Cores of Ariki-Koike algebras

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NICOLAS JACON AND CÉDRIC LECOUVEY

Abstract. We study a natural generalization of the notion of cores for l-partitions attached with a multi-charge $\mathbf{s} \in \mathbb{Z}^l$: the $(e, \mathbf{s})$-cores. We rely them both to the combinatorics and the notion of weight defined by Fayers. Next we study applications in the context of the block theory for Ariki-Koike algebras.

1. Introduction

Let $\mathbb{F}$ be a field of characteristic p. Let l and n be positive integers and $\mathbf{s} := (s_1, \dots, s_l) \in \mathbb{Z}^l$. Fix $\eta \in \mathbb{F}^*$. The Ariki-Koike algebra $\mathbb{F}\mathcal{H}_n^{\mathbf{s}}(\eta)$ associated with this datum is the unital associative $\mathbb{F}$-algebra with a presentation by:

- generators: T_0, T_1,..., T_{n-1},
- relations:

$$\begin{aligned}
&T_0T_1T_0T_1 = T_1T_0T_1T_0,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \ (i = 1, ..., n-2),\\
&T_iT_j = T_jT_i \ (|j-i| > 1),\\
&(T_0 - \eta^{s_1})(T_0 - \eta^{s_2})...(T_0 - \eta^{s_l}) = 0,\\
&(T_i - \eta)(T_i + 1) = 0 \ (i = 1, ..., n-1).
\end{aligned}$$

Let $e \geq 2$ be minimal such that $1 + \eta + \ldots + \eta^{e-1} = 0$ in $\mathbb{F}$ so that $e \in \{2, 3, \ldots\} \cup \{\infty\}$ and $e = p$ if and only if $\eta = 1$.

The Ariki-Koike algebra , also called Hecke algebra of the complex reflection group $G(l, 1, n)$, has been intensively studied during the last past decades. It is in relation with various important objects (e.g. rational Cherednik algebras, quantum groups, finite reductive groups etc.) and has a deep representation theory. Recently, the interest on these algebras have even grew up thanks to the introduction of the quiver Hecke algebras which has strengthened their relations with the theory of quantum groups and has allowed the study of their graded representation theory.

When $l = 1$, the Ariki-Koike algebra is nothing but the Hecke algebra of the symmetric group. When $\eta = 1$ (and thus $e = p$), it is the group algebra of the symmetric group over $\mathbb{F}$. More generally, in the case where e is a prime number (and even in the case where $p = 0$), the representation theory of this algebra presents strong analogies with the modular representation theory of the symmetric groups (in characteristic e): both structures admit a class of remarkable finite dimensional modules indexed by the set of partitions of n: the Specht modules. The simple modules are indexed by the set of e-regular partitions and the decomposition matrices, which control their representation theories, can be connected using an adjustment matrix. Using these decomposition matrices, one can obtain a natural partition of the set of Specht modules into smaller subsets called blocks. To each block, one can also associate another notion: the weight. Roughly speaking, this positive integer measures how "complicated" this block is. Remarkably, one can describe the blocks and the weights quite easily using well known combinatorial notions. In particular, the most "simple" blocks, the blocks with weight 0, can be described explicitly: they are singleton consisting of a unique Specht module labeled by an e-core partition. Moreover, any block with a given weight w may be obtained from this simple blocks by adding w times e-hooks to the given e-core partition. Importantly, all these properties still make sense when e is an arbitrary postive integer (strictly greater than 1) on the side of the Hecke algebra.

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When $l > 1$, one can also define analogues of Specht modules. They are now indexed by the set of l-partitions of n. The simple modules are then naturally indexed by certains generalizations of e-regular partitions which depend on $\mathbf{s}$: the Uglov l-partitions. A notion of weight has also been provided by Fayers in [1] which generalizes the case $l = 1$. Thanks to this definition, many properties known in the case $l = 1$ have been extended to the general case $l \in \mathbb{N}$. In particular in [8], Lyle and Mathas have given a necessary and sufficient condition for two Specht modules for being in the same block. However, the generalization of e-core partitions and a generalization of the above process of adding e-hooks were missing in this picture (even if, as explained in §4.1, a non explicit definition of core multipartitions has nevertheless been given by Fayers in [3]).

The aim of this paper is to study in details the $(e, \mathbf{s})$-core l-partitions, as introduced in a recent paper by the authors [5]. We show that this notion gives the right generalization of the e-core partitions: they correspond to the elements with weight 0 (with respect to Fayers definition of weight), and all l-partitions with a given weight may be obtained from them by adding analogues of e-hooks. As a consequence, we obtain a direct and simple generalization of what happen in the case $l = 1$. The only difference with this latter case is that, in our definition, the core of an l-partition associated with a multicharge is also a multipartition associated with a multicharge but, this last multicharge may be different from the initial one. To do this, the strategy is to show that essentially all the theory can be derived from the case $l = 1$ by introducing a weight-preserving map, inspired by the work of Uglov, from the set of $(e, \mathbf{s})$-core l-partitions to the set of e-core partitions.

The paper will be organized as follows. We first recall the definition of our main object of study: the $(e, \mathbf{s})$-cores and provide some of their combinatorial properties. The third part studies the weights of the l-partitions as defined by Fayers. We show how this notion can be interpreted in the theory of Fock spaces and computed via a combinatorial procedure detailled in our last section. This section will also explore some consequences of our results and will explain how our approach can simplify the block theory for Ariki-Koike algebras.

2. Generalized cores and abaci

In this section, after recalling certain classical combinatorial definitions regarding the partitions, we introduce the notion of $(e, \mathbf{s})$-core multipartition. Then we use abaci to associate to each $(e, \mathbf{s})$-core a certain core partition. This section will be purely combinatorial.

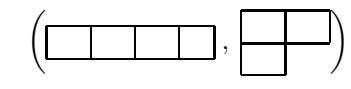
2.1. **Partitions and multipartitions.** A *partition* is a nonincreasing sequence $\lambda = (\lambda_1, \cdots, \lambda_m)$ of nonnegative integers. One can assume this sequence is infinite by adding parts equal to zero. The rank of the partition is by definition the number $|\lambda| = \sum_{1 \leq i \leq m} \lambda_i$. We say that λ is a partition of n. By convention, the unique partition of 0 is the empty partition $\emptyset$.

More generally, for $l \in \mathbb{Z}_{>0}$, an *l-partition* $\boldsymbol{\lambda}$ of n is a sequence of l partitions $(\lambda^1, \ldots, \lambda^l)$ such that the sum of the ranks of the λ^j is n. The number n is then called the rank of $\boldsymbol{\lambda}$ and it is denoted by $|\boldsymbol{\lambda}|$. The set of l-partitions is denoted by Π^l. The *nodes* of $\boldsymbol{\lambda}$ are by definition the elements of the *Young diagram* of $\boldsymbol{\lambda}$:

$$[\boldsymbol{\lambda}] := \{(a, b, c) \mid a \geq 1,\ c \in \{1, \ldots, l\},\ 1 \leq b \leq \lambda^c_a\} \subset \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \ldots, l\}.$$

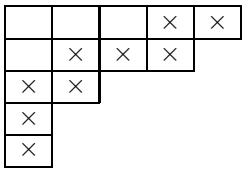
(in the case of partition, the third coordinate which is always equal to 1 will be sometimes omitted.) Each l-partition will be identified with its Young diagram. We say that a node of $\boldsymbol{\lambda}$ is *removable* when one can remove it from the Young diagram of $\boldsymbol{\lambda}$ and still get the Young diagram of an l-partition $\boldsymbol{\mu}$. In this case, this node is called an *addable node* for $\boldsymbol{\mu}$.

Example 2.1. For $l = 2$, the 2-partition $((4), (2, 1))$ of 7 is identified with its Young diagram:



Now let us come back to the case $l = 1$ (we refer to [9] for details). Let $e \in \mathbb{N}_{>1}$. A *rim e-hook* (or simply an e-hook) of a partition λ is a connected subset of the rim of λ with exactly e nodes and which can be removed from λ to obtain another partition μ as in the following example.

Example 2.2. Let $\lambda := (5,4,2,1,1)$ and $e = 3$. The Young diagram of λ is:



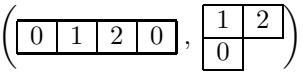
Starting from above, we can successively remove three rim 3-hooks (indicated with the symbol $\times$ above)

By definition an *e-core* is a partition which does not admit any rim e-hook. The set of e-core partitions is denoted by $\mathfrak{C}(e)$. If λ is an arbitrary partition, the *e-weight* $\omega_e(\lambda)$ of λ is the number of consecutive e-hooks which can be removed from λ before obtaining an e-core, which is then denoted by $\mathrm{Core}_e(\lambda)$. These notions are well-defined since both $\omega_e(\lambda)$ and $\mathrm{Core}_e(\lambda)$ do not depend on the order in which the rim e-hooks are removed from λ.

Example 2.3. Keeping the above example, we obtain $\omega_3(\lambda) = 3$ and $\mathrm{Core}_3(\lambda) = (3,1)$.

Let $\mathbf{s} = (s_1, \ldots, s_l) \in \mathbb{Z}^l$. This is called a *multicharge* (a charge if $l = 1$). For an l-partition $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^l)$, one can associate to each node (a,b,c) of the Young diagram its *residue* $b - a + s_c + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$. Such a residue will be identified with its representant in $\{0, \ldots, e-1\}$. If $i \in \mathbb{Z}/e\mathbb{Z}$, we denote by $c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})$ the number of nodes with residue i in the l-partition. We moreover denote $\mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda}) := (c_0(\boldsymbol{\lambda}), \ldots, c_{e-1}(\boldsymbol{\lambda}))$.

Example 2.4. For $l = 2$, $\mathbf{s} = (0,1)$ and $e = 3$ the residues of the nodes of the 2-partition $((4),(2,1))$ of 7 are given as follows:



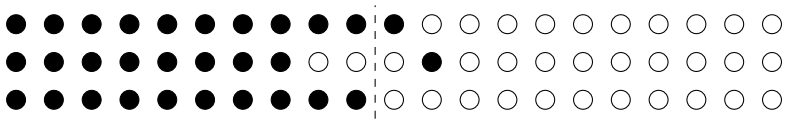
Here we have $\mathcal{C}_{e,\mathbf{s}}((4),(2,1)) = (3,2,2)$.

2.2. **Abaci.** The notion of abacus is convenient to read the weight of a partition and to obtain its e-core. Let $s \in \mathbb{Z}$. An *abacus* is a subset $\mathcal{A}$ of $\mathbb{Z}$ such that $-i \in \mathcal{A}$ and $i \notin \mathcal{A}$ for all i large enough. In a less formal way, each $i \in \mathcal{A}$ corresponds to the position of a black bead on the horizontal abacus which is full of black beads on the left and empty on the right. One can associate to λ and $s \in \mathbb{Z}$ an abacus $L_s(\lambda)$ such that $k \in \mathcal{A}$ if and only if there exists $j \in \mathbb{N}$ such that $k = \lambda_j - j + s$ (Note that λ is assumed to have an infinite number of zero parts). Given an abacus L, one can easily find the unique partition λ and the integer $s \in \mathbb{Z}$ such that $L_s(\lambda) = L$. Indeed, each part corresponds to a black bead of the abacus with length given by the number of empty position at its left, the integer s is equal to $x + 1$ where x is the position of the rightmost black bead in the abacus obtained after sliding all the black beads as much as possible at the right in L.

Example 2.5. Let us take the partition $\lambda := (5,4,2,1,1)$ and $s = 0$. The associated abacus $L_0(\lambda)$ may be represented as follows, where the positions at the right of the dashed vertical line are labelled by the non negative integers:

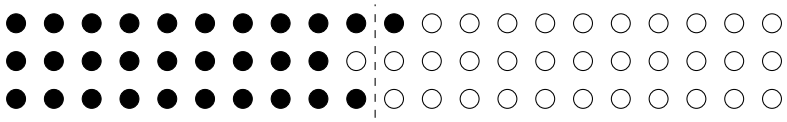
To $L_s(\lambda)$, one can associate an e-tuple of abacus $\mathcal{L}_s^e(\lambda) := (L^0, \ldots, L^{e-1})$. This is done as follows: for each black bead in position k in $L_s(\lambda)$, we write $k = q.e + r$ where $q \in \mathbb{Z}$ and $r \in \{0, \ldots, e-1\}$ and we set a black bead in position q of the abacus L_r. To picture this, write first the abacus L_0, then immediately above the abacus L_1 and so on, so that all the beads associated with the entry 0 of each abacus appear in the same vertical line.

Example 2.6. For the partition (5.4.2.1.1) and $e = 3$, we get the following:



For each runner, sliding one black bead from right to left is equivalent to remove an e-rim hook in the associated partition. As a consequence, after performing this procedure as much as possible, we obtain an e-abacus which can be transformed (by reversing the previous procedure) into an abacus representing the e-core of λ. The number of moves of the black beads gives the e-weight of λ.

Example 2.7. If we do the above procedure for our example, we obtain:



The associated abacus is

whose associated partition is $(3,1)$ and we have $\omega_3(\lambda) = 3$ as in Example 2.3.

Last, we will need an additional notation. For two abaci L and L', we write $L \subset L'$ if we have the following property: for each black bead in position i of the abacus L, there is a black bead in position i in L'.

Let now consider $\mathbf{s} \in \mathbb{Z}^l$ and consider an l-tuple of abaci $(L_{s_1}, \dots, L_{s_l})$. This l-abacus is, as above, conveniently pictured as follow: take first the abacus L_{s_1} and then juste above the abacus L_{s_2} and so on, so that all the beads in position 0 of each abacus appear in the same vertical line.

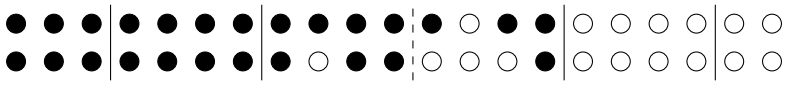
Definition 2.8. Under the above notations, we say that the l-tuple of abaci $(L_{s_1}, \dots, L_{s_l})$ is $(e, \mathbf{s})$-*complete* if:

(1) $l = 1$ and $L_{s_1}(\lambda^1) \subset L_{s_1+e}(\lambda^1)$,
(2) or $l > 1$ and

$$L_{s_1}(\lambda^1) \subset L_{s_2}(\lambda^2) \subset \dots \subset L_{s_l}(\lambda^l) \subset L_{s_1+e}(\lambda^1).$$

To a multicharge $\mathbf{s} \in \mathbb{Z}^l$ and an l-partition $\boldsymbol{\lambda}$, is associated its l-abacus defined as the l-tuple $(L_{s_1}(\lambda^1), \dots, L_{s_l}(\lambda^l))$. It can be pictured exactly as above and will be called the $(e, \mathbf{s})$-*abacus* of $\boldsymbol{\lambda}$. In fact, it does not depends on e but we have chosen here a notation similar to that for the $(e, \mathbf{s})$-core below.

Example 2.9. Let $\mathbf{s} = (0, 3)$ and $e = 4$. We consider the 2-partition $((4,1,1),(1,1))$. Its associated $(e, \mathbf{s})$-abacus $(L_0(4.1.1), L_1(1.1))$ can be represented as follows:



2.3. **The notion of $(e, \mathbf{s})$-cores.** The notion of $(e, \mathbf{s})$-core has been introduced in [5, Def. 5.7] (the definition below is slightly different but it is an easy exercice to show the equivalence). This is a generalization of the notion of e-core partitions in the context of l-partitions associated with a multicharge. First let us introduce the notion of reduced $(e, \mathbf{s})$-core:

Definition 2.10. Assume that $\mathbf{s} \in \mathbb{Z}^l$ then we say that the l-partition $\boldsymbol{\lambda}$ is a *reduced* $(e, \mathbf{s})$-*core* if is $(e, \mathbf{s})$-abacus $(L_{s_1}(\lambda^1), \dots, L_{s_l}(\lambda^l))$ is $(e, \mathbf{s})$-complete.

To give a first study of this notion, let us introduce the following two sets:

$$\overline{\mathcal{A}}_e^l := \{(s_1, \dots, s_l) \in \mathbb{Z}^l \mid \forall (i,j) \in \{1, \dots, l\},\ i < j,\ 0 \leq s_j - s_i \leq e\},$$

$$\mathcal{A}_e^l := \{(s_1, \dots, s_l) \in \mathbb{Z}^l \mid \forall (i,j) \in \{1, \dots, l\},\ i < j,\ 0 \leq s_j - s_i < e\}.$$

Proposition 2.11. *Assume that $\mathbf{s} \in \mathbb{Z}^l$ then if $\boldsymbol{\lambda}$ is a reduced $(e, \mathbf{s})$-core, we have $\mathbf{s} \in \overline{\mathcal{A}}_e^l$.*

Proof. Assume that $(L_{s_1}(\lambda^1), \dots, L_{s_l}(\lambda^l))$ is $(e, \mathbf{s})$-complete then for each $i = 1, \dots, l-2$, we have $L_{s_i}(\lambda^i) \subset L_{s_{i+1}}(\lambda^{i+1})$ which implies that $s_{i+1} \geq s_i$. We also have $L_{s_l} \subset L_{s_1+e}$ and this implies that $s_l \leq s_1 + e$. This concludes the proof.

□

We now give the definition of our main object of interest. Let $\mathbf{s} \in \mathbb{Z}^l$ and $e \in \mathbb{N}_{>0}$, denote by $\widetilde{\mathbf{s}} := (s'_1, \dots, s'_l) \in \{0, \dots, e-1\}^l$ the multicharge such that $s'_i \equiv s_i (\mathrm{mod}\ e)$. Then we define $\sigma_{\mathbf{s}} \in \mathfrak{S}_l$ to be the unique permutation as follows.

$$s'_{\sigma_{\mathbf{s}}(1)} \leq s'_{\sigma_{\mathbf{s}}(2)} \leq \dots \leq s'_{\sigma_{\mathbf{s}}(l)}$$

with the additional property that if $s_{\sigma_{\mathbf{s}}(i)} = s_{\sigma_{\mathbf{s}}(i+1)}$ for $i \in \{1, \dots, l-1\}$ then $\sigma_{\mathbf{s}}(i) < \sigma_{\mathbf{s}}(i+1)$. We set:

$$\widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}} := (s'_{\sigma_{\mathbf{s}}(1)}, s'_{\sigma_{\mathbf{s}}(2)}, \dots, s'_{\sigma_{\mathbf{s}}(l)}). \tag{1}$$

We then clearly have $\widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}} \in \mathcal{A}_e^l$.

Definition 2.12. Let $\mathbf{s} \in \mathbb{Z}^l$, we say that the l-partition $\boldsymbol{\lambda}$ is a $(e, \mathbf{s})$-*core* if the l-partition $\boldsymbol{\lambda}^{\sigma_{\mathbf{s}}} := (\lambda^{\sigma_{\mathbf{s}}(1)}, \dots, \lambda^{\sigma_{\mathbf{s}}(l)})$ is a reduced $(e, \widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}})$-core. We denote by $\mathfrak{C}^l(e, \mathbf{s})$ the set of all $(e, \mathbf{s})$-cores.

As already noted in the previous paragraph, for $l = 1$, the (e, s)-core are exactly the e-cores. Thus, the set $\mathfrak{C}^1(e, s)$ does not depend on $s \in \mathbb{Z}$ and is exactly given by the set of e-cores $\mathfrak{C}(e)$. One can also easily see that if $\boldsymbol{\lambda}$ is a $(e, \mathbf{s})$-core, each component λ^j is an e-core.

Remark 2.13. Assume that $\mathbf{s} \in \overline{\mathcal{A}}_e^l$ and there exists $i \in \{1, \dots, l-1\}$ such that $s_i = s_{i+1}$. Then if $\boldsymbol{\lambda}$ is a reduced $(e, \mathbf{s})$-core we must have $\lambda^i = \lambda^{i+1}$.

We need to check that the reduced $(e, \mathbf{s})$-cores are always $(e, \mathbf{s})$-cores. This is clear if $\mathbf{s} \in \mathcal{A}_e^l$ but not if $\mathbf{s} \in \overline{\mathcal{A}}_e^l \setminus \mathcal{A}_e^l$. So let us assume that $\mathbf{s} \in \overline{\mathcal{A}}_e^l$ but $\mathbf{s} \notin \mathcal{A}_e^l$ and let $\boldsymbol{\lambda}$ be a reduced $(e, \mathbf{s})$-core, this implies that there exists $j \in \{2, \dots, l\}$ such that $s_j = s_{j+1} = \dots = s_l = s_1 + e$. Then the abacus $(L_{s_j - e}(\lambda^l), \dots, L_{s_l - e}(\lambda^l), L_{s_1}(\lambda^1), \dots, L_{s_{j-1}}(\lambda^{l-1}))$ is $(e, (s_j - e, \dots, s_l - e, s_1, \dots, s_{j-1}))$-complete. By the above remark, we thus obtain $\lambda^j = \dots = \lambda^l = \lambda^1$. We so conclude that in the case where $\mathbf{s} \in \overline{\mathcal{A}}_e^l$, the $(e, \mathbf{s})$-cores are exactly the reduced $(e, \mathbf{s})$-cores.

Remark 2.14. The above definition can be formulated in terms of β-numbers and symbols (see [5, §5.1]), which gives an equivalent definition of the set of $(e, \mathbf{s})$-cores. We get that $\boldsymbol{\lambda}$ is a $(e, \mathbf{s})$-core if and only if

- for all $c = 1, \dots, l-1$ and $j \in \mathbb{Z}_{>0}$, there exists $i \in \mathbb{Z}_{>0}$ such that

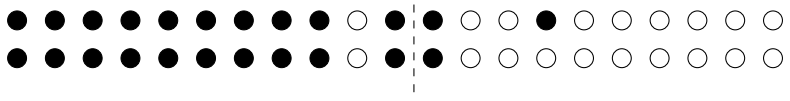
$$\lambda_j^{\sigma_{\mathbf{s}}(c)} - j + s'_{\sigma_{\mathbf{s}}(c)} = \lambda_i^{\sigma_{\mathbf{s}}(c+1)} - i + s'_{\sigma_{\mathbf{s}}(c+1)},$$

- for all $j \in \mathbb{Z}_{>0}$, there exists $i \in \mathbb{Z}_{>0}$ such that

$$\lambda_j^{\sigma_{\mathbf{s}}(l)} - j + s'_{\sigma_{\mathbf{s}}(l)} = \lambda_i^{\sigma_{\mathbf{s}}(1)} - i + s'_{\sigma_{\mathbf{s}}(1)} + e.$$

Remark 2.15. As already noticed, the irreducible representations of the Ariki-Koike algebras associated with the datum $(e, \mathbf{s})$ are naturally labeled by a distinguished set of l-partitions called Uglov l-partitions. In the particular case where $\mathbf{s} \in \mathcal{A}_e^l$, these l-partitions are called FLOTW l-partitions and it is easy to check that any $(e, \mathbf{s})$-core is then a FLOTW l-partition in the sense of [4, Th. 5.8.5]. Now for an arbitrary choice of $\mathbf{s}$, there is an explicit bijection between the set of FLOTW partitions associated with $(e, \widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}})$ and the set of Uglov l-partitions associated with $(e, \mathbf{s})$ (this bijection is described in [6]). It is easy to see that this bijection restricted to the set of $(e, \mathbf{s})$-cores sends $\boldsymbol{\lambda}$ to $\boldsymbol{\lambda}^{\sigma_{\mathbf{s}}^{-1}}$. This implies that $(e, \mathbf{s})$-cores are always Uglov l-partitions. This fact has a representation theoretic meaning as we will see in the following.

Example 2.16. Let $l = 2$, $e = 3$ and $\mathbf{s} = (0, 1)$. Consider the 2-partition $((1, 1), (3, 1, 1))$. With the above notation, we have $\sigma = \mathrm{Id}$ and $\mathbf{s}' = \mathbf{s} = \mathbf{s}^{\sigma_{\mathbf{s}}}$. The associated 2-abacus is



and we see that we here have a $(e, \mathbf{s})$-core. As a consequence, taking $\mathbf{s} = (10, 0)$, we have that the 2-partition $((3, 1, 1), (1, 1))$ is a $(e, \mathbf{s})$-core.

2.4. **Uglov map.** Let $\mathbf{s} \in \overline{\mathcal{A}}_e^l$. We now show how to associate to a reduced $(e, \mathbf{s})$-core $\boldsymbol{\lambda}$ a certain e-core partition that we denote by $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ and conversely. This construction uses a map defined by Uglov [10, §4.1] (see also [11, §3.1]) which associates to any l-partition associated to a multicharge, a certain partition. We will be interested in the restriction of this map to the set of reduced $(e, \mathbf{s})$-cores.

Let $\boldsymbol{\lambda}$ be an l-partition. We consider the l-abacus $(L_{s_1}(\lambda^1), \dots, L_{s_l}(\lambda^l))$. Then we construct an associated 1-abacus as follows. For each $c = 1, \dots, l$ and for each black bead in position k of the abacus L_{s_c}, we write

$$k = q.e + r$$

with $q \in \mathbb{Z}$ and $r \in \{0, \dots, e-1\}$. Then we set a black bead in our new abacus in position $(l-c)e + qel + r$. We then define $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ to be the partition associated with this resulting abacus. We obtain a map

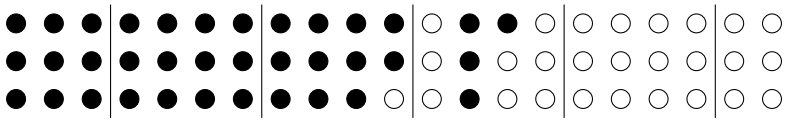
$$\tau_{e,\mathbf{s}} : \Pi^l \to \Pi^1$$

which will be called the *Uglov map*. Let us illustrate the computation of the Uglov map by two following examples.

Example 2.17. We resume the example 2.9. The above procedure gives the following abacus:

We thus get $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}) = (5, 2, 2, 1, 1, 1)$.

Example 2.18. Let $\mathbf{s} = (0, 1, 2)$ and $e = 4$. We consider the 3-partition $((2), (1), (1, 1))$, the associated 3-abacus $(L_0(2), L_1(1), L_2(1, 1))$ can be written as:



The above procedure gives the following abacus:

We thus get $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}) = (7, 4, 2, 2)$.

The map $\tau_{e,\mathbf{s}}$ is not surjective in general but it is clearly injective.

Proposition 2.19. *Let $\mathbf{s} \in \overline{\mathcal{A}}_e^l$, then $\tau_{e,\mathbf{s}}(\emptyset)$ is an e-core.*

Proof. This is clear by the characterization of e-cores with abaci in the last section.

□

Proposition 2.20. *The map*

$$\begin{array}{rccc} \tau_e^l : & \{(\boldsymbol{\lambda}, \mathbf{s}) \mid \mathbf{s} \in \overline{\mathcal{A}}_e^l,\ \boldsymbol{\lambda} \in \mathfrak{C}^l(e, \mathbf{s})\} & \to & \{(\lambda, s) \mid s \in \mathbb{Z},\ \lambda \in \mathfrak{C}^1(e)\} \\ & (\boldsymbol{\lambda}, \mathbf{s}) & \mapsto & (\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}), \sum_{1 \le i \le l} s_i) \end{array}$$

is bijective.

Proof. First, the map is well defined. Indeed, assume that $\boldsymbol{\lambda} \in \mathfrak{C}^l(e, \mathbf{s})$. From Definition 2.12 (2), we get that the partition $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ satisfies (1) of Definition 2.8. This implies that it is an e-core as desired. Now let us prove that the map is bijective. Let $s \in \mathbb{Z}$ and $\lambda \in \mathfrak{C}^1(e)$. Then we have an associated (e, s)-abacus associated with this datum and by construction, there exists a unique $\boldsymbol{\lambda} \in \Pi^l$ and $\mathbf{s} \in \overline{\mathcal{A}}_e^l$ such that $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \lambda$ and $\sum_{1 \le i \le l} s_i = s$. It thus suffices to prove that $\boldsymbol{\lambda} \in \mathfrak{C}^l(e, \mathbf{s})$. But it follows from the fact that its $(e, \mathbf{s})$-abacus is complete. Indeed, we obtain that $\boldsymbol{\lambda}$ is a reduced $(e, \mathbf{s})$-core and thus a $(e, \mathbf{s})$-core.

□

Remark 2.21. If we consider $\mathbf{s} \notin \overline{\mathcal{A}}_e^l$ and a $(e, \mathbf{s})$-core $\boldsymbol{\lambda}$ then we have $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}) \notin \mathfrak{C}^1(e)$ in general.

We now give two important results showing remarkable links between $\boldsymbol{\lambda}$ and $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$. The first one compare the number of nodes in the two Young diagrams with a given residue.

Proposition 2.22. *Let $\boldsymbol{\lambda} \in \Pi^l$ and $\mathbf{s} \in \overline{\mathcal{A}}_e^l$. Set $s = \sum_{1 \leq i \leq l} s_i$. For all $i = 0, 1, \ldots, e-1$, we have:*

$$c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) - c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\emptyset)) = c_i^{e,\mathbf{s}}(\boldsymbol{\lambda}) + l.c_0^{e,\mathbf{s}}(\boldsymbol{\lambda})$$

Proof. We will argue by induction on the rank of $\boldsymbol{\lambda}$. If this rank is 0 then $\boldsymbol{\lambda}$ is the empty l-partition and the result is trivial. Assume now that $\boldsymbol{\lambda}$ is an l-partition of rank $n > 0$. Let $\boldsymbol{\mu}$ be an l-partition of rank $n-1$ which is obtained from $\boldsymbol{\lambda}$ by removing a removable i-node for some $i \in \mathbb{Z}/e\mathbb{Z}$. Assume first that $i \neq 0 (\mathrm{mod}\ e)$. Then it is easy to see that $\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$ is obtained from $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ by removing a removable i-node. As a consequence, we have $c_j^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) = c_j^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu}))$ and $c_j^{e,\mathbf{s}}(\boldsymbol{\lambda}) = c_j^{e,\mathbf{s}}(\boldsymbol{\mu})$ if j is different from i modulo $e\mathbb{Z}$. Thus, we get $c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) = c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) + 1$ and $c_i^{e,\mathbf{s}}(\boldsymbol{\lambda}) = c_i^{e,\mathbf{s}}(\boldsymbol{\mu}) + 1$. So the formula is still true by induction.

Assume now that $i = 0 (\mathrm{mod}\ e)$. In this case, we still have $c_j^{e,\mathbf{s}}(\boldsymbol{\lambda}) = c_j^{e,\mathbf{s}}(\boldsymbol{\mu})$ if $j \neq 0$ and $c_0^{e,\mathbf{s}}(\boldsymbol{\lambda}) = c_0^{e,\mathbf{s}}(\boldsymbol{\mu}) + 1$. Now, we need to see how $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ is obtained from $\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$. The node that we add to $\boldsymbol{\mu}$ to obtain $\boldsymbol{\lambda}$ corresponds to a black bead in the abacus of $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ and to another in the abacus of $\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$. Let us denote by m the number of black beads between theses two positions (not including these two) in the abacus (the number is the same in both abaci). Then $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ is obtained by removing a part of length $x > 0$ ending by a node with residue $e-1$, adding one node to the m parts above and adding one part of length $x + l.e - m + 1$ which ends with a node with residue 0. This thus consists in $x + l.e + 1$ consecutive nodes. More precisely, to obtain $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ from $\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$ we add $l+1$ nodes with residue 0, and l nodes of residue j for all $j \neq 0$. Thus we obtain

$$c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) = c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) + l$$

if $i \neq 0$ and

$$c_0^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) = c_0^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) + l + 1$$

Now we have by induction for all $i \in \{0, \ldots, e-1\}$:

$$c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) - c_i^{(e,s)}(\tau_{e,\mathbf{s}}(\emptyset)) = c_i^{e,\mathbf{s}}(\boldsymbol{\mu}) + l.c_0^{e,\mathbf{s}}(\boldsymbol{\mu})$$

which permits to conclude. □

Recall the notation $\mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda})$ introduced in Subsection 2.1 for the multiset of residues of a multipartition.

Corollary 2.23. *Let $\boldsymbol{\lambda} \in \Pi^l$, $\boldsymbol{\mu} \in \Pi^l$ and $\mathbf{s} \in \overline{\mathcal{A}}_e^l$. We have*

$$\mathcal{C}_{e,s}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) = \mathcal{C}_{e,s}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) \iff \mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\mu})$$

Proof. This directly follows from the previous proposition. □

Last, we will need a useful property which permits to compare the number of removable and addable i-nodes of λ and $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$. To do this, we denote by $M_i^{\mathbf{s}}(\boldsymbol{\lambda})$ the number of addable nodes of $\boldsymbol{\lambda}$ minus the number of removable nodes of $\boldsymbol{\lambda}$.

Proposition 2.24. *For all $\boldsymbol{\lambda} \in \Pi^l$, $\mathbf{s} \in \overline{\mathcal{A}}_e^l$ and $i \in \mathbb{Z}/e\mathbb{Z}$, we have:*

$$M_i^{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{cases} M_i^s(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) & \text{if } i \neq 0, \\ M_i^s(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) + l - 1 & \text{if } i = 0. \end{cases}$$

Proof. First, consider a partition λ and a charge s and write its associated 1-abacus. Let $i \in \mathbb{Z}/e\mathbb{Z}$. Let $x \in \mathbb{Z}$ be such that $x \equiv i (\mathrm{mod}\ e)$. Note that each black bead in the abacus corresponds to a part λ_i of the partition λ (the position of this bead being given by $\lambda_i - i + s$).

- If we have a black bead in position x and a black bead in position $x-1$, this does not correspond to any removable nor addable i-node.
- If we have a black bead in position x and no black bead in position $x-1$, this does correspond to one removable i-node.
- If we have no black bead in position x and a black bead in position $x-1$, this does correspond to one addable i-node.

Of course no black bead in position x and no black bead in position $x-1$ give no associated addable or removable i-node.

Let us fix $r << 0$ and let us now consider all the black beads in position greater (or equal) than $r.e$ in the abacus, for each $i \in \mathbb{Z}/e\mathbb{Z}$, write $B_i^r(\lambda, s)$ the number of such black beads in position x in the abacus with $x \equiv i (\text{mod } e)$. This number is finite by assumption. The above discussion shows that:

$$M_i^s(\lambda) = \begin{cases} B_{i-1}^r(\lambda, s) - B_i^r(\lambda, s) & \text{if } i \neq 0 \\ B_{i-1}^r(\lambda, s) - B_i^r(\lambda, s) + 1 & \text{if } i = 0 \end{cases}$$

(the last equality comes from the fact that we have a black bead in position $r.e-1$).

Now let $(\boldsymbol{\lambda}, \mathbf{s}) \in \Pi^l \times \mathbb{Z}^l$. We fix again $r << 0$, by the discussion above, for each $c \in \{1, \dots, l\}$ and $i \in \mathbb{Z}/e\mathbb{Z}$, we have:

$$M_i^{s'_{\sigma_{\mathbf{s}}(c)}}(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}) = \begin{cases} B_{i-1}^r(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}, s'_{\sigma_{\mathbf{s}}(c)}) - B_i^r(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}, s'_{\sigma_{\mathbf{s}}(c)}) & \text{if } i \neq 0 \\ B_{i-1}^r(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}, s'_{\sigma_{\mathbf{s}}(c)}) - B_i^r(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}, s'_{\sigma_{\mathbf{s}}(c)}) + 1 & \text{if } i = 0 \end{cases}$$

Now by construction, we obtain for all $i \in \mathbb{Z}/e\mathbb{Z}$

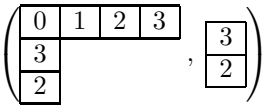
$$B_i^{lr}(\tau_{e,\mathbf{s}}(\lambda), s) = \sum_{c=1,\dots,l} B_i^r(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}).$$

As in addition, we also have:

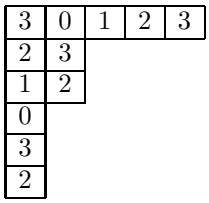
$$M_i^{\mathbf{s}}(\boldsymbol{\lambda}) = \sum_{c=1,\dots,l} M_i^{s'_{\sigma_{\mathbf{s}}(c)}}(\lambda^{s'_{\sigma_{\mathbf{s}}(c)}}),$$

we can conclude. □

Example 2.25. Let us illustrate the proof with the 2-partition $((4,1,1),(1,1))$ and the multicharge $(0,3)$ of Example 2.17 (here $e=4$). The Young diagram with its residues is:



We have seen that $\tau_{e,\mathbf{s}}((4,1,1),(1,1)) = (5,2,2,1,1,1)$ with $s = 0+3 = 3$. Thus the associated Young diagram with residues is:



On the one hand, we have $M_0^{(0,3)}((4,1,1),(1,1)) = 3$ and $M_0^3(5,2,2,1,1,1) = 2$. On the other hand, we get $M_1^{(0,3)}((4,1,1),(1,1)) = M_1^3(5,2,2,1,1,1) = 2$, $M_2^{(0,3)}((4,1,1),(1,1)) = M_2^3(5,2,2,1,1,1) = -2$ and $M_3^{(0,3)}((4,1,1),(1,1)) = M_3^3(5,2,2,1,1,1) = -1$.

3. Cores and weights for Ariki-Koike algebras

In this part, we review the notion of weight for Ariki-Koike algebras as introduced by Fayers in [1]. To avoid a possible confusion with the notion of weight for the type A affine Kac-Moody algebra, Fayers's weights will be refeered as core-weights in the sequel. We will notably interpret them in the representation theory of the type A affine Kac-Moody algebra.

3.1. **Block weights for Ariki-Koike algebras and relations with Fock spaces.** The block weight of an l-partition for a given multicharge is defined in [1] as follows.

Definition 3.1. Let $\mathbf{s} \in \mathbb{Z}^l$, $e \in \mathbb{Z}_{>0}$ and $\boldsymbol{\lambda} \in \Pi^l$, then the block $(e, \mathbf{s})$-weight (or simply block weight) of $\boldsymbol{\lambda}$ is

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) = \sum_{1 \leq i \leq l} c^{e,\mathbf{s}}_{s_i}(\boldsymbol{\lambda}) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c^{e,\mathbf{s}}_i(\boldsymbol{\lambda}) - c^{e,\mathbf{s}}_{i-1}(\boldsymbol{\lambda}))^2.$$

Remark 3.2. From this definition, it is immediate to see that, under the notation of (1), we have for all $\boldsymbol{\lambda} \in \Pi^l$,

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) = p_{(e,\widetilde{\mathbf{s}}^{\sigma_\mathbf{s}})}(\boldsymbol{\lambda}^{\sigma_\mathbf{s}})$$

We can thus again restrict ourselves to the case $\mathbf{s} \in \mathcal{A}^l_e$.

This notion of weight has a natural interpretation in the representation theory of Kac-Moody algebras that we shall now make explicit. Consider the Kac-Moody algebra $\mathfrak{g}$ of type $A^{(1)}_{e-1}$. Let $\mathfrak{h}$ be a $\mathbb{Q}$-vector space with basis $\{h_0, \ldots, h_{e-1}, D\}$. Let $\{\Lambda_0, \ldots, \Lambda_{e-1}, \delta\}$ be the dual basis with respect to the pairing:

$$\langle ., . \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Q}$$

defined by:

$$\langle \Lambda_i, h_j \rangle = \delta_{ij},\ \langle \Lambda_i, D \rangle = \langle \delta, h_i \rangle = 0,\ \langle \delta, D \rangle = 1\ (0 \leq i, j \leq e-1).$$

The Λ_i with $0 \leq i \leq e-1\}$ are called the *fundamental weights*. The *simple roots* α_i with $1 \leq i \leq e-1$ are the elements of $\mathfrak{h}^*$ defined by:

$$\alpha_i := -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} + \delta_{i,0}\delta$$

where the subscript have to be understood modulo e. For $0 \leq i, j \leq e-1$, we denote by a_{ij} the coefficient of Λ_j in α_i. Then the matrix $A := (a_{ij})_{0 \leq i,j \leq e-1}$ is the *Cartan matrix* of $\widehat{\mathfrak{sl}}_e$. As $(\Lambda_0, \alpha_0, \ldots, \alpha_{e-1})$ is a basis of $\mathfrak{h}^*$, one can define a symmetric non degenerate bilinear form on $\mathfrak{h}^*$ by setting:

$$(\alpha_i, \alpha_j) = a_{ij},\ (\Lambda_0, \alpha_i) = \delta_{i,0}, (\Lambda_0, \Lambda_0) = 0\ (0 \leq i, j \leq e-1\}.$$

We then derive

$$(\Lambda_i, \alpha_j) = \delta_{i,j},\ (\delta, \alpha_i) = 0\ (0 \leq i, j \leq e-1),$$

where $\delta = \alpha_0 + \ldots + \alpha_{e-1}$ is the null root. We have $(\delta, \delta) = 0$ and $(\delta, \Lambda_i) = 1$ for all $0 \leq i \leq e-1$.

Let now consider v an indeterminate and write $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ for the *quantum affine algebra* of type $A^{(1)}_{e-1}$. This is an algebra over $\mathbb{Q}(q)$ with generators e_i, f_i, $t_i^{\pm 1}$ $(0 \leq i \leq e-1)$ and δ, the relations will be omitted (see [4, Def. 6.1.3]). Fix $\mathbf{s} \in \mathbb{Z}^l$ and consider the associated Fock space

$$\mathcal{F}_\mathbf{s} := \bigoplus_{\boldsymbol{\lambda} \in \Pi^l} \mathbb{Q}(v)\boldsymbol{\lambda}$$

with basis the l-partitions. There is a simple $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-action on $\mathcal{F}_\mathbf{s}$ (depending on $\mathbf{s}$) which endows it with the structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-module (see [10, Th. 2.1]). In particular, this means that $\mathcal{F}_\mathbf{s}$ is the direct sum of its weight subspaces. The elements of the basis $\boldsymbol{\lambda} \in \Pi^l$ are weight vectors whose weights can easily be calculated as follows:

$$\alpha^{e,\mathbf{s}}(\boldsymbol{\lambda}) := -\Delta_\mathbf{s}\delta + \Lambda_{s_1} + \ldots + \Lambda_{s_l} - \sum_{0 \leq i \leq e-1} c^{e,\mathbf{s}}_i(\boldsymbol{\lambda})\alpha_i,$$

where $\Lambda_\mathbf{s} := \Lambda_{s_1} + \ldots + \Lambda_{s_l}$ and

$$\Delta_\mathbf{s} := \frac{1}{2} \sum_{1 \leq i \leq l} \left((\frac{s_i^2}{e} - s_i) - (\frac{s_i'^2}{e} - s_i') \right),$$

with s_i' is the representant modulo e of s_i in $\{0, 1, \ldots, e-1\}$. Then we set:

$$\|\boldsymbol{\lambda}\|^{(e,\mathbf{s})} := \frac{(\alpha^{e,\mathbf{s}}(\boldsymbol{\lambda}), \alpha^{e,\mathbf{s}}(\boldsymbol{\lambda}))}{2},\ \|\Lambda_\mathbf{s}\| = \frac{(\Lambda_\mathbf{s}, \Lambda_\mathbf{s})}{2},$$

so that

$$\begin{aligned}\|\emptyset\|^{(e,\mathbf{s})} &= \frac{1}{2}(-\Delta_{\mathbf{s}}\delta+\Lambda_{\mathbf{s}},-\Delta_{\mathbf{s}}\delta+\Lambda_{\mathbf{s}})\\ &= -\Delta_{\mathbf{s}}l+\|\Lambda_{\mathbf{s}}\|.\end{aligned}$$

Example 3.3. For $l=1$ and $s=0$ we have $\Delta_s=0$ and $\|\Lambda_s\|=0$ so that $\|\emptyset\|^{(e,0)}=0$.

There is easy way to calculate $\|\boldsymbol{\lambda}\|^{(e,\mathbf{s})}$. The proof is in fact contained in [12, Lemme 4.13] and is similar to [7, Prop 8.1]. We give it below for the convenience of the reader.

Proposition 3.4. *Let $\mathbf{s}\in\mathbb{Z}^l$ and let $\boldsymbol{\lambda}\in\Pi^l$. Assume that $\boldsymbol{\mu}\in\Pi^l$ is such that one can add an addable i-node to $\boldsymbol{\mu}$ to obtain $\boldsymbol{\lambda}$. Then we have*

$$\|\boldsymbol{\mu}\|^{(e,\mathbf{s})}-\|\boldsymbol{\lambda}\|^{(e,\mathbf{s})}=M_i^{\mathbf{s}}(\boldsymbol{\mu})-1$$

where $M_i^{\mathbf{s}}(\boldsymbol{\mu})$ is the number of addable nodes of $\boldsymbol{\mu}$ minus the number of removable nodes of $\boldsymbol{\mu}$.

Proof. Under the aboce notations, we have that:

$$\begin{aligned}\|\boldsymbol{\mu}\|^{(e,\mathbf{s})}-\|\boldsymbol{\lambda}\|^{(e,\mathbf{s})} &= (1/2)\left((\alpha^{e,\mathbf{s}}(\boldsymbol{\mu}),\alpha^{e,\mathbf{s}}(\boldsymbol{\mu})-(\alpha^{e,\mathbf{s}}(\boldsymbol{\mu})-\alpha_i,\alpha^{e,\mathbf{s}}(\boldsymbol{\mu})-\alpha_i)\right)\\ &= (1/2)\left(2(\alpha^{e,\mathbf{s}}(\boldsymbol{\mu}),\alpha_i)-(\alpha_i,\alpha_i)\right)\\ &= (\alpha^{e,\mathbf{s}}(\boldsymbol{\mu}),\alpha_i)-1\end{aligned}$$

Now, by the previous definition of the weight $\alpha^{e,\mathbf{s}}$, we have $\alpha^{e,\mathbf{s}}(\boldsymbol{\mu})=\sum_{0\le i\le e-1}a_i\Lambda_i+d\delta$ if and only if $\delta.\boldsymbol{\mu}=d\boldsymbol{\mu}$ and $t_i\boldsymbol{\mu}=v^{a_i}\boldsymbol{\mu}$. As by [10, Th. 2.1], it is known that $t_i\boldsymbol{\mu}=v^{M_i^{\mathbf{s}}(\boldsymbol{\mu})}\boldsymbol{\mu}$, we can conclude. □

It is now easy to compute the block weight $p_{(e,\mathbf{s})}$.

Proposition 3.5. *Let $\mathbf{s}\in\mathbb{Z}^l$ and let $\boldsymbol{\lambda}\in\Pi^l$. We have*

$$\|\boldsymbol{\lambda}\|^{(e,\mathbf{s})}=\|\emptyset\|^{(e,\mathbf{s})}-p_{(e,\mathbf{s})}(\boldsymbol{\lambda}).$$

Proof. We can write:

$$\begin{aligned}(\alpha^{e,\mathbf{s}}(\boldsymbol{\lambda}),\alpha^{e,\mathbf{s}}(\boldsymbol{\lambda})) &= (-\Delta_{\mathbf{s}}\delta+\Lambda_{\mathbf{s}}-\sum_{0\le i\le e-1}c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})\alpha_i,-\Delta_{\mathbf{s}}\delta+\Lambda_{\mathbf{s}}-\sum_{0\le i\le e-1}c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})\alpha_i)\\ &= 2\|\emptyset\|^{(e,\mathbf{s})}-2\sum_{0\le i\le e-1}c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})(\Lambda_{\mathbf{s}},\alpha_i)+\sum_{0\le i,j\le e-1}c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})c_j^{e,\mathbf{s}}(\boldsymbol{\lambda})(\alpha_i,\alpha_j)\\ &= 2\|\emptyset\|^{(e,\mathbf{s})}-2\sum_{1\le i\le l}c_{s_i}^{e,\mathbf{s}}(\boldsymbol{\lambda})+\sum_{0\le i,j\le e-1}(-c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})c_{i-1}^{e,\mathbf{s}}(\boldsymbol{\lambda})+2c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})^2-c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})c_{i+1}^{e,\mathbf{s}}(\boldsymbol{\lambda}))\\ &= 2\|\emptyset\|^{(e,\mathbf{s})}-2\sum_{1\le i\le l}c_{s_i}^{e,\mathbf{s}}(\boldsymbol{\lambda})+\sum_{0\le i,j\le e-1}(c_i^{e,\mathbf{s}}(\boldsymbol{\lambda})-c_{i-1}^{e,\mathbf{s}}(\boldsymbol{\lambda}))^2\\ &= 2\|\emptyset\|^{(e,\mathbf{s})}-2p_{(e,\mathbf{s})}(\boldsymbol{\lambda}).\end{aligned}$$

□

Combining these two propositions lead to:

Proposition 3.6. *Let $\mathbf{s}\in\mathbb{Z}^l$ and $\boldsymbol{\lambda}\in\Pi^l$. Assume that $\boldsymbol{\mu}\in\Pi^l$ is such that one can add an addable i-node to $\boldsymbol{\mu}$ to obtain $\boldsymbol{\lambda}$. Then we have*

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda})-p_{(e,\mathbf{s})}(\boldsymbol{\mu})=M_i^{\mathbf{s}}(\boldsymbol{\mu})-1.$$

The above proposition will be a crucial ingredient in the proof of one of our main results in the next section.

3.2. **Computation of weights.** We here want to prove the following theorem. It mainly asserts that the block weight for an l-partition associated with a multicharge can always been computed in term of the usual block weight for a partition. This result uses the map $\tau_{e,\mathbf{s}}$ defined in the previous section only for the multicharge in $\overline{\mathcal{A}}_e^l$ (see Proposition 2.20).

Theorem 3.7. *Let $\mathbf{s}\in\overline{\mathcal{A}}_e^l$ and $\boldsymbol{\lambda}\in\Pi^l$. We have:*

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda})=p_{(e,s)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}))$$

where $s=\sum_{1\le i\le l}s_i$

Proof. We argue by induction on the rank n of $\boldsymbol{\lambda}$. Assume that $n = 0$. Then $p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) = 0$ and by Prop. 2.19, $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ is an e-core so its weight is equal to 0. Assume now that $n > 0$. Let $\boldsymbol{\mu}$ be an l-partition obtained from $\boldsymbol{\lambda}$ by deleting a removable i-node for some $i \in \mathbb{Z}/e\mathbb{Z}$. By Proposition 3.6, we have

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) - p_{(e,\mathbf{s})}(\boldsymbol{\mu}) = M_i^{\mathbf{s}}(\boldsymbol{\mu}) - 1.$$

Now we have two cases to consider.

- Assume that $i \neq 0$ then $\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$ is also obtained from $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ by deleting a removable i-node. We have

  $$p_{(e,\mathbf{s})}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})) - p_{(e,\mathbf{s})}(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) = M_i^s(\tau_{e,\mathbf{s}}(\boldsymbol{\mu})) - 1.$$

  We can thus conclude by induction using Proposition 2.24
- Assume that $i = 0$ then to simplify the notation write $\mu := \tau_{e,\mathbf{s}}(\boldsymbol{\mu})$ and $\lambda := \tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$. We assume that the removable node corresponds to a black bead of the 1-abacus of $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ in position x. By hypothesis, there is no black bead in position $x-(l-1)e-1$, because $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ and $\tau_{e,\mathbf{s}}(\boldsymbol{\mu})$ have the same abacus except that the black beads between position x and $x-(l-1)e-1$ are exchanged (and so are the empty position in the remaining one). Again, we will consider two cases:
  - Assume that there is no black bead in position $x-(l-1)e$. Then one can consider ν the partition defined by the abacus obtained by moving the bead in position x from the abacus of λ to the position $x-(l-1)e$. Its weight $p_{(e,\mathbf{s})}(\nu)$ is equal to $p_{(e,\mathbf{s})}(\lambda) - (l-1)$ because ν is obtained from λ by removing $l-1$ hooks from λ. Now we have by Proposition 3.6:

    $$p_{(e,\mathbf{s})}(\nu) - p_{(e,\mathbf{s})}(\mu) = M_0^s(\mu) - 1.$$

    We conclude that

    $$p_{(e,\mathbf{s})}(\lambda) - p_{(e,\mathbf{s})}(\mu) = M_0^s(\mu) + l$$

    that is, by Proposition 2.24

    $$p_{(e,\mathbf{s})}(\lambda) - p_{(e,\mathbf{s})}(\mu) = M_0^{\mathbf{s}}(\boldsymbol{\mu}) - 1.$$

  - Assume that there is no black bead in position $x-(l-1)e$. Then we proceed in the opposite way: we define ν to be the partition obtained from λ by moving the bead in position $x-(l-1)e$ to the position $x-(l-1)e$. Then μ is obtained from ν by moving the bead in position x to the position $x-(l-1)e$ (which consists in removing $(l-1)$ e-hooks). We conclude exactly as in the previous case.

□

What can we do in the case where $\mathbf{s} \notin \overline{\mathcal{A}}_e^l$? In fact, one can use the procedure in §2.3 and associate to $\boldsymbol{\lambda}$ and $\mathbf{s}$ a multicharge $\widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}} \in \mathcal{A}_e^l$ and a multipartition $\boldsymbol{\lambda}^{\sigma_{\mathbf{s}}}$. It is clear from the definition that:

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) = p_{(e,\widetilde{\mathbf{s}}^{\sigma_{\mathbf{s}}})}(\boldsymbol{\lambda}^{\sigma_{\mathbf{s}}})$$

which thus gives an effective way to compute the weight in all cases.

Remark 3.8. A (maybe more direct) proof might also be obtained using Proposition 2.22 but the above one has the advantage to avoid cumbersome computations.

4. Further remarks and applications

In this section, we show how our main results simplify the block theory for Ariki-Koike algebras. In particular we show the relations of our work with some results by Fayers. By the definitions of cores and block weights, one can assume that $\mathbf{s} \in \mathcal{A}_e^l$ in this section. However, we will try to explain how all our results can be adapted to the general case $\mathbf{s} \in \mathbb{Z}^l$.

4.1. **Cores of multipartitions.** Let us start with an easy corollary of Theorem 3.7.

Corollary 4.1. *Assume* $\mathbf{s} \in \overline{\mathcal{A}}_e^l$. *Then, the reduced* $(e, \mathbf{s})$*-core are exactly the elements of block weight* 0.

Proof. Let $\boldsymbol{\lambda} \in \Pi^l$, by Theorem 3.7, we have:

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) = p_{(e,\sum_{1\leq i\leq l} s_i)}(\tau_{e,\mathbf{s}}(\boldsymbol{\lambda}))$$

so $\boldsymbol{\lambda}$ is of block weight 0 if and only if $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$ is of block weight 0. Now, we know that the e-cores are exactly the partitions with block weight 0 and we can thus conclude thanks to Proposition 2.20. □

If we take $\mathbf{s} \in \mathbb{Z}^l$, then we have already noticed that:

$$p_{(e,\mathbf{s})}(\boldsymbol{\lambda}) = p_{(e,\widetilde{\mathbf{s}}^{\sigma_\mathbf{s}})}(\boldsymbol{\lambda}^{\sigma_\mathbf{s}}).$$

Since in addition $\boldsymbol{\lambda}^{\sigma_\mathbf{s}}$ is a reduced $\widetilde{\mathbf{s}}^{\sigma_\mathbf{s}}$-core if and only if $\boldsymbol{\lambda}$ is a $(e, \mathbf{s})$-core, we conclude that in the general case, the $(e, \mathbf{s})$-cores are exactly the elements of core weight 0.

In [3], Fayers has also introduced a notion of core for an l-partition associated with a multicharge. His definition is the following one. Let $\mathbf{s} \in \mathcal{A}_e^l$ then an l-partition $\boldsymbol{\lambda}$ is a $(e|\mathbf{s})$-core if there is no other l-partition $\boldsymbol{\mu}$ such that $\mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\mu})$. In fact this coincides with our notion of $(e, \mathbf{s})$-cores. Indeed, by the results in [1], the $(e|\mathbf{s})$-core multipartitions are exactly the elements of weight 0 (see [3, Rem 2.3.1]) which are exactly the $(e, \mathbf{s})$-cores by the above corollary. In other words, Definition 2.8 thus reveals the combinatorial structure of the $(e|\mathbf{s})$-cores introduced by Fayers. Let us explain the consequences concerning the block theory of Ariki-Koike algebras and especially, the similarities and the differences with the case $l = 1$ that is, the case of the symmetric group.

Let $\mathbb{F}\mathcal{H}_n^\mathbf{s}(\eta)$ be the Ariki-Koike algebra as defined in the introduction. The representation theory of $\mathbb{F}\mathcal{H}_n^\mathbf{s}(\eta)$ is controlled by its decomposition matrix which we now briefly define. For all l-partition $\boldsymbol{\lambda}$; one can define a certain finite dimensional $\mathbb{F}\mathcal{H}_n^\mathbf{s}(\eta)$-module $S^{\boldsymbol{\lambda}}$ called a Specht module. For each $M \in \mathrm{Irr}(\mathbb{F}\mathcal{H}_n^\mathbf{s}(\eta))$, we have an associated composition factor $[S^{\boldsymbol{\lambda}} : M]$. The matrix:

$$\mathcal{D} := ([S^{\boldsymbol{\lambda}} : M])_{\boldsymbol{\lambda}\in\Pi^l(n), M\in\mathrm{Irr}(\mathbb{F}\mathcal{H}_n^\mathbf{s})(\eta)}$$

is the *decomposition matrix*. By definition, two l-partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie in the *same block* if there exists a sequence $(M_1, \dots, M_r)$ of simple $\mathbb{F}\mathcal{H}_n^\mathbf{s}(\eta)$-modules and a sequence of l-partitions $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{r+1})$ with $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}$, $\boldsymbol{\lambda}_{r+1} = \boldsymbol{\mu}$ and for all $i \in \{1, \dots, r\}$, we have $[S^{\boldsymbol{\lambda}_i} : M_i] \neq 0$ and $[S^{\boldsymbol{\lambda}_{i+1}} : M_i] \neq 0$. When $l = 1$, we know that two partitions are in the same block if and only if they have the same e-core and that their common weight is the number of e-hooks that can be removed to obtain this e-core. For $l > 1$, a criterion has been provided by Lyle and Mathas [8] but it does not consist in any notion of hook or cores. It asserts that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are in the same block of $\mathbb{F}\mathcal{H}_n^\mathbf{s}(\eta)$ if we have $\mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\mu})$.

Let $\boldsymbol{\lambda}$ be an l-partition of rank n. To describe the blocks, one can restrict ourselves to the case $\mathbf{s} \in \mathcal{A}_e^l$ (as usual the general case is derived by using the transformations in §2.3). We consider the $(e, \mathbf{s})$-abacus $(L_{s_1}, \dots, L_{s_l})$ of $\boldsymbol{\lambda}$. An *elementary operation* on this abacus is defined as a move of one black bead from one runner of the abacus to another satisfying the following rule.

(1) If this black bead is not in the top runner, then we can do such an elementary operation on this black bead only if there is no black bead immediately above (that is in the same position on the runner just above). In this case, we slide the black bead from its initial position, in a runner i, to the runner $i+1$ located above in the same position. The resulting l-abacus corresponds to an l-partition of rank $n - s_{i+1} + s_i - 1$. Indeed, when we add a black bead in the runner $i + 1$ the rank becomes $n + N - s_{i+1} - 1$ for a certain integer N and when we remove a black bead from the runner i in the same position, the rank becomes $n + N - s_{i+1} - 1 - (N - s_i)$ that is $n - s_{i+1} + s_i - 1$.

(2) If this black bead is in the top runner in position x, then we can do such an elementary operation only if there is no black bead in position $x - e$ on the lowest runner. In this case, we slide the bead to the position $x - e$ of the lowest runner. As above, the rank of the resulting l-partition is $n - (s_1 - s_l + e + 1)$

Note that, after this procedure, the resulting multicharge associated with the l-abacus may not be in $\mathcal{A}_e^l$ but this is not a problem: we can still perform it in the resulting abacus. At the end, by construction, we obtain an l-abacus

$$(L_{v_1}, \dots, L_{v_l})$$

satisfying:

$$L_{v_1} \subset L_{v_2} \subset \ldots \subset L_l \subset L_{v_l+e}.$$

This abacus is complete. Thus, by Proposition 2.11, this corresponds to an l-partition $\boldsymbol{\mu}$ and a multicharge $\mathbf{v} \in \overline{\mathcal{A}}_e^l$ such that $\boldsymbol{\mu}$ is a reduced $(e, \mathbf{v})$-core.

Definition 4.2. The *core* of the l-partition $\boldsymbol{\lambda}$ associated with a multicharge $\mathbf{s}$ is the pair $(\boldsymbol{\mu}, \mathbf{v})$ attached to $\boldsymbol{\lambda}$ and $\mathbf{s}$ by the previous procedure.

Doing an elementary operation on the $(e, \mathbf{s})$-abacus of $\boldsymbol{\lambda}$ as above is equivalent to remove one e-hook on the Young diagram of $\tau_{e,\mathbf{s}}(\boldsymbol{\lambda})$. As a consequence, by Theorem 3.7 and by the definition of the Uglov map, $p_{(e,\mathbf{s})}(\boldsymbol{\lambda})$ is the number of elementary operations we have made in this process to obtain our final abacus. The rank of the multipartition can also been computed thanks to the above remarks. The fact that this does not depend on the order in which the elementary operations are performed follows from the case $l = 1$.

Remark 4.3.

(1) Take a black bead in the runner i of the l-abacus of $\boldsymbol{\lambda}$ in position x such that there is no black bead in position $x-e$ in the same runner. Then one can always perform a series of l elementary operations (as defined in the previous procedure) to obtain the same abacus except that the black bead in position x moves to the position $x-e$ of the same runner. Indeed, let us denote by b_1 the bead in position x in runner i, and consider all the beads $b_2, \ldots, b_k$ in position x and runner $i_2, \ldots, i_k$ with $i < i_2 \ldots < i_k$. Consider also the beads $b_{k+1}, \ldots, b_r$ in position $x-e$ and runner $i_{k+1}, \ldots, i_r$ with $i_{k+1} \ldots < i_r < i$. Then we can slide the bead b_r in position $x-e$ in runner i, and then slide the bead b_{r-1} to the position previously occupied by b_r and so on. At the end, we obtain the desired abacus and we have made l elementary operations to do that. In this case, the rank of resulting l-partition is equal to $n-(s_{i+1}-s_i+1)-(s_{i+2}-(s_{i+1}+1)+1)-\cdots-(s_1-(s_l+1)+e+1)-\cdots-(s_i-1-(s_{i-1}+1)+1)$, that is $n-e$. The l-partition so obtained is just the l-partition $\boldsymbol{\lambda}$ where a rim e-hook has been removed in λ^i. This is thus consistent with our result. Nevertheless, this shorter hook removal procedure does not suffice to produce the core of $\boldsymbol{\lambda}$ for it can only yield a sequence of l cores, that is a multicore.

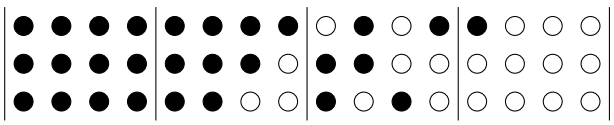
(2) In [1], the notion of multicore is used instead of our notion of core. From an arbitrary l-partition $\boldsymbol{\lambda}$, one can indeed associate another l-partition, with a smaller block weight, which may be seen as an "intermediate" between the given l-partition and its e-core in the sense of Definition 4.2. To do this, we can simply take the e-core of each partition or apply a sequence of elementary operations as we have just explained. We have already seen that the $(e, \mathbf{s})$-cores are multicores but the converse is not true in general.

Corollary 4.4. *Two l-partitions with the same rank have the same core if and only if they belong to the same block of $\mathbb{F}\mathcal{H}_n^{\mathbf{s}}(\eta)$.*

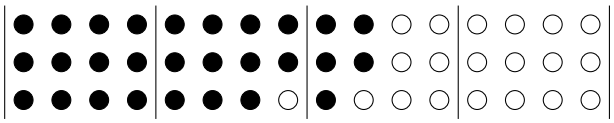
Proof. This directly follows from Corollary 2.23 together with the Lyle-Mathas characterization of blocks. □

Example 4.5. Let us take $\mathbf{s} = (0, 1, 3)$ and $e = 4$. We consider the two 3-partitions $\boldsymbol{\lambda} = ((3, 2), (1, 1), (2, 2, 1))$ and $\boldsymbol{\mu} = ((1), (4, 2), (3, 2))$ with Young diagrams:

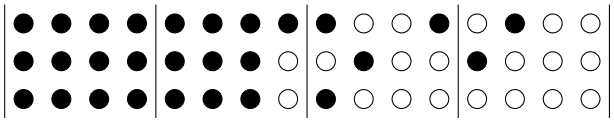
They are in the same block because $\mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\lambda}) = \mathcal{C}_{e,\mathbf{s}}(\boldsymbol{\mu})$. Now the 3-abacus of $\boldsymbol{\lambda}$ is



To determine its core, we perform the above procedure and we obtain the following 3-abacus:



the associated $(e, \mathbf{s})$-core is the 3-partition $((1), \emptyset, \emptyset)$ together with the multicharge $(0, 2, 2)$ and the weight is 8 because we perform 8 moves of beads to obtain this core. Now if we consider $\boldsymbol{\mu}$ whose 3-abacus is



and apply our procedure, one can check that we obtain the same core.

Remark 4.6. When l=1 and given an e-core λ, one can obtain directly all the partitions in a fixed block with a given core weight w by adding w hooks to λ while we stay in the set of partitions. This process is less direct if $l > 1$. Let $\boldsymbol{\lambda}$ be a $(e, \mathbf{s})$-core. We can assume that $\mathbf{s} \in \mathcal{A}_e^l$. Then if we perform w "inverse" elementary moves on its l-abacus, we obtain an l-partition $\boldsymbol{\mu}$ associated with a multicharge $\mathbf{s}'$ and the core of $\boldsymbol{\mu}$ in $\mathbb{F}\mathcal{H}_n^{\mathbf{s}'}(\eta)$ is $(\boldsymbol{\lambda}, \mathbf{s})$. Now, still starting from the e-core, if we do w other "inverse" elementary moves on its l-abacus, one may obtain another l-partition $\boldsymbol{\nu}$ but also another multicharge $\mathbf{s}''$. Thus $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ will be in the same block of $\mathbb{F}\mathcal{H}_n^{e,\mathbf{s}}(\eta)$ if and only if $\mathbf{s}' = \mathbf{s}''$. This means, one can obtain all the l-partitions in a fixed block of $\mathbb{F}\mathcal{H}_n^{\mathbf{s}'}(\eta)$ as in level 1 except we have to keep only those with associated multicharge $\mathbf{s}'$.

4.2. **Multipartitions of small (block) weights.** As already noted in [1], in level $l > 1$, each block of block weight 0 contains exactly a $(e, \mathbf{s})$-core and thus is a simple block, as in the case $l = 1$. This implies in particular that the Specht modules labeled by these l-partitions are irreducible and that they coincide with their projective cover. This shows that the $(e, \mathbf{s})$-cores are always Uglov l-partitions. This is consistent with remark 2.15.

In [1, Th. 4.4], Fayers has given a description of the blocks of block weight 1. Using our approach, we here give an explicit characterization of these blocks. When $l = 1$, such bocks always contain exactly e partitions. We will see that when $l > 1$, this will depend on the multicharge we choose. Let $\mathbf{v} \in \mathbb{Z}^l$ and consider an l-partition $\boldsymbol{\mu}$ with block weight 1. The core af $\boldsymbol{\mu}$ is the same as the core of the l-partition $\boldsymbol{\mu}^{\sigma_{\mathbf{v}}}$ associated with the multicharge $\widetilde{\mathbf{v}}^{\sigma_{\mathbf{v}}} \in \mathcal{A}_e^l$. This means that we can in fact assume that $\mathbf{v} \in \mathcal{A}_e^l$.

Now the l-abacus of a l-partition $\boldsymbol{\mu}$ with block weight 1 for the multicharge $\mathbf{v}$ can be derived from a reduced $(e, \mathbf{s})$-core $\boldsymbol{\lambda}$ where $\mathbf{s} \in \overline{\mathcal{A}}_e^l$ by performing one inverse elementary operation on the abacus of $\boldsymbol{\lambda}$ (that is by inversing the procedure described in §4.1). This consists in moving a black bead in position x from a runner $i \in \{2, \dots, l\}$ to the position x of the runner $i - 1$, or from the runner 1 in position x to the runner l in position $x + e$, if possible.

All the l-partitions $\boldsymbol{\mu}$ of weight 1 are then obtained as follows:

- For all $i \in \{1, \dots, l-1\}$, if $\mathbf{s} := (v_1, \dots, v_{i-1} - 1, v_i + 1, \dots, v_l)$ is such that $\mathbf{s} \in \overline{\mathcal{A}}_e^l$, they are obtained from a $(e, \mathbf{s})$-core $\boldsymbol{\lambda}$ by doing one inverse elementary operation in its abacus from the runner i to the runner $i - 1$. By definition of our notion of core, we can exactly do $v_i + 1 - (v_{i-1} - 1)$ inverse elementary operations between the runner $i-1$ and the runner i. Thus, we have exactly $v_i - v_{i-1} + 2$ multipartitions obtained from a given such core and they are all of the same rank $|\boldsymbol{\lambda}| + v_{i+1} - v_i + 1$.
- If $\mathbf{s} := (v_1 + 1, \dots, v_l - 1)$ is such that $\mathbf{s} \in \overline{\mathcal{A}}_e^l$, they are obtained from a $(e, \mathbf{s})$-core $\boldsymbol{\lambda}$ by doing one inverse elementary operation in its abacus from the runner 1 to the runner l. We have exactly $v_1 - v_l + 2 + e$ multipartitions obtained from a given such core and they are all of the same rank $|\boldsymbol{\lambda}| + v_1 - v_l + e + 1$.

Remark 4.7. By [8], the procedure describes in this paper also gives the description of the blocks for affine Hecke algebras of type A.

Remark 4.8. It is likely that the results of this paper may be used to study the block theory for the cyclotomic Hecke algebras of type $G(r, p, n)$. Besides, Theorem 3.7 gives a correspondance between $(e, \mathbf{s})$-core and e-cores which could induce similarities between blocks of Ariki-Koike algebras and blocks of Hecke algebras of type A. We will come back to these questions in future works.

4.3. **Examples.** We end this section with an example of computation of block weights and cores. We here take $n = 4$, $e = 4$ and $\mathbf{s} = (0, 1)$. Here is a table giving the block weight and the core of each 2-partition.

2-partition	core	block weight	2-partition	core	block weight
$((4), \emptyset)$	$(\emptyset; (0, 1))$	2	$((1, 1), (2))$	$(\emptyset; (0, 1))$	2
$((3), (1))$	$((\emptyset, (1, 1)); (0, 3))$	1	$((1), (2, 1))$	$(((1), (2, 1)); (0, 1))$	0
$(\emptyset, 4)$	$(\emptyset; (0, 1))$	2	$((1, 1), (1, 1))$	$(((2), \emptyset); (0, 3))$	1
$((3, 1), \emptyset)$	$(\emptyset; (0, 1))$	2	$(\emptyset, (3, 1))$	$(\emptyset; (0, 1))$	2
$((2), (2))$	$((\emptyset, 1.1); (0, 3))$	1	$((1, 1, 1), (1))$	$(\emptyset; (0, 1))$	2
$((1), (3))$	$(\emptyset; (0, 1))$	2	$(\emptyset, (2, 2))$	$((\emptyset, (1, 1)); (0, 3))$	1
$((2, 2), \emptyset)$	$(((2), \emptyset); (0, 3))$	1	$((1, 1, 1, 1), \emptyset)$	$(\emptyset; (0, 1))$	2
$((2, 1), (1))$	$(((2, 1), 1); (0, 1))$	0	$(\emptyset, (2, 1, 1))$	$(\emptyset; (0, 1))$	2
$((2, 1, 1), \emptyset)$	$(\emptyset; (0, 1))$	2	$((1), (1, 1, 1))$	$(((2), \emptyset); (0, 3))$	1
$((2), (1, 1))$	$(((2), (1, 1)); (0, 1))$	0	$(\emptyset, (1, 1, 1, 1))$	$(\emptyset; (0, 1))$	2

Note that the core of the blocks of block weight 1 are always associated with the same multicharge, which is $\mathbf{v} = (0, 3)$ and there is two different cores which gives $3 = v_2 - v_1$ elements in the same block in both cases. The rank of this core is the $n - (s_2 - s_1 + 1) = 2$. This is consistent with the results of the previous section.

Let us consider now the multicharge $\mathbf{s} := (0, 1, 3)$ with $e = 4$ and $n = 4$. Then

- we have seven $(e, \mathbf{s})$-cores: $(\emptyset, (3, 1))$, $(\emptyset, (1), (1, 1, 1))$, $((1), (2, 1), \emptyset)$, $((2), (1, 1), \emptyset)$, $(\emptyset, (2), (1, 1))$, $((1, 1), \emptyset, (2))$, $((1), \emptyset, (2, 1))$.
- We have three blocks of clock weight 1 which are:
  - $\{((2, 1), \emptyset, (1)), ((1, 1), (1), (1)), (\emptyset, (1, 1, 1), (1))\}$ with $(((1), \emptyset, (1)), (-1, 2, 3))$ as a core.
  - $\{((3), (1), \emptyset), ((1, 1), (1), (1)), (\emptyset, (2, 2), \emptyset)\}$ with $(\emptyset, (1, 1), \emptyset), (-1, 2, 3))$ as a core.
  - $\{(\emptyset, (1), (3)), (2, 1, 1), ((2, 1), (1), \emptyset)\}$ with $((1), (1), \emptyset), (1, 1, 2))$ as a core.

Again, this is consistent with the results of the previous section.

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