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► **To cite this version:**

Dave Hewitt. “Never carry out any arithmetic”: the importance of structure in developing algebraic thinking. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02416275

**HAL Id: hal-02416275**

**<https://hal.science/hal-02416275>**

Submitted on 17 Dec 2019

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# **“Never carry out any arithmetic”: the importance of structure in developing algebraic thinking**

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*The use of tasks which ask learners to find general rules for growing figural patterns is widely reported in the algebra research literature. Such tasks are sometimes seen as a way to develop early algebra thinking. This paper looks at examples of such activities from the literature and presents a theoretical argument against the common practice of learners creating a table of values and seeking patterns within the numbers. Instead an argument is made for learners to focus on more complex examples where learners are discouraged from counting and turning the original figures into numbers. Instead, it is suggested that learners seek structure within complex examples and express what they see without carrying out any arithmetic but instead just writing down what arithmetic they would do.*

*Keywords: Early algebra, pattern spotting, algebraic thinking, structure.*

## **Introduction**

The difference between arithmetic and algebra has been discussed at some length in the literature. For example, Filloy and Rojano (1989) talked about a didactic cut existing when a letter appears on both sides of an equation. This is, of course, an attribute of the equation rather than a statement about ways of thinking which might differentiate between arithmetic and algebraic thinking. Herscovics and Linchevski (1994) spoke about the need for learners to work with the unknown for something to be considered as algebraic activity. This places a greater emphasis on the learner rather than the object, such as the form of an equation, with which they are working. With the emphasis on a learner, Radford (2010a) talked of a *zone of emergence of algebraic thinking* where expressions of generalisation can be expressed in a variety of forms such as gesturing and actions as well as writing. For Radford, there was not a need for letters to be present for algebraic thinking to be emerging. Kieran (2004) allowed the possibility of thinking algebraically about something even if that something is not explicitly on the list of algebra content within the curriculum. This allows for algebraic thinking to happen in many different areas of the curriculum, one of which is number. In this paper I argue for the case to place structure at the forefront of developing the algebraic thinking of learners. Several researchers have included structure as a key aspect of algebra work, including Kieran (2004). Usiskin (1988) included the study of structures within his four conceptions of algebra; Kaput (1995) did the same within his five aspects of algebra; and NCTM's (1998) discussion document included structure within four organizing themes for school algebra. Yet, I will argue that a number of reported teaching practices do not assist learners in placing their attention on structure; indeed often practices do the opposite and turn attention away from structure and onto what I would consider to be more arithmetic thinking.

Radford (2000) posited that algebraic thinking differed from arithmetic thinking in the form of the mathematical practice in which learners were engaged. For him it was about the focus being on, for example, investigating and expressing the general term of a pattern. At one

level, arithmetic is involved in generality in that  $3+4=7$  is an expression of generality: it does not matter what we are talking about, 3 things plus 4 more of the same things result in 7 of those things. This is true no matter what those things are. However, algebra, as generalised arithmetic, requires a second level of generalisation. It stresses the structure of the given situation rather than a 'result'. Thus,  $3+4=4+3$  can be judged to be correct through arithmetic thinking, by carrying out calculations and stressing the similarity of the results from both sides of the equation (both sides result in 7). Algebraic thinking will stress the structure inherent within the operation of addition, that of commutativity; the two sides are the same because it does not matter which way round you do addition. Thus algebraic thinking stresses operations as objects of attention and structure whereas arithmetic thinking stresses processes and with the results being the object of attention. Linchevski and Livneh (1999) introduced the notion of structure sense and since then other authors have developed the notion further (e.g., Hoch & Dreyfus, 2004; Mulligan & Mitchelmore, 2009). In particular, they have in common the idea of pattern, connections and relationships. Mulligan and Mitchelmore (2009) offered an example of seeing a 3 by 5 rectangle as three rows of five columns or as five columns of three rows. Each of these views placed a structured way of seeing the object rather than seeing it just as a whole. My own definition of algebraic structure which will be used in this paper is:

*A way of viewing an object or expression such that it is seen as a combination of recognizable parts along with recognizable patterns which connect those parts together. Such a way of viewing results in an expression of the object in terms of the parts and connections which places the object as a particular example of a more general type.*

The use of figural patterns (either based upon quasi-real life situations, such as the number of chairs around a line of tables, or purely geometrical, such as the length of the perimeter of a line of regular hexagons joined together) is widely reported in the literature (e.g., Montenegro, Costa & Lopes, 2018; Blanton et al., 2015b). I will analyse some examples of such tasks and consider ways in which the students have worked towards trying to find a functional rule for a given initial figural growth pattern. I will argue that taking students' attention away from the original figural situation and onto numbers generated from those patterns reduces the likelihood of students finding functional rules. A key to generalising is to keep attention with the figurative context and to focus on noticing structure. Carrying out arithmetic hides any inherent structure which had been noticed and thus reduces still further the opportunity to express generality. I start off by looking at the common practice of looking at the first few cases of a figural grown pattern and creating a table of values.

## **Tables of values**

Asking learners to create a table of values for the first few cases is a common strategy which is either suggested by teachers or included in the task instructions. Transferring learners' attention from the original problem situation to a table of values means that they can lose contact with three important elements to assist with generality. Firstly the structural properties inherent within the original situation can be lost as learners stare only at numbers. Secondly, the nature of a classic table of values is that it is structured to be in term order from

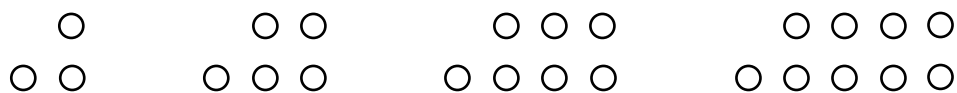
1 onwards. This brings a natural temporal aspect where recursive rules are more likely to be articulated. Thirdly, if a functional rule is found for the numbers, there can never be certainty by looking at the table of values that the rule will apply to the original situation. I will now consider each of these three elements in turn.

### **Losing connection with the context**

In Blandon et al.'s (2015b) study, one task involved finding the number of people sitting round a certain number of square tables which were joined in a line. Drawings of the first two cases were given and then the learners were asked to record their results in a table of values. They were asked to find patterns in the table. One 1<sup>st</sup> grade learner noticed a pattern in their table of values (Figure 3) where the numbers were the same along diagonals. Whilst this is, indeed, a pattern in the table, Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015a) report that this could have been any symbol set where the symbols were the same along a diagonal and was not an observation which necessarily related to a sense of number let alone the contextual situation. Indeed, the creation of a table of values takes away the context within which those numbers were generated.

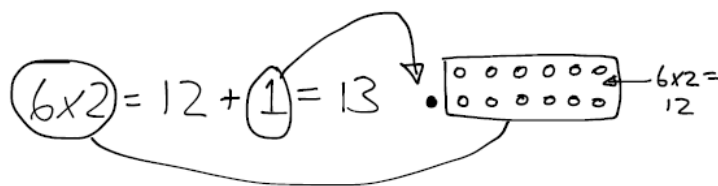
I observed a group of lower attaining 12-13 year olds spending half an hour building examples of the first few cases of a ring of tiles surrounding a 1 by  $n$  rectangle. The task was to find a general rule for how many tiles would be needed to surround such a rectangle. They began by making the first few cases with multi-link cubes and constructed a table of values for these first few cases. They then spent a long time looking at the numbers in the table and not getting further than noticing the recursive rule of the numbers going up in twos. They did not turn their attention back to the multi-link cubes but instead kept staring at the numbers in the table of values. I will say more about how this activity was concluded in a later section. The point here is that they no longer paid attention to the original situation.

These examples are in contrast to when learners' attention remains with the original context and where a structured way of viewing that context can lead to finding a functional rule, as can be seen from the following example. For example, Ferrara and Ferrari (2017) discussed a group of 10-11 year olds working on finding the number of circles in further terms of the sequence shown in Figure 1.



**Figure 1: A figural pattern of dots**

They were also asked to explain to someone from another class how to find the number of circles in any term in this sequence. An example of work from a group of three students showed how they justified their rule by seeing the figural patterns in a structured way. For example, Figure 2 shows a link between the notational expression and the pictorial situation for the 6<sup>th</sup> term.



**Figure 2: Re-drawn version of a learner’s response from Ferrara & Ferrari (2017, p. 28)**

They also showed an alternative way of structuring the geometric figure when one of them explained that “The figure is made of two rows: First row, second row, which, the second row is equal to the first row, plus 1” (p. 28). The written explanation for the 6<sup>th</sup> term (Figure 2) and the more verbal articulation in terms of the two rows, reveal two ways of imposing a structured way of viewing the geometric figure rather than seeing it as a single collection of circles. Both of these ways of viewing reveal a sense of generality where I feel confident they would be able to calculate the number of circles for any term in this sequence.

### **The seduction towards finding recursive rules**

The second concern I have about tables of values is that their usual sequential nature can seduce students into noticing recursive rules rather than functional rules. Recursive rules give a way in which the next term can be generated from the previous term. This can involve the carrying out of a long series of arithmetic operations if a ‘far’ term in a sequence is to be found. Learners can also find it difficult to make use of a recursive rule in order to generate a functional rule; recall, for example, the case of the learners trying to find a rule for a ring of tiles surrounding a 1 by  $n$  rectangle. They had a recursive rule but this did not help them towards finding a functional rule.

Montenegro et al. (2018) described a class of 18 Portuguese students aged between 10 and 13 working on a sequence of growing geometric figures in the shape of an ‘L’. Two of the four groups were not able to find a functional rule even though they had found a recursive rule. All groups had turned their attention to working with numbers from the first six cases as this had been encouraged from the questioning of the task. The teacher decided that the students had missed the potential within the visual representation of the ‘L’ shapes and focused their attention on the shapes; asking students to say what was the same and what was different about the figurative representations. This led to students making comments such as “Figure 15 has one common square and 15 on each side” and “figure  $n$  will have a common square and  $n$  squares on each side”. This led to an exclamation of “Ahhhhh!” (p. 102) and the students were able to quickly find the functional rule. The focus being turned from trying to find patterns in numbers to looking for structure within the original geometric context was significant in the success obtained by those students.

The sequential nature of a table of values encourages learners to see a recursive rule, which is often ‘easier’ than a functional rule as it is usually additive rather than multiplicative, linear rather than quadratic, etc. It also does not always help with finding a functional rule.

### Is the rule correct?

The third concern I have about tables of values is that even if a functional rule is found for the data in the table, students can never know from the table alone if that rule is correct for the original contextual situation. For example, without me telling you, you would not know the context which the table in Figure 4 represented. The contextual situation has disappeared and the activity becomes one of pure number pattern spotting. There is a danger that this can lead to little awareness of the mathematical situation from which those numbers originated (Hewitt, 1992). Some patterns spotted may not actually apply to the general situation from which the numbers originated. The table in Figure 4 gives a strong sense of the rule  $2^{n-1}$ , however if the numbers came from the first five cases of looking at the maximum number of regions formed within a circle when  $n$  points on the circumference are joined to each other by chords, then this rule would not be correct. The case for six points produces a maximum number of 31 regions and not the 32 which might be expected from looking only at the table of values.

Justification or proof is never possible when attention is solely with a table of values. Palatnik and Koichu (2017) reported a pair of 9<sup>th</sup> grade students who obtained formulae connected with the problem of finding the maximum number of pieces a pizza can be cut into with  $n$  straight ‘cut’ lines. The students had a sense that their formulae were correct as they fitted with the numbers in the table of values. However, when challenged to work on why the formulae worked, the students had to return to the context of the problem to articulate reasons for why they were correct. So, the common practice of encouraging learners to construct a table of values (e.g., Blanton et al., 2015a; Blanton et al., 2015b) restricts any pattern spotting to the level of conjecture at best. Any sense of certainty of generality requires attention to be on the structure of the original problem situation, not a table of numbers.

<b>p</b>	<b>x</b>
3	4
4	5
5	6
6	7

Figure 3: Numbers on a diagonal are the same

1	1
2	2
3	4
4	8
5	16

Figure 4: A table of values. What is the rule?

### Never carry out any arithmetic

I will now return to the story of the lower attaining students who were trying to find a functional rule for the number of tiles surrounding a 1 by  $n$  rectangle. They spent a long time looking at their table of values without getting anywhere towards finding a functional rule. After a long while observing them, I asked them how many tiles there would be in a situation where the rectangle was 47 long. They replied quickly that it would be two lots of 47 plus 3 for each end. Within one minute

they had expressed the rule of  $2n+3+3$ . The articulation of  $2 \times 47 + 3 + 3$  rather than the single number of 100, which would be the result of carrying out the arithmetic of that expression, meant that the structured way in which they viewed the tiles was contained within that expression. The stressing of the structure allowed them to see the generality that, however long the rectangle was, they needed two lots of that length plus another two sets of three tiles. A significant aspect of this anecdote is that the rectangle was long enough so that they would not attempt to build this and count the number of tiles. As Radford (2010b) noted, it is important to work with a case where the numbers involved are too big for a learner to carry out arithmetic or to count. This forces the need to seek structure. Indeed I go further and say that in order to head towards algebra in such situations, all arithmetic should be avoided.



**Figure 5: Chairs round tables**

Chimoni, Pitta-Pantazi & Christou (2018) give an indicative answer from a group of 10-13 year olds who were successful in solving all the tasks given to them. In this case they were asked to find how many children could be seated around a group of 10 trapezoidal tables (see Figure 5 for the first two cases). The answer was presented similar to  $N_{10} = 5 + (9 \times 3) = 32$ . The number 32 may be seen as the answer to the question but it is the expression  $5 + (9 \times 3)$  which reveals the structural way of seeing the situation. It is this expression, not the 32, which offers the insight to go on to express a general rule of  $5 + ((n-1) \times 3)$ , which may be written in the conventional form of  $5 + 3(n-1)$ . I argue that in order to head towards general algebraic statements *learners should never do any arithmetic, just write down the arithmetic they would do*. This is another reason why a table of values is often not as helpful as it might initially be considered to be; arithmetic is carried out in order to enter each value into the table. This means that any structural insights obtained are lost. Ferrara and Sinclair (2016) describe a teacher who worked with a class of grade 3 learners to see if they could work out the position number of 26 in the sequence 2, 4, 6, ... The teacher stressed that this was not to be done by counting. After a while several children were clear that doubling was involved. The teacher asked what number would be in position 99. One learner answered “Hum, I don’t know the double of 99” (p. 13). For algebra, I argue that you do not need to know the double of 99, you just need to express the arithmetic you would do (i.e.  $2 \times 99$ ). This then leads towards an articulation of  $2 \times n$ , or  $2n$ . In some ways a learner might struggle to carry out some arithmetic whilst just writing down what arithmetic they would do can feel easier. In that sense expressing an algebraic structure in written form can feel easier than having to carry out any arithmetic involved.

## Summary and implications

The research literature has shown a frequent practice of presenting learners with figural diagrams and encouraging them to construct a table of values for the first few terms. This results in learners’ attention being focused on to numbers rather than the context from which those numbers arose. The activity for learners then becomes number pattern spotting and there are examples of students

finding patterns in the numbers which can be detached from the contextual situation. The nature of a table of values is such that recursive rules are likely to be found and students do not always find it easy to translate such rules into a general functional rule. The argument I have made is that it may be preferable to stay with the contextual situation and encourage learners to find ‘structural ways of seeing’. This is supported by Strømskag (2015), who identified staying with the context as important for a successful pattern-based approach to algebra. Indeed, learners who stayed focused on the original figures were more likely to take a functional approach (El Mouhayar & Jurdak, 2016) and find several important *connected* features (El Mouhayar, 2018), whereas those who focused on numbers were more likely to take a recursive approach and noticed features which were *disconnected* with each other.

The principle of *never carrying out any arithmetic but just writing down what arithmetic you would do* can allow the structured way of seeing to be expressed within a written expression. Such expressions, and their associated ways of seeing, are a step towards seeing generality through a particular example (Mason, 1987). To aid this process of not carrying out any arithmetic and stressing structure instead, it is worth teachers considering offering just one example for learners to work on. This example should be one which is sufficiently complex so that learners are not able to count and are unlikely to be able to do any necessary arithmetic (Radford, 2000). This will increase the likelihood of learners expressing the structure they observe and writing it in a form which preserves that structure. Since a large number will be involved, this acts as a quasi-variable (Fujii & Stephens, 2001) in which the number acts as if it were a variable as learners have a sense that this could be any other number and the expression would still be true.

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