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Investigating early algebraic thinking abilities: a path model

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The introduction of algebra in the elementary school mathematics is expected to navigate students from concrete, arithmetical thinking to increasingly complex, abstract algebraic thinking required in secondary school mathematics and beyond. Yet, empirical studies exploring this idea are relatively scarce. Drawing on a sample of 684 students from grades 4, 5, 6, and 7, this study explored a path model which tested associations between students' abilities in solving different types of early algebraic tasks: generalized arithmetic, functional thinking, and modeling languages. Results emerging from latent path analysis showed that students were more successful in generalized arithmetic tasks and only when this was achieved they were able to solve functional thinking tasks; once these were achieved, they could proceed to solve modeling languages tasks. Qualitative analysis of students' solutions provided further insight into these findings.

Keywords: Early algebraic thinking, generalized arithmetic, functional thinking, modeling languages, path model.

Introduction

While the traditional arithmetic-then-algebra curricula have been proved to be unsuccessful in developing deep understanding of abstract algebra in the secondary grades, it has been suggested that algebra should become a cohesive thread in mathematics curriculum from K to 12 grades (Carragher & Schliemann, 2007). Yet, introducing algebra in the early grades does not mean to move the content of secondary algebra courses to elementary mathematics. Instead, it means setting the ground for students to develop ways of thinking that can support the later learning of algebra. To make this idea more clear, different approaches have been pursued, ranging from the identification of core algebra content strands, such as generalized arithmetic, functional thinking, and modeling languages (Kaput, 2008), to the examination of fundamental algebraic concepts, such as the equals sign (Matthews et al., 2012), to the analysis of essential algebraic processes, such as generalization and representation (Blanton & Kaput, 2005). Moreover, several researchers examined developmental paths where intuitive forms of algebraic thinking become more sophisticated through the mediation offered by appropriate instruction in specific areas of early algebra (e.g. Radford, 2014; Stephens et al., 2017).

The findings of these studies imply that the development of early algebraic thinking lies in the extension of arithmetic that is typically taught in elementary school so that students become aware of its underlying structure and properties and gradually develop the ability to identify and describe functional relationships between quantities that co-vary. Yet, more studies are needed to empirically test this idea. Towards this end, this study aimed to investigate the validity of an algebraic thinking path model which describes early algebraic thinking abilities that students seem to develop first and the way these are associated to other abilities that students seem to develop later on.

The rest of this manuscript is structured as follows: The next section provides an overview of studies that explored the nature of early algebraic thinking, its processes and mathematical content. Following

that, the research question is presented and the hypothesis tested. The methodology, including information about the participants, tasks, and analyses, is outlined next. After presenting the main findings, conclusions and implications for practice are discussed.

Theoretical Framework

Algebraic processes that mark the emergence of early algebraic thinking

A critical issue within the field of early algebra has been the way in which early algebraic thinking might occur and be expressed by students. Several researchers tend to agree that students exhibit early algebraic thinking by the time they begin to formulate generalizations about mathematical relationships, properties, and structure (Blanton & Kaput, 2005). Particularly, the process of generalizing is considered as an undisputed characteristic of early algebraic thinking. Generalizing is interrelated with the process of representing, as students make efforts to articulate and represent their generalizations (Blanton et al., 2015). At the same time, a variety of other processes frame generalizing and representing, such as noticing, hypothesizing, and validating (Blanton et al., 2011).

Algebraic processes are not developed or expressed by students in a mere single way (Stephens et al., 2017). For example, students' use of natural language, gestures or diagrams might be an indicator for the emergence of early algebraic thinking, which will in turn mediate the long-term development of alphanumeric symbolism. Particularly, Radford (2003) exemplified that the way students communicate generalizations might differ, varying from concrete numerical actions (*factual* generalization), to situated descriptions of the objects of the actions (*contextual* generalization), to actions with symbols or signs (*symbolic* generalization). Summing up, algebraic processes seem to be central to students' engagement with early algebraic tasks. The following section describes further the content areas in which these processes and forms of thinking are expected to emerge and grow.

Core algebra content strands and concepts

Kaput (2008) suggested that three core content strands involve algebraic thinking from K to 12 grades: generalized arithmetic, functional thinking, and application of modeling languages. These strands, which are interconnected with the processes described in the previous section, provide a unified framework for analyzing the multidimensional and complex nature of early algebraic thinking (Chimoni, Pitta-Pantazi & Christou, 2018).

Generalized arithmetic refers to the study of structure and relationships that are inherent to numbers and arithmetical operations. The emphasis in this strand is about looking at the structure of arithmetical expressions and computations rather than their results (Kaput, 2008). In generalized arithmetic, students are expected to use the properties of operations and relationships on classes of numbers, interpret the equals sign relationally, and solve single-variable equations and inequalities.

Functional thinking involves generalizing relationships between quantities that co-vary and representing them through multiple tools, such as natural language, alphanumeric symbols, pictures, function tables, and graphs (Blanton et al., 2017). Central in this strand are the concepts of variable and functional relationships, such as recursive patterns, co-variational relationships, and correspondence relationships (Stephens et al., 2017).

Modeling languages refers to the application of a cluster of modeling languages both inside and outside mathematics (Kaput, 2008). In this strand students begin with problem situations and make attempts to mathematise the regularities that are implicitly presented through the situation, using appropriate models, such as verbal expressions, equations, and graphs.

Research question and hypothesis

The research question that guided this study was the following: *Is there a consistent trend in the level of difficulty across generalized arithmetic, functional thinking, and modeling languages tasks that describes students' early algebraic thinking abilities?* In addressing this question and based on theoretical implications described in the previous section, this study tested the following hypothetical path model: *Students are able to solve generalized arithmetic tasks first. Once this is achieved, they proceed to solve functional thinking tasks. Solving modeling languages tasks is achieved last.*

Participants

A convenience sample of 684 students was involved in the study. The sample consisted of 170 fourth-graders (10 years old), 164 fifth-graders (11 years old), 184 sixth-graders (12 years old) and 166 seventh-graders (13 years old). The selection of these grades was done purposefully, since some of the concepts embedded in the three algebra content strands are not covered in earlier grades; at the same time, formal algebra is taught in later grades. Moreover, the existence of four age-groups enabled the examination of the stability of the path model or possible changes across age-groups.

Test

The test used to measure students' early algebraic thinking abilities consisted of 18 tasks that were adapted from previous research studies (e.g. Blanton & Kaput, 2005). The test included generalized arithmetic, functional thinking, and modeling languages tasks. Table 1 presents indicative examples.

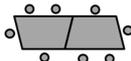
Example	Strand
<p>Is the value of the following expressions the same or not? Justify your answer.</p> <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="border: 1px solid black; padding: 5px; margin: 2px;">$7 + 5$</div> <div style="border: 1px solid black; padding: 5px; margin: 2px;">$7 + 5 + 2 - 2$</div> </div>	Generalized Arithmetic
<p>At a table that has the shape of a trapezium, 5 children can be seated. If two tables are connected, then 8 children can be seated.</p> <p>Table 1 </p> <p>Table 2 </p> <p>(a) How many children can be seated at 3 tables? Justify your answer. (b) How many children can be seated at 10 tables? Justify your answer.</p>	Functional thinking
<p>Joanna will take computers lessons twice a week. Which is the best offer? Justify your answer.</p> <p>Offer A: €8 for each lesson Offer B: €50 for the first 5 lessons of the month and then €4 for every additional lesson</p>	Modeling languages

Table 1: Examples of tasks included in the algebraic thinking test

Considering that students had adequate time to complete the test during administration, tasks with no response were graded with 0 marks. For the multiple choice tasks which had four alternative

responses, one mark was given to each correct response and zero marks were given to each incorrect response. For the tasks that had sub-questions, partial credit was given, considering the maximum sum of the marks of the sub-questions to be equal to 1. In the tasks where students had to justify their answers, the scoring was as follows: 0 mark for fully incorrect responses, 0.50 for giving a correct answer without justifying the answer or giving a wrong justification and 1 mark for giving a correct answer and a proper justification. In some tasks, where students were expected to use various strategies to solve them, a second identification mark was given based on the strategy used. Yet, the variation in students' strategies is not under investigation in the current paper.

Analysis

To analyze the quantitative data collected from the test, Latent Path Analysis was used. This kind of analysis is part of a more general class of techniques called Structural Equation Modeling (SEM). Specifically, Latent Path Analysis is used for testing whether a set of factors that are measured from multiple indicators (latent factors) are connected via directional regression paths. In this study, Latent Path Analysis was used for testing whether the three algebra content strands are connected via a specific path model. The analysis was conducted separately for each age-group and for the whole sample as well.

Firstly, the model described in the study's hypothesis was examined (Model 1). Moreover, two alternative models were also examined (Models 2 & 3). Model 2 assumed that students are able to solve both generalized arithmetic and functional thinking tasks first, and then they proceed to solve modeling languages tasks. Model 3 assumed that students are able to solve modeling languages tasks first, then they proceed to solve functional thinking tasks, and finally they are able to solve generalized arithmetic tasks.

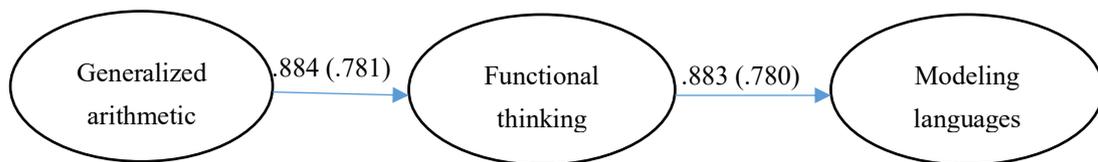
Three fit indices were computed for each model: the chi-square to its degree of freedom ratio (χ^2/df), the comparative fit index (CFI) and the root mean-square error of approximation (RMSEA). For a model to be validated, the observed values for χ^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be close to or lower than .08. The analysis was carried out using the MPLUS software (Muthén & Muthén, 1998).

In order to better interpret the quantitative results and illustrate examples of students' thinking in each strand, qualitative information from students' tests was analyzed: ten students were selected and their solutions were examined. The selection of these students was based on the method of purposeful sampling (Patton, 2002) which suggests the selection of cases that provide rich information about the phenomenon under investigation.

Results

The results of the Latent Path Analysis suggested that the best fitting model was Model 1, since it had the best fitting indices and high regression coefficients for each of the three latent factors. This result was the same for each of the four age-groups and for the whole sample. Figure 1 presents the structure of the model and the regression coefficients, as resulted from the Latent Path Analysis conducted to the data of the whole sample. Specifically, the fitting indices were adequate to provide evidence that supports the structure implied in the model (CFI=.959, $\chi^2=201.853$, $\chi^2/df=1.73$, RMSEA=.033). This,

supports the hypothesis of the existence of a consistent trend. Particularly, it appears that students are first able to solve generalized arithmetic tasks, such as using properties of numbers and operations, solving equations and inequalities, and analyzing expressions that involve symbols and numbers. Once they develop these abilities, they are able to solve functional thinking tasks, such as identifying distant terms in patterns, analyzing tables that represent correspondence relationships, and interpreting graphs. Generalized arithmetic and functional thinking appear as stepping stones for students to move on to modeling languages, where they are able to identify relationships between quantities that are presented through a situation and mathematize the situation using an appropriate model.



*Note. *p<.01, The first number indicates the regression coefficient and the number in parenthesis indicate the proportion of variability that can be explained (r²)*

Figure 1: Associations between students’ abilities in different types of early algebraic tasks

The results of descriptive statistics analyses showed that the students’ highest performance mean was in generalized arithmetic tasks (M=.530). The second highest mean score was in functional thinking tasks (M=.507). Students’ lowest mean score was in modeling languages tasks (M=.303), indicating that these tasks were more difficult for them.

The qualitative analysis of students’ responses to specific tasks provided insights into the way students develop and apply the processes of generalizing and representing in each content strand. At the same time, these data allow further interpretation of the trend outlined from the latent path analysis. Figure 2 presents indicative examples of students’ responses to three different tasks.

In the generalized arithmetic task (Fig. 2a), the student seems to directly recognize that the value of each expression is the same, without performing computations and comparing their results. The student expresses verbally the observed structure and seems to use the numbers -2 and +2 in a general way, as “quasi-variables” (Fujii & Stephens, 2001), to represent a regularity. Student’s response shows that a familiar context, as the one of arithmetical expressions, enables noticing structure and regularities, generalizing from these regularities, and using means other than alphanumeric symbols to represent these generalizations, such as verbal descriptions.

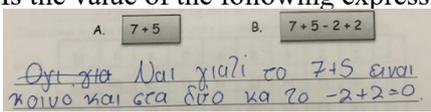
The abilities developed within a generalized arithmetic perspective seem to facilitate students to begin making “factual” generalizations from concrete actions (Radford, 2003). Based on the quantitative data, these abilities seem to prompt students to look for structure and relationships in more complex contexts. As shown in Figure 2b, in a figural pattern, the student approaches the first question based on numerical actions, by drawing the next table and seats and probably counting the number of seats. The student seems to focus on one variable (the number of seats) and finds the answer based on the identification of a recursive pattern. Yet, the second question prompts the student to reason about two quantities that co-vary, since drawing and counting would not be a convenient strategy for finding a far term. The student seems to notice the pattern’s structure and the relationship between the number

of tables and the number of seats. In this case, the student recognizes that tables at the edges always have 4 seats each and tables in the middle always have 3 seats each, and writes two distinct expressions to reach the final answer. However, the student seems to acknowledge that there is a general rule, despite the fact that there is no use of alphanumeric symbols. The student's generalization at this point seems to be, in terms of Radford (2003), a “contextual” generalization based on a situated description.

a. Generalized arithmetic task

Is the value of the following expressions the same or not? Justify your answer.

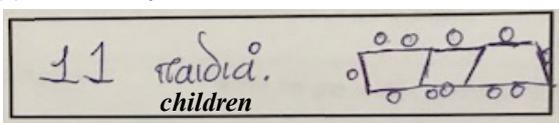
A. $7+5$ B. $7+5-2+2$



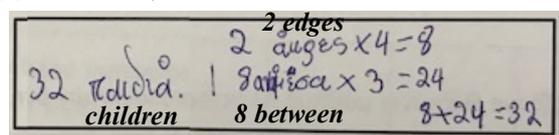
Yes because 7+5 is the same in both expressions and -2+2=0.

b. Functional thinking task

(a) How many children can be seated at 3 tables? Justify your answer.



(b) How many children can be seated at 10 tables? Justify your answer.

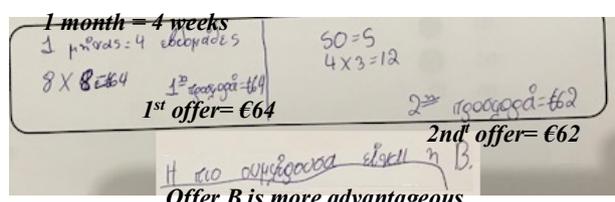


c. Modeling languages task

Joanna will take computers lessons twice a week. Which is the best offer? Justify your answer.

Offer A: €8 for each lesson

Offer B: €50 for the first 5 lessons of the month and then €4 for every additional lesson



Offer B is more advantageous

Figure 2: Indicative examples of students' responses

In the modeling languages task (Fig. 2c), the student identifies first the quantities involved: the dependent variable (the monthly cost of the lessons) and the independent variable (the number of lessons during a month); the student considers the number of monthly lessons to be 8. Furthermore, the student formulates expressions for Offer A and B, taking into consideration all variables described in the problem. In this sense, the student develops the processes of generalizing and representing in order to establish a general mathematical relationship that fits the situation. The student's response implies that arithmetical expressions are used to build a “contextual” generalization (Radford, 2003).

Discussion and Conclusions

In this study, we attempted to validate a path model which describes early algebraic thinking abilities that students seem to develop first and the way these are associated to other abilities that students seem to develop later on. The abilities examined were extracted from Kaput's (2008) theoretical framework about the three core algebra content strands: generalized arithmetic, functional thinking, and modeling languages. The results of the quantitative analyses provide evidence that there is a consistent trend in the level of difficulty across the three strands. It appears that students find it first easiest to solve generalized arithmetic tasks, and then functional thinking tasks. Once this is achieved, they can proceed to solve modeling languages tasks. These results complement but also extend the findings of previous studies that reported the significant role of content strands, such as generalized arithmetic and functional thinking in setting foundations in elementary mathematics for the longterm development of students' algebraic thinking (e.g. Carraher et al., 2006). Particularly, the findings of this study verify the ideas of a number of researchers (e.g. Fujii & Stephens, 2001) that the development of early algebraic thinking lies in the arithmetic that is taught in elementary school so that students pay attention to its underlying structure and properties and gradually develop the ability to identify and describe functional relationships between quantities that co-vary.

The data from the qualitative analysis exemplify the nature of students' thinking while they try to solve these types of tasks. Moreover, these data provide a plausible explanation for the idea that, in generalized arithmetic, students' pre-acquired arithmetical knowledge serves as a turning point for introducing them to early algebraic thinking. Generalized arithmetic encompasses familiar contexts that allow students to develop the processes of generalization and representation, without necessarily using alphanumeric symbols. The strand of functional thinking seems to strengthen the interplay between arithmetic and algebra, by introducing students to the concept of functions (Radford, 2014). Modeling languages appear to involve more advanced forms of algebraic thinking (Blanton & Kaput, 2005) that require the use of algebra as a tool in order for students to generalize, model, and justify relationships that are implicitly presented through problem situations.

Overall, this study provides empirical data that document students' capability to develop essential forms of early algebraic thinking that are foundational for the development of formal and abstract algebraic thinking. Moreover, the results imply that a comprehensive and sustained early algebra approach must build on developing students' abilities not only on one specific content strand but across the strands of generalized arithmetic, functional thinking and modeling languages. It should be noted that the model proposed in this study was formulated based on Kaput's theoretical framework and involved a sample of students with specific mathematical experiences. Future studies may examine the stability of this model in earlier or later grades of schooling. Following recent suggestions (e.g. Blanton et al., 2017), future studies might also confront generalized arithmetic and equation solving as two separated strands and test the associations between them as well as with the strands of functional thinking and modeling languages.

References

Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36(5), 412–446.

- Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). Developing essential understanding of algebraic thinking for teaching mathematics in grades 3-5. In B.J. Dougherty & R.M. Zbieck (Eds), *Essential understandings series*. NCTM: Reston, VA.
- Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics*, 95(2), 181-202. doi: <https://doi.org/10.1007/s10649-016-9745-0>
- Carraher, D.W. & Schliemann, A. D. (2007). Early Algebra and Algebraic Reasoning. In F. Lester (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning: A project of the NCTM* (Vol II, p. 669-705). Charlotte, NC: Information Age Publishing.
- Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37(2), 87-115.
- Chimoni, M., Pitta-Pantazi, D. & Christou, C. (2018). Examining early algebraic thinking: insights from empirical data. *Educational Studies in Mathematics*, 98(1), 57–76. doi: <https://doi.org/10.1007/s10649-018-9803-x>
- Fujii, T. & Stephens, M. (2001). Fostering understanding of algebraic generalisation through numerical expressions: The role of the quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The Future of the Teaching and Learning of Algebra* (Vol. 1, pp. 258–264). Melbourne, Australia: The University of Melbourne.
- Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. W. Carraher, & M.L. Blanton (Eds), *Algebra in the early grades* (pp. 5-17). New York: Routledge.
- Matthews, P. G., Rittle-Johnson, B., McEldoon, K. L., & Taylor, R. T. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, 43(3), 316–350. doi: 10.5951/jresmetheduc.43.3.0316
- Muthén, L. K., & Muthén, B. O. (1998). *Mplus user's guide*. Los Angeles, CA: Muthén & Muthén.
- Patton, M. Q. (2002). Two decades of developments in qualitative inquiry. *Qualitative Social Work: A Personal, Experiential Perspective*, 1(3), 2261-2283.
- Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical thinking and learning*, 5(1), 37-70. doi: https://doi.org/10.1207/S15327833MTL0501_02
- Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai. (Ed.), *The compendium for research in mathematics education*. Reston: NCTM.