respond correctly to this task. The reason for this may be that those in 8th grade have worked more with equations and in so doing they might have developed a relational understanding of the equal sign. It may also be the case that they have acquired a mechanical way of solving this kind of tasks (which maybe Lisa has).

The results from the task-based interviews showed that there were students who displayed an operational understanding of the equal sign, both in the way they solved this equal sign task and in the explanation they gave of what the equal sign means. What is more interesting is what was revealed in the interviews with the students Leif and Arne. In solving this task, Leif showed an operational understanding of the equal sign. But when asked what the sign means, he came up with an explanation saying that the left- and right-hand sides of the equal sign must be equal. For Arne, it is the opposite. He explained the equal sign with a concrete example, indicating an operational meaning of the equal sign. But when he solved the equal sign task, he seemed to confuse the subtraction sign with an addition sign on the left-hand side of the equal sign. That gave him a wrong answer, but the explanation he gave would be correct with the correct calculation sign. Leif and Arne were the two interviewed students in 5th grade who seemed to be closest to a relational level. With these two students we saw there was a difference between using a relational strategy when solving equal sign tasks and expressing a relational understanding of the equal sign.

Based on the construction map for knowledge of the equal sign as indicator of mathematical equality, developed by Rittle-Johnson et al. (2011), we saw that all the students who completed the first task in the interviews were at least at the rigid operational level. They were able to solve a task with an operation on the left-hand side and the answer on the right-hand side of the equal sign. Of the four interviewed in the 8th grade, three were able to solve the equal sign task and two of them used argumentation about the two sides being equal. These two students were probably at a relational level (either level 3 or 4). They are able to solve a task with operations on both sides of the equal sign and they reveal a relational understanding of the equal sign. Two of the 5th grade and one of the 8th grade students interviewed may be in the transition between operational and relational understanding of the equal sign.

Despite many years of research on the importance of students at early age developing a relational understanding of the equal sign, this study shows that it does not appear that there has been any progress in Norway with regard to students' understanding of the equal sign. In today's curriculum, the equal sign is not explicitly mentioned. But in Norway, a new curriculum will come into effect in 2020 that applies to 1st to 13th grade. A first draft of the new curriculum was out for consultation in the autumn of 2018. One of the competence aims for 3rd grade states that the students should be able to use equality and inequality in comparing quantities, expressions and numbers, using the equal sign and unequal sign and giving reasons for their choice. If the proposal for a new competence goal is passed, then perhaps we can hope that more students will acquire a relational understanding of the equal sign.

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