



HAL
open science

Color-coding as a means to support flexibility in pattern generalization tasks

Per Nilsson, Andreas Eckert

► To cite this version:

Per Nilsson, Andreas Eckert. Color-coding as a means to support flexibility in pattern generalization tasks. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02416406

HAL Id: hal-02416406

<https://hal.science/hal-02416406>

Submitted on 17 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Color-coding as a means to support flexibility in pattern generalization tasks

Per Nilsson¹ and Andreas Eckert²

¹Örebro University, School of Science and Technology, Örebro, Sweden; per.nilsson@oru.se

²Örebro University, School of Science and Technology, Örebro, Sweden; andreas.eckert@oru.se

This study investigates how color-coding can support processes of flexibility in figural pattern generalization tasks. A lesson from a Grade 8 class serves the case for our investigation. The lesson is part of a larger research project, which is based on the iterative research methodology of design experiments and involves a total of six lessons, distributed over two classes (three lessons in each class). The study shows how coloring can encourage students to move from recursive strategies, like successive addition, and support processes of flexibility in linking algebraic expressions and the meaning of n to visual structures of an expanding figural pattern.

Keywords: Pattern generalization, visualization, color-coding, time-limitation, flexibility.

Introduction

In the development of algebraic thinking, students need to learn to see connections between the structure of a pattern and how the pattern can be described by a general algebraic expression (Rivera, 2010). It is crucial to discern a pattern unit when making such connections, around which a pattern structure can be modeled in relation to the place order of the elements in a pattern series (Nilsson & Juter, 2011).

Based on a constructivist methodology, educational research in mathematics has significantly contributed to our developing understanding of how students may obtain algebraic generalizations involving figural and numerical patterns. However, our knowledge is less extensive on how teaching can support the learning of pattern generalization (Rivera, 2010). In other words, if students need to learn to connect visual and symbolic elements effectively in relation to a structural unit, there is a need for further investigations on learning activities that can support that coordination to take place (Rivera, 2010). In this study we aim at making a contribution to such investigations by investigating how *color-coding* in task design (Watson & Ohtani, 2015) can support students' pattern flexibility in the visualization of structures in figural pattern generalizations (Figure 1). We address the research question: How can color-coding be implemented in task design to support flexibility in students' visual structure reasoning, in order to reach a general expression of a figural pattern?

Theoretical background

Individuals see patterns differently depending on how they perceive the structure of the pattern and conceptualize pattern-units (Rivera, 2010). Investigating 8-13 year old students' generalization of linear pattern Stacey (1989) found that the constant difference property was largely recognized and used as a unit to find the n^{th} element by a recursive process of successive addition of this property. Stacey also found that some students were using proportional reasoning, according to the constant difference property. However, when adopting proportional reasoning, Stacey saw that students may

tend to ignore additive elements of a pattern; a significant number of students used a direct proportional method, determining the n^{th} element as the n^{th} multiple of the difference. For instance, the constant difference is three in Task 5 (Figure 1). Applying a direct proportional method of the n^{th} multiple of the difference results in the generalization $3n$, which excludes the extra dot that must be added.

Successive addition of the constant difference property is limited to linear patterns. To develop

Task 5. Miriam was thinking according to the picture. Pick the expression you believe best matches her picture.

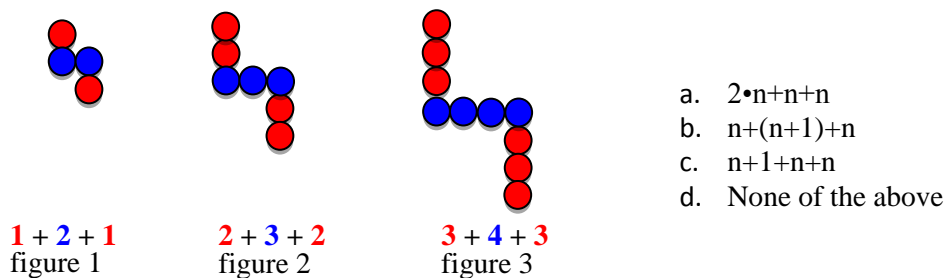


Figure 1: Color-coding in the expansion of a chair

general expressions of patterns where the difference between subsequent elements in the pattern is not constant (e.g. quadratic patterns) one needs to adopt visual structure reasoning (cf. Rivera, 2010; Stacey, 1989). In Task 5, the constant difference is three and, applying the strategy of successive addition, the number of dots for the fourth figure is found from $4+3+3+3$. If we generalize this strategy we reach the expression $3n+1$. However, in visual structure reasoning we define the unit by partitioning the figures in appropriate parts, in relation to the place order of the figures (El Mouhayar & Jurdak, 2015). In Task 5 we can visually structure the third figure in threes. In other words, the third figure of the pattern is composed by three threes plus one extra dot. Similarly we find out that the fourth figure of the pattern consists of three fours, plus one extra dot. Obviously we come to the same general expression, $3n+1$, as in the additive situation. However, important to note is that n has a different meaning in the two situations. In successive addition n gains meaning as the number of constant differences, related to the place value of the n^{th} element in the series, whilst in visual structure reasoning n is a structural feature of a figure.

Generalization tasks are often grouped in near generalization tasks and far generalization tasks (El Mouhayar & Jurdak, 2015). Near generalization tasks involve finding the value of a step that is close to previous steps and far generalization tasks consist of determining the value of a step that is relatively far from given steps. Previous research reports that while near generalization tasks are accessible to a majority of the students, solving far generalization tasks is not. In the present study we will use both near and far generalization tasks, to challenge pattern flexibility, towards the formulation of a general expression, which is based on visual structure reasoning.

It has been observed that, once students have fixed on a pattern in a certain way, it can be hard for them to give up their perception (Lee, 1996). The key to success of seeing an algebraically recordable pattern seems to be at the first stage of pattern perception, where a certain flexibility is necessary to hit on a mathematically recordable pattern (Zazkis & Liljedahl, 2002). Visualizations

of a pattern can provide strong support for the symbolic representation of a pattern (Healy & Hoyles, 1999). Perception, or visualization, in pattern generalization is however not a passive process (Nilsson & Juter, 2011). Getting insight to a problem situation includes an operative, analytical activity (Deliyianni, Elia, Gagatsis, Monoyiou, & Panaoura, 2009; Duval, 1998), like structuring a figure into appropriate parts and/or transforming a figure into another figure (Rivera, 2010). In the present study we will investigate how a teacher can use color-coding as a visual means for supporting flexible, visual structure reasoning, to reach a general expression for a figural pattern.

Method

Design research methodology

This study is part of a larger research project that investigates teaching and learning of pattern generalization. The project follows the methodology of design experiments (Cobb, Confrey, Lehrer, & Schauble, 2003), involving a lesson series of three consecutive lessons. The project team consists of two researchers, the authors of the paper, and two teachers (Teacher A and Teacher B). Teacher A taught a class of 13-year old students and Teacher B taught a class of 14-year old students. All lessons were developed and analyzed according to *prospective*, *reflective* and *retrospective* analyzes (Cobb et al., 2003). Each lesson was conducted twice: First in Teacher A's class and then in Teacher B's class. The changes made to Teacher B's teaching were based on reflective analyses of what happened in the corresponding lesson of Teacher A's teaching. In the present study we focus exclusively on Lesson 1.

Lesson 1 involved seven tasks, distributed through Socrative¹, and aimed to support flexibility in visual structure reasoning in pattern generalization. All tasks in Lesson 1 were situated in the task context of the growing chair-pattern (Figure 1) (Rivera, 2010). In Lesson 1, a time-limited far generalization task and color-coding were at the heart of the lesson. We focus, particularly, on Lesson 1B (Teacher B's class). However, the results of the study are based on a retrospective analysis of both Lesson 1A and Lesson 1B, and the relationship between them. All lessons were video-recorded from three different positions; one placed at back of the classroom, one at front of the classroom and one zooming in on a specific group.

The answers of each task were made available to both the teacher and the students, which were intended to reveal students' reasoning and create an opportunity for whole class discussion where students could react on each other's reasoning. No color-coding was added to the first four tasks. The first task was a near generalization task. Here the students were asked to figure out the number of dots in the 4th figure of the chair-pattern. The second task was a semi-far generalization task, where students were asked to figure out the number of dots in the 9th figure. The third task and follow up question were far generalization tasks about the number of dots in figure 1000. In the third task the students had a limited amount of time. They were asked to reflect on whether it was

¹ Socrative is an online student response system where teachers post questions that show up on students' devices and allows them to post answers. Answers become instantly available to the teacher, which s/he can choice to make available to the students.

possible for them to complete the task within a minute with the method they had been using in the second task. Until now the students were expected to reason according to the constant difference property, since that was how they had solved similar tasks in the past. Task four asked students about the meaning of n in pattern tasks. The task was supposed to challenge the students to start to reflect on how n can be connected to place order in a pattern. Figure 1 shows the fifth task, with a color-coded visual structure, designed to stimulate students' visual structure reasoning. The sixth and seventh task flipped task 5 around and asked the students to color-code different versions of algebraic expressions. First the students were asked to apply $n+(n+1)+n$ on the seventh figure; then, in task seven, they color-coded for example $(n+1)+(n-1)+(n+1)$. The tasks, and their relationship, aimed at furthering the students' flexible reasoning; to be flexible in ways of discerning visual structure units by reversing and expanding their reasoning in task 5. To open up for whole class discussions, the students were asked to share their color-coding by publishing them in Padlet².

Method of analysis

The analysis aimed to investigate how color-coding can support flexibility in students' visual structure reasoning, in order to reach a general expression of a figural pattern.

The analysis followed in chronological order, according to the sequence of the seven tasks. In the first step, we coded and grouped the data according to the ways in which students distinguished and built their reasoning on a pattern unit and connected a pattern unit to n , to the place order of the figure in question. Next, we investigated how they changed their way of reasoning about the chair-pattern during the lesson. Particularly, we looked in detail at how color-coding supported students to move from the strategy of successive addition to discern a unit in the visual structure of a pattern, connected to n , i.e. to the place order of the figure in question.

Result and analysis

Successive additive reasoning and n as a symbol for place order of any figure in the pattern

In the first task, 75% of the students claimed that the fourth figure contained 13 dots. Many students discerned three as the constant difference property of the pattern and used this property in successive addition. In order to challenge successive addition and to prompt proportional reasoning the teacher introduced Task 2 and 3. In Task 2 the students were asked to determine the number of dots in the ninth figure. While the students were working on the task, the teacher walked around in the classroom and listened to the groups. He noted that most students conclude that the ninth figure contains 28 dots and that they reached this by successively adding up by three. In order to challenge successive additive reasoning, the students were asked figure out the number of dots in the 1000th figure within one minute, based on the strategy they used in Task 2. This task stimulated monitoring processes (Stylianou, 2002); it challenged the students to reflect on their strategies and how efficient they were. The students submitted their individual answers in Socrative and about 40% of the students believed that they could not determine the number of dots in figure number 1000 within one minute. Strategies that the student found too time-consuming were, for instance, to draw all

² Padlet is an online virtual pinboard that allows students to publish contributions that can be instantly available for the whole class on the white board or on students' own devices.

figures and count the dots or to successively add threes. The formula $3n+1$ had not yet been expressed by the teacher or any student. But from students' group-work the teacher observed that some students expressed aspects of such reasoning. The teacher asked Charlie to explain:

Charlie: Because, as it [the pattern] constantly increases by three, you need to multiply thousand by three and then plus one.

Charlie gives further reasons for his strategy by showing how it works in an example were the number of dots of a figure is known to the class or can be visually confirmed:

Charlie: For instance, if we look at figure 3 [within Figure 1 above], it [the pattern] constantly increases by three, and we take three times three it will be nine and then, as it is ten [dots] there [pointing at the board], you need to add one more. And, then it is the same with the other; thousand multiplied by three plus one.

After the three first tasks we have reason to believe most students in the class came to understand that they can multiply the number of a given figure with three and then add one to reach the number of dots of any figure. However, a general algebraic expression involving n had not yet been formulated. In Task 4 the students were asked to post the meaning of n in Socrative. The most common posts were expressions of $3n+1$ and that n stands for any figure. The teacher asked, "Do we agree, everyone who agrees raise their hands, that n stands for any figure?" Almost all students raised their hands. That the students had a sense of n , as a symbol that stands for any figure, was assumed to be necessary for moving on to Task 5. Task 5 introduced color-coding as a means for supporting students to discern a unit in the visual structure of a figure, in relation to the place order of the figure in question.

Supporting structural flexibility by color-coding

The symbol n was now understood by most students in the class, as a symbol for place order of any figure in the pattern. However, up until now, n was seen as a variable, determining how many constant differences one should multiply to reach the number of dots in any figure. Tasks 5, 6 and 7 were designed to change perspective on n . The tasks were designed to stimulate students' pattern flexibility by pushing them to make sense of n in the visual structure of the expanding chair-pattern.

The students posted individual answers on Task 5 in Socrative. The result were rather even between alternative b) and c) (Figure 1). The students were then asked to discuss in their groups and to explain to each other how they were reasoning. After the group-discussion the students were asked to post an individual answer again and now the majority of students chose alternative b). George was asked to present his reasoning at the whiteboard:

George: I thought I would figure this out first [pointing to $(n+1)$ in the expression]. First I was thinking n plus one. It is this [pointing to the two blue dots in the first figure of the pattern]. This is the blue, one could say. n plus one. And, both of them [pointing to the two red dots in the first figure] are n .

According to flexible processes, George then said that alternative c), displaying $n+1+n+n$, is the same as alternative b). The only difference is the order of the signs. He implicitly manipulated the expression by visualizing a bracket in the expression $(n+1)+n+n$.

A number of students answered a), displaying $2 \cdot n + n + n$. The teacher asked why alternative a) was chosen. Charlie responded:

Charlie: It works for the first figure but not for the other.

We believe this is a relevant interpretation made by Charlie and, supported by his observation, we also suggest that deciding on alternative a) is based on calculations rather than on visual structural reasoning like the one George showed an example of. The color-coding implemented in the design of Task 5 helped George to externalize and show in public his way of reasoning. The design supported George to make sense of n in the visual structure of the pattern and to communicate his meaning making to the class by visually linking parts of the expression to the corresponding parts of the figure.

The class turned to Task 6, to color the seventh figure of the pattern according to $n + (n + 1) + n$. In Task 5 the class observed and followed how George linked an algebraic expression to the color-structure of the pattern figures. Task 6 aimed at letting students make sense of and generalize this idea according to a new figure. The students worked in pairs. As they finished, they posted and made their solutions public at the whiteboard. The group-solutions witnessed the fact that several students had changed perspective of the meaning of n , to see n in the structure of the figures. The teacher picked out Rikki's solution to Task 6, where the link between the algebraic expression and the coloring was evident and proper according to the task (Figure 2).

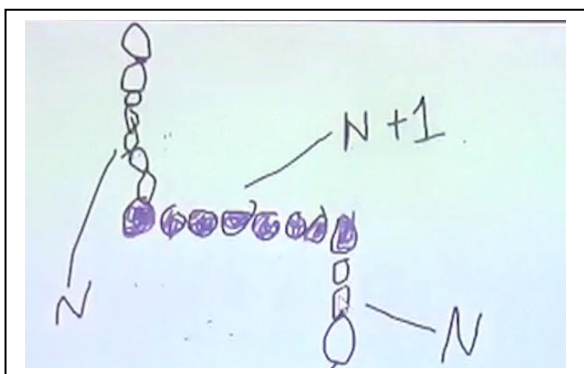


Figure 2: Rikki's color-solution to Task 6

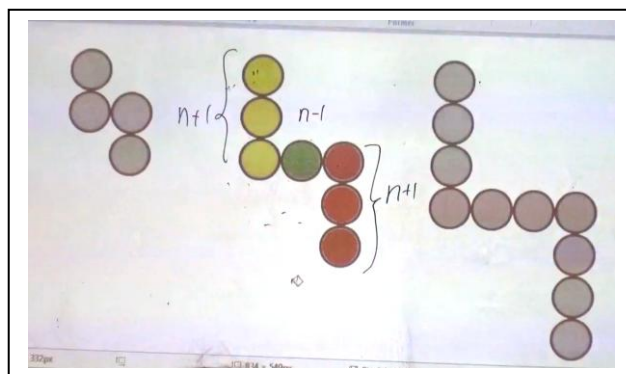


Figure 3: Rikki's color-solution to Task 7

Rikki: First I was taking $n+1$ [saying something audible] and, then I draw that in the middle [The teacher points to the vertical line of eight dots that Rikki has colored in blue]. n is the number of the figure, thus seven [the teacher is nodding in confirmation]. But then it was plus one so, then I did eight. And then it was n on the other sides. Seven down and seven up [the teacher moves his hand over the picture to illustrate down and up of the seven dots Rikki is referring to].

In Task 7 the students were working in pairs and used Paint to color other expressions in the pattern. The teacher noted that Rikki solved the first subtask of Task 7, coloring $(n+1) + (n-1) + (n+1)$. For some reason the teacher did not succeed in presenting Rikki's own solution on the board. Instead, he showed an uncolored picture of the three first figures of the pattern and asked Rikki to guide him through the coloring of $(n+1) + (n-1) + (n+1)$ in the second figure of the pattern (Figure 3). That the teacher failed in up-loading Rikki's solution became something positive from a teaching

perspective. Now Rikki was forced to give details of her reasoning so the teacher was able to understand how he should color the figure. Moreover, in the dialogue between the teacher and Rikki, the rest of the class was provided an opportunity to follow, in detail, how the solution was constructed.

Concluding discussion

In this paper we addressed the research question: How can color-coding be implemented in task design to support flexibility in students' visual structure reasoning, in order to reach a general expression of a figural pattern? In answering the research question, we have shown how coloring in pattern generalization can stimulate processes of flexibility (Stacey, 1989) in linking algebraic expressions and, particularly, the meaning of n to visual structures of expanding figural patterns.

The present study sheds new light on task-design to support reflection on visual structures (Rivera, 2010) and pattern flexibility (Nilsson & Juter, 2011). In pattern generalization, the first stage of pattern perception is crucial, where a certain flexibility is necessary to hit on a mathematically recordable pattern (Zazkis & Liljedahl, 2002). To move beyond linear pattern generalizations one needs to discern a unit from the visual structure of a pattern, connected to the place order of the pattern-figures (El Mouhayar & Jurdak, 2015). The present study shows how color-coding can support such processes of flexibility by turning students' attention to the relationship between an algebraic expression and visual structures of the pattern. The color-coding prompted the students to change perspective of n , from a number variable (how many threes to multiply) to a visual unit in the structure of a figure. In particular, the analysis shows how implementing coloring in pattern generalization task, supported students in discerning and making sense of n in the relationship between n in the structure of a general expression and n in the structure of a figure.

The role of far generalization tasks, and the added condition of time-limitation, has not been the main focus in the present study because of limited empirical evidence. However, the analysis indicates that it might be worth re-considering the role of far generalization tasks in the learning of pattern generalization. Previously, research has focused on students' difficulties to account for far generalization tasks and how students find them more difficult than near generalization tasks (El Mouhayar & Jurdak, 2015). Near generalization tasks can be handled by recursive reasoning, which is not as applicable to far generalization tasks. Recursive reasoning necessitates that the preceding value of the pattern is known. So, to determine the value of the 1000th figure, one needs to know the value of figure number 999. The present study also shows that students find a far generalizing task more difficult than near generalization task due to recursive reasoning, coming in the form of successive addition. Challenging students to a far-generalization task did not support multiplicative, proportional, reasoning in a direct sense. However, the analysis shows how it made students reflect on and examine limitations of a recursive, additive approach and come to be responsive to proportional reasoning. Hence, instead of just accounted for students' difficulties, the far generalization task provided the teacher a context for exposing the limitation of successive additive reasoning and for highlighting and emphasizing multiplicative structures. In the present study the teacher created such a context by making Charlie's reasoning in Task 2 public.

The outcome of the present study calls for further research in the realm of task and lesson design (Watson & Ohtani, 2015). Still many questions are open on a patterning approach to algebra. We need to know more about tasks that challenge students' flexible reasoning and engage students in visual structure reasoning. We encourage research to further explore coloring as principles for

designing tasks in authentic practices of algebra teaching. The current study also indicates the need to carefully sequence tasks in pattern activities, in order to encourage processes of reflection and flexibility that move students' understanding towards multiplicative, structural, reasoning, which can be used to handle patterns that go beyond linear patterns.

References

- Cobb, P., Confrey, J., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32, 9-13.
- Deliyianni, E., Elia, I., Gagatsis, A., Monoyiou, A., & Panaoura, A. (2009). A theoretical model of students' geometrical figure understanding. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 696-706). Lyon, France: CERME.
- Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st century* (pp. 37-51). Dordrecht: Kluwer Academic.
- El Mouhayar, R., & Jurdak, M. (2015). Variation in strategy use across grade level by pattern generalization types. *International Journal of Mathematical Education in Science and Technology*, 46, 553-569.
- Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers? *Mathematical Thinking and Learning*, 1, 59-84.
- Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 87-106). Dordrecht: Kluwer Academic Publishers.
- Nilsson, P., & Juter, K. (2011). Flexibility and coordination among acts of visualization and analysis in a pattern generalization activity. *The Journal of Mathematical Behavior*, 30, 194-205.
- Rivera, F. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics*, 73, 297-328.
- Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, 20, 147-164.
- Stylianou, D. A. (2002). On the interaction of visualization and analysis: the negotiation of a visual representation in expert problem solving. *The Journal of Mathematical Behavior*, 21, 303-317.
- Watson, A., & Ohtani, M. (Eds.). (2015). *Task design in mathematics education: An ICMI study 22*. Berlin: Springer.
- Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49, 379-402.