Linear figural patterns as a teaching tool for preservice elementary teachers – the role of symbolic expressions

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Figural patterns connect several aspects of mathematical activity central to the work of teaching mathematics. In this pilot study, we investigated the solutions of 16 preservice elementary teachers to linear figural patterns of different levels of complexity after the completion of a series of six teaching sessions of a course in mathematics education. We found that a) most students were able to generalize and find the figural number of an arbitrary figure in the sequence; b) about half of the students produced mathematically imprecise formulas when translating from an arbitrary number into a general algebraic expression; c) the formulas students produced frequently lacked structural correspondence with the figural patterns and d) students had difficulties in interpreting figural patterns that are more complex. These results indicate that although the course successfully trains students to generalize with linear figural patterns, more attention to precisely formulating mathematical ideas and to interpretation of more difficult patterns can further improve the training of preservice elementary teachers.

Keywords: generalization, prospective elementary teachers, algebraic thinking, figural patterns.

Introduction

Research on students' algebraic thinking has drawn considerable interest over the past decades. For Blanton and Kaput (2011), experiences in building, expressing and justifying mathematical generalizations constitute the heart of algebra and algebraic thinking. Mason, Burton & Stacey (2010) highlight the importance of the generalization process in mathematics when stating that "generalizations are the life-blood of mathematics. Whereas specific results may in themselves be useful, the characteristically mathematical result is the general one" (p.8). The placement of algebraic content within the mathematics curriculum has received considerable attention. For example, documents from the National Council of Teachers of Mathematics (2000) in the United States, the Department for Education (2014) in England and The Norwegian Directorate for Education and Training (2013) recommend the development of algebraic ideas at elementary and middle school levels through activities such as generalizing number and figural patterns. Several authors have researched algebraic generalizations, but in most cases the target group were pupils in primary or middle school (e.g. Carpenter, Franke & Franke, 2003; Becker & Rivera, 2005; Rivera & Becker, 2009; Rivera, 2010). There is less research on generalization of pre-service elementary teachers (PSETs). A study by Yeúilderea & Akkoç (2010) indicated that PSETs, when trying to find the general term of number patterns, described number pattern rules in relation to differences between terms and used visual models without a purpose. The study of Rivera & Becker (2003) suggests that PSETs who used figural reasoning acquired a better understanding of the generalizations they constructed than PSETs who reasoned numerically, even if relationships among numerical values had greater contribution to similarity between the compared entities than the figural patterns. Måsøval (2011) studied the factors that constrain PSETs' establishment and justification of formulae and mathematical statements that represent generality in different quadratic shape patterns. Her findings indicate three such constraints: The first constraint is related to a limited feedback potential in situations where the students are supposed to solve the mathematical tasks without teacher intervention. The second constraint is related to obstacles the students face when they transform informal mathematical statements expressed in natural language into formal algebraic notation. The third constraint is related to challenges with justification of formulae and mathematical statements that the students have proposed. Hallagan, Rule & Carlson (2009) found that after a problem solving-based teaching intervention, PSETs improved in their ability to generalize, however, they encountered more difficulty with determining the algebraic generalization for items arranged in squares with additional single items as exemplified by x^2+1, than with multiple sets of items, as exemplified by 4x. Callejo & Zapatera (2016) characterised profiles of the teaching competence "noticing students' mathematical thinking" in the context of pattern generalization. PSETs named various mathematical elements to describe the students' answers but did not always use them to interpret the understanding of pattern generalization of each student. Their findings allow one to generate descriptors of the development of this teaching competence and provide information for the design of interventions in teacher education addressed to support the recognition of evidence of students' mathematical understanding.

To meet the goals of teaching in the elementary school curriculum, we need to understand more about how to better prepare PSETs for this undertaking. As part of a formative evaluation of the mathematics education course design, we set out to identify aspects within the course module about generalization that require increased attention and emphasis. Specifically, we asked: Which are the most common challenges that PSETs still face when solving problems with generalizations of linear figural patterns after completing our six-session course module about algebraic generalization?

Theoretical framework

According to Radford (1996), the goal of generalizing spatial or numerical patterns is to find an expression representing the conclusion derived from the observed facts (concrete numbers). Radford claims that the obtained expression is in fact a formula, which is constructed on the basis, not of the concrete numbers in the sequence, but on the idea of a general number. Radford asserts that "general number" appears as preconcepts to the concept of variable. Hence, he claims that the notion of letter as variable is consistent with a generalizing approach to algebra, aiming at establishment of relations between numbers. The point is constructing formulae where the symbols represent generalized numbers (Radford, 1996). He highlights that one of the most significant characteristics of generalization is its logical nature, which makes the conclusion possible. This means that the process of generalization is closely connected to that of justification and proof. The underlying logic of generalization can be of various types, depending on the student's mathematical thinking. Radford (2008) distinguish between arithmetic and algebraic generalization of the pattern. While in both domains some generalizations do certainly occur, in algebra, a generalization will lead to results that cannot be reached within the arithmetic domain. Algebraic pattern generalization involves the students in (1) grasping a commonality, (2) generalizing this commonality to all the terms of the sequence, and (3) providing a rule that allows them to directly determine any term of the sequence (Figure 1; Radford, 2008).

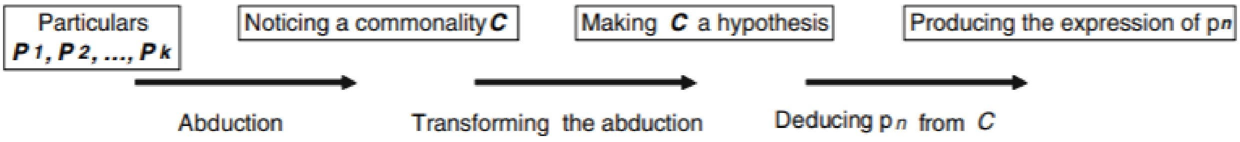


Figure 1: Radford's (2008) architecture of algebraic pattern generalizations

Rivera and Becker (2009) extended Radford's definition by including the necessity of justification at the middle school level. "Students have to provide some kind of explanation that their algebraic generalization is valid by a visual demonstration that provides insights into why they think their generalization is true." (p.213-214) Rivera (2010) claims that meaningful pattern generalization involves the coordination of two interdependent actions, as follows: (1) abductive–inductive action on objects, which involves employing different ways of counting and structuring discrete objects or parts in a pattern in an algebraically useful manner; and (2) symbolic action, which involves translating (1) in the form of an algebraic generalization.

Methodological approach

After a six-session module on algebraic generalization as part of a course in mathematics education, 16 PSETs completed a digital survey with six tasks about generalization of figural patterns. Before data collection, the background of the study and research questions were presented to the PSETs. Participation was voluntary and anonymous. Answers were in the form of multimodal digital texts including text and freehand drawings. Subsequent to data collection, PSETs' responses were downloaded from the server and sorted according to task. A content analysis was performed independently by both authors in order to identify categories of common challenges in the PSETs' algebraic pattern generalization process.

Course content

The Mathematics 2 course is an optional course which integrates mathematics and didactics. The content of the course previous to data collection was: Rich mathematics conversations, Argumentation and proof, Representations, Algebraic thinking, Generalization, Figural numbers, Equality, Relational thinking, Models for negative numbers, Realistic Mathematics Education, Fractions – multiplication and division, Decimal numbers, Percent, Difficulties in Mathematics.

A module with six sessions of 180 minutes was devoted to these topics: algebraic thinking, generalization, figural numbers, equality and relational thinking. These six sessions were taught in English and gave the PSETs the opportunity to discuss several tasks about generalization (not only with figural patterns), their different solutions, and challenges for pupils. PSETs also discussed several research papers on generalization, equality and relational thinking (e.g. Carpenter, Franke & Franke, 2003; Kaput & Blanton, 2005; Becker & Rivera, 2006; Mason, 1996) under the guidance of the course instructor.

Prior to the Mathematics 2 course, in 2016, the PSETs had completed a two-semester compulsory Mathematics 1 course whose main content was: Numbers and the number line; Counting; The position system; Addition and subtraction; Multiplication and division; Fractions – models, comparison, estimation, addition, subtraction; Probability and statistics in primary school.

Survey

The survey consisted of six tasks; for the purpose of this paper we focus only on three of them (Figures 2, 3, 4).

1. Monika began designing the pattern with short sticks. Each day she continues the pattern.
a) Describe how she would proceed to make her design on 5th day.
b) How many sticks will she need to make her design on 8th day?
c) How do you calculate how many sticks she will need to make her design on day number 100? Write down the corresponding formula.
d) Write down the general formula for total number of sticks she will need to make her design on day number n. Explain how you got the formula and how do you know that it is correct.

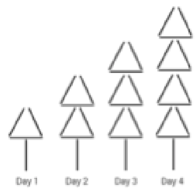


Figure 2: Task 1

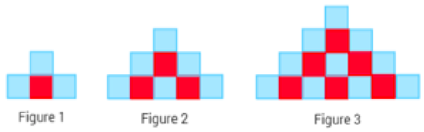
Figure 3: Task 2

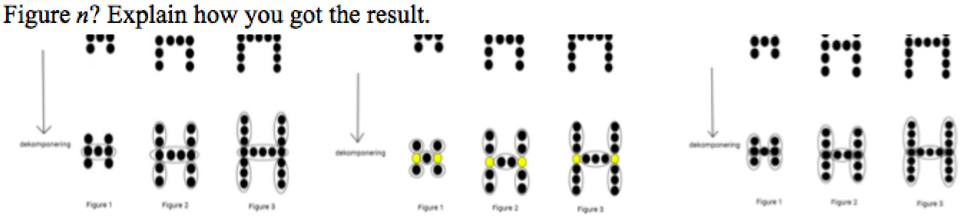
Figure 4: Task 4

Participants

PSETs were in their third year of a 4-year undergraduate teacher education program for Grades 1-7

2. Having the sequence of figures composed of blue and red squares,
a) how many blue squares there would be in Figure 4? Explain how you got the result.
b) consider Figure 7. How many more blue squares than red squares will there be in Figure 7? Explain how you got the result.
c) consider Figure number n. What is the relationship between number of blue and red squares in Figure n? Explain how you got the result.





Decomposition a) Decomposition b) Decomposition c)

at Department of Teacher Education, NTNU in Trondheim, Norway. The whole group consisted of 67 students, but only 20 were present in the last session. For technical reasons we only received answers from 16 students. In the third teaching session, PSETs were given a figural pattern task to solve on their own. Most PSETs did not complete an answer to the first question about near generalization, and all of them gave up answering any of the following questions. All PSETs who answered the survey were present at almost all teaching sessions and the majority of them was participating quite actively during the course. Hence, the subjects in this study were likely among the most highly motivated students in the class.

Findings

Pattern generalization

The majority of students gave satisfactory answers to most of the figural pattern questions, requiring both finding the first few figural numbers (near generalization), as well as generalizing to figure number 50 or 100 (far generalization). This requires successful abductive-inductive action and shows that the PSETs in the study had learned to generalize the figural patterns. This is indeed an encouraging result considering that none of the PSETs could solve figural pattern tasks at the beginning of the course. At the same time, it is likely that the sixteen PSETs who participated in the survey were also among the highest attaining students in the class.

Imprecise symbolic action

For items involving the construction of formulas, typically around half of the subjects formed symbolic expressions containing mathematical imprecisions, indicating a potential for improvement in symbolic action.

Answers from task 1 illustrate this point well. While in task 1 (Figure 2) 75 % of PSETs provided valid generalizations to pattern number 100, only 50 % of PSETs wrote accurate algebraic formulas. Answers from the subjects who did not provide mathematically precise formulas fell into two categories: a) inaccurate use of variables, and b) improper use of the equal sign.

Although the problem text specified that the letter *n* denotes the figure number, 4 PSETs chose to use the letter *x* for their formula without defining what *x* denotes. As for subject 3:

S3: General formula: N=3*x+1; Because we always want to start with 3 since this is the "bush" [that makes up the base of the continuing pattern], then we have to multiply this "bush" with x since we do not know which day we want to find. Finally we must add 1 since all these shapes have only one "stem".

The notation *F(n)* could be used to denote the figural number while the letter *x* was used instead of *n* to denote the figure number. Subject 11:

S11: 100* 3 + 1 Corresponding formula: fn= x*3 +1

These inaccuracies indicate an inflexibility in the students' appropriation of variables, which might be rooted in an experience with solving equations where *x* stands for the unknown.

In some answers, PSETs gave the same variable name to the figural number and the figure number, as for subjects 5 and 10 below:

S5: General formula: N = (Nx3)+1; This expression is correct because figure number N will have N number of triangles. These triangles have to be multiplied by 3, since each triangle consists of three sticks. Then one has to add 1, which corresponds to the stick at the bottom.

S10: General formula: n = n x 3 + 1; Because in each triangle one needs 3 matches, so to make 4 triangles one needs 4x3 (12) matches, and to make 100 triangles one needs 100x3 (300) matches. Therefore n x 3. In the end one needs to add the one match which stands, and then it finally becomes +1.

These responses indicate a lack of attention to equality and the meaning of the equal sign. It appeared not only in solutions of task 1, but also in task 4, as shown in the example below for subject 15 (task 4c, Figure 4).

S15 Here he divides into parts where he circles the same dot twice. n = n + ((n + 1) x 4) – 2. Have to subtract 2 since this is counted twice.

Lack of structural correspondence with figural pattern

Finally, both for correct and incorrect answers, the structure of the formulas given by the PSETs often did not correspond to the figural pattern.

For instance, subjects 5 and 13 responded with the same formula for all three different decompositions of the figure in task 4 (Figure 4):

S13: N = n+2+(nx4); n+2 is the body; nx4 is the arms

Decomposition c) in task 4 contained overlapping regions that were to be subtracted. Still, subject 7 provided the formula in reduced form, reinterpreting the figure from the second decomposition of the same figure:

S7: n*5+2; n is the figural number, 5 is the number that it increases by for each figure, + 2 are the circles that are overlapping encircled, that are constant

While all students wrote some formula for decompositions in task 4a) and 4b), in task 4c) six students wrote “I don’t know”. We could see a similar phenomenon in task 2 (Figure 3). While all students gave adequate answers to tasks 2a) and 2b), five students answered task 2c with “I don’t know”. Slightly more difficult structures of figural patterns (overlapping regions or the two colors, which add a layer of complexity) seem to be challenging for PSETs.

Discussion and conclusion

In this paper we investigated sixteen preservice elementary teachers’ solutions to linear figural pattern generalization tasks. The analysis of these solutions showed that although PSETs typically recognized the underlying structure of linear figural patterns, their algebraic notation and syntax of algebra was often imprecise. This observation corresponds to the second constraining factor identified in Måsøval (2011), related to formalizing mathematical ideas expressed in natural language. Most of the formulas that PSETs provided were meaningful, i.e. the formulas conveyed the underlying structure in the figural patterns. However, several solutions were characterized by imprecise mathematical language. Variable names commonly were not defined or not used consistently, and indicated a lack of attention to equality and to what a variable represented. The fact that it wasn’t an interview, where PSETs were not asked for further and detailed explanations of their answers, might support the assumption that this is how their notation and argumentation will look like in the classroom.

A meaningful pattern generalization must be accompanied by a symbolic action (Rivera, 2010). In the study of Hill et al. (2008), mathematical errors, including errors in language (conventional notation, technical language, general language for expressing mathematical ideas), proved the most strongly related to teacher knowledge. In further research we will investigate whether increased attention to equivalence, symbolic notation, dependent and independent variables, variable names and the accuracy of the mathematical language during teaching sessions on generalization might benefit the development of PSETs' teacher knowledge and better prepare PSETs for teaching generalization in real classrooms. The recent decision to incorporate programming into the Norwegian mathematics curriculum is also an invitation to investigate the role of programming in developing the concept of variables and symbolic action in pattern generalization.

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