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IX ECCOMAS Thematic Conference on Smart Structures and Materials
SMART 2019
A. Benjeddou, N. Mechbal and J.F. Deü (Eds)

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ABSTRACT

To predict the wave characteristics of the periodic media in the presence of fuzzy uncertainties, the wave finite element method in conjunction with fuzzy logic and algebra has been applied. For one-dimensional wave propagation, firstly, the most significant input parameters such as Young's modulus and mass density are identified and then *fuzzified* using the membership functions. Then, the fuzzy variable is propagated through the numerical model of interval analysis. The dispersion curves for flexural and longitudinal waves with fuzzy parameters have been used to illustrate the generality of the proposed approach also looking at the possibility to have frequency region in which those waves cannot propagate (frequency band gaps). The triangular membership functions have been used in the numerical examples and the obtained results are compared against the classical Monte Carlo simulations (MCS). The approach was presented for very simplified test-cases but it is found to be more efficient when compared with the conventional MCS approach in terms of computational cost.

Key words: WFEM, uncertainty modeling, periodic media, bandgap, fuzzy analysis.

1 INTRODUCTION

In recent years the propagation of elastic waves in periodic media with periodic material and geometrical properties has been extensively studied. These periodic media offer property known as band gaps where the propagation of the elastic wave is restricted in all the directions. Such properties are demanded for the vibroacoustic performance and dynamics of the structure and especially sought in aeronautics, transport, energy and space for vibration reduction, acoustic blocking, acoustic channeling and acoustic cloaking [1].
Three different frequency ranges identified: at the low-frequency range, because of the low order modes and low sensitivity to variability generally, the finite element approaches are applicabe; and high frequency, energy-based methods (e.g. Statistical Energy Analysis (SEA))

are well suited because of the involvement of many high order modes; finally the mid-frequency range affected by the variability of the model parameter and wave-based approaches has been developed by increasing computational efficiency [2].
The design of periodic media is generally based on deterministic models without considering the effect of intrinsic uncertainties existing in these media. In general, the design is aimed at controlling as much as possible the mechanical waves; however, inherent uncertainty may affect their characteristics. Periodic media are diffused in all the transportation engineering and demand a high level of robustness, which can be ensured with the careful consideration of the presence of uncertainty in the numerical models. The uncertainties, in terms of material properties and geometrical parameters, are mostly exhibited in both the manufacturing and assembly processes. To address this unavoidable actuality, the effects of uncertainties need to be considered when analyzing band structures (pass and stop bands). Generally, the stochastic characteristics of the periodic media can be determined by studying the design parameter uncertainties which are often modeled by random variables with consideration for spatial variability of the material and geometrical properties.

Manohar et al. studied the randomness in the wave propagation in waveguides using spectral element analysis [3]. Ichchou et al. proposed a numerical approach using the Wave Finite Element Method (WFEM) considering spatially homogeneous variability in waveguides using a first and second order perturbations are proposed [4][5]. Recently Rayleigh quotient method introduced to investigate the symplectic eigenvalue problem of random symplectic matrix [6]. The material or geometrical uncertainty often displays spatial correlation, in layered structures. There, random fields theory can be used to model spatially distributed variability with probability measure [7]. Also, this would be an approximate representation for the spatially varying system [8].

The probabilistic models to account for these uncertainties employ probabilistic methods which require a wealth of data on probabilistic parameters. The exact sources of uncertainty are rarely found since their identification represents a difficult task. Furthermore, even small inaccuracies in the data can lead to large errors in the computed probability of response [9].

When faced with incomplete information such as, "*Young's modulus of the material lies between 70 GPa to 72.4 GPa*," "*mass ration is about 4%*," and "*lenght of side slightly larger than 1.2m*," cannot be accomdated by deterministic model[10]. In fact, statements as ''*the mean are approximately equal to*.. ." and ''*the variance lies in the range*.. ." are typical when handling real mechanical data and by virtue of their subjective nature they deal with fuzzy uncertainties[11]; the adoption of the probabilistic approach can result in very challenging evaluations. In this scenario, the fuzzy set theory offers a way to approximating the uncertainty distribution in the form of the confidence interval through fuzzy membership functions. These are equivalent representations for the characterization of the linguistic, vague and missing data uncertainties.

The work reported so far in the literature with the best of the author's knowledge, is limited to wave propagation in elastic media and random media with probability method. In this work, a possibilistic method developed for the fuzzy uncertainty modeling and propagation in conjunction with WFEM and named as FWFEM (fuzzy wave finite element method). The approach was presented for a very simplified test-case 1D periodic bar and beam model with

fuzzy uncertainty. It is modeled using triangular membership function and uncertainty bound are computed with fuzzy arithmetic steps and α-cut method. The results of FWFEM formulation are compared to those obtained from MCS.

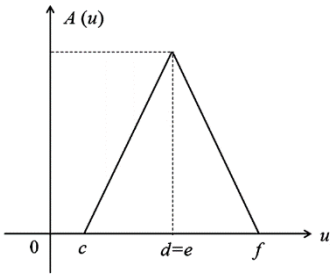
2 FORMULATION OF FWFEM

Firstly the definition of the fuzzy membership function is revisited [12]. Let U is a classical set of objects, it is called the universe. The generic elements of U are denoted u. Then the membership in a classical subset A of U can be viewed as a characteristic function μA from U to {0, 1} such that

$$\mu_A(u) = \begin{cases} 1, & \text{if} \quad u \in A \\ 0, & \text{if} \quad u \notin A \end{cases} \tag{1}$$

The set {0, 1} is called a valuation set. A set A is called a fuzzy set if the valuation set is allowed to be the real interval [0, 1]. The fuzzy set A is characterized by the set of pairs

$$A = \{[u, \mu_A(u)], \ u \in U\}$$



where $\mu_A(u)$ is called the grade of the membership function.

Figure 1: Triangular membership function

α - Cuts: When an element $u \in U$. The α-cut $A\alpha$ of A, which is a fuzzy set of crisp values U and is the original set of membership values greater than some threshold $\alpha \in [0\ 1]$.

If triangular fuzzy set A shown in Fig.1 is denoted by A= < a, b, d > then the α-cuts of the A can be shown as:

$$\left[a + \alpha(b-c), d - \alpha(d-b)\right], \qquad \alpha \in [0\,1] \tag{2}$$

To propagate the fuzzy uncertainty in the dynamic model, the α-cut method is adopted. In this method, fuzzy membership function is descritized to different iInterval using α-cut level concept. For each α-cut, the interval analysis is performed with Sobol sequence to get the máximum and mínimum bound of the response.

One dimensional periodic media obtained by formulating the unit cell and then repeating in the propagation direction. Then, the study of this media can be converted into a study of unit cell based on the Floquet-Bloch theorem [13]. The schematic representation is shown in Fig. 2.

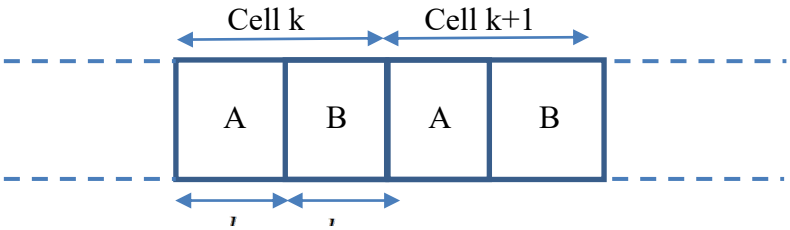


Figure 2: Schematic representation of periodic structure

The WFEM is used to study the behavior of periodic media. This is extended to apply the WFEM for the fuzzy uncertainty modeling. This starts with consideration of the fuzzy membership function as a supplementary dimension through the fuzzy arithmetic using fuzzy set theory. Which start from the discretization of one sub-element of length (d). This discretization leads to fuzzy dynamic equilibrium of any substructure. The imprecise or vague variables are modeled using the triangular membership function, such that the dynamical equilibrium is expressed as:

$$[-\omega^2 M(+)K(1+i\eta)](.)\begin{pmatrix} q_L \\ q_R \end{pmatrix} = \begin{pmatrix} F_L \\ F_R \end{pmatrix} \tag{3}$$

The (.) symbol denotes the fuzzy arithmetic.

The dynamic stiffness matrix is condensed into the left and right side denoted by L and R respectively.

$$\begin{pmatrix} D_{LL} & D_{LR} \\ D_{RL} & D_{RR} \end{pmatrix}(.)\begin{pmatrix} q_L \\ q_R \end{pmatrix} = \begin{pmatrix} F_L \\ F_R \end{pmatrix} \tag{4}$$

By defining two state vectors in the left and right side of each element and are related by symplectic matrix S as:

$$u_R = S(.)u_L \tag{5}$$

Where

$$S = \begin{pmatrix} -D_{LR}^{-1}(.)D_{LL} & D_{LR}^{-1} \\ D_{RL} - D_{RR}(.)D_{LR}^{-1}(.)D_{LL} & -D_{RR}(.)D_{LR}^{-1} \end{pmatrix}$$

Then using the periodicity and Bloch's theorem, spectral eigenvalue problem is expressed as

$S(.)\Phi_i = \mu_i(.)\Phi_i$

$$|S - \mu_i(.)I_{2n}| = 0 \tag{6}$$

The solution of the above equation leads to identify the propagation constant and mode shapes. The considered structure is dissipative and the wave classified as incident ($|\mu_i^{inc}| < 1$) and reflected ones ($|\mu_i^{ref}| > 1$). Considering the above classification the wave basis $\boldsymbol{\Phi}$ composed by eigenvectors in following form:

$$\Phi = \begin{pmatrix} \Phi_q^{inc} & \Phi_q^{ref} \\ \Phi_F^{inc} & \Phi_F^{ref} \end{pmatrix} \tag{7}$$

Where q and F are the displacement and force component of eigenvectors with $\Phi_q^{inc}, \Phi_F^{inc}, \Phi_q^{ref}, \Phi_F^{ref}$ of the size of nxn matrices.

To obtain the harmonic responce, The amplitude of wave at the first element obained using

$$\begin{pmatrix} Q^{inc(1)} \\ Q^{ref(1)} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & \mu^N \end{pmatrix} (.) \begin{pmatrix} \Phi_F^{inc} & \Phi_F^{ref}\mu^N \\ \Phi_q^{inc}\mu^N & \Phi_q^{ref} \end{pmatrix}^{-1} (.) \begin{pmatrix} -F \\ q \end{pmatrix}$$

Projection of kinematic variable on wave basis and compute the displacement at the mth element by relation

$$(Q)^m = \begin{pmatrix} \mu^{inc} & 0 \\ 0 & \mu^{ref} \end{pmatrix}^{(m-1)} (.) \begin{pmatrix} Q^{inc(1)} \\ Q^{ref(1)} \end{pmatrix} \tag{8}$$

3 RESULTS

This section shows the validation of FWFEM formulation presented in the previous section. The analysis is performed for the periodic rod and beam. Periodic rod and beam consist of section A of length l_1 and section B of length l_2 with different materials as depicted in the Fig. 3. Firstly, the longitudinal wave studied for the band gap and frequency response function (FRF) then flexural wave in the beam is studied.

3.1 Periodic rod with fuzzy uncertainty

In this subsection, the FWFEM is used to study the effect of fuzzy parametric uncertainty on the band diagram and FRF of a longitudinal wave in the periodic rod. The section of periodic

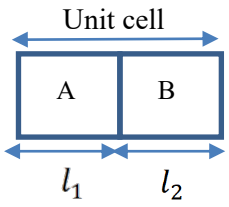


Figure 3: Symmetric unit cell

rod namely A and B made of aluminum and steel respectively. The length of section A is l_1 and section B is l_2 with 1m each with circular cross section of radius of 0.0644m.

To validate FWFEM results, a comparison made between the result from the proposed method and upper and lower bound obtained from MCS.

The uncertainty effect is studied considering fuzzy uncertain material properties modeled with triangular fuzzy membership function with lower and upper variation fixed at ±10 percent of the nominal value. The considered frequency range for the numerical experiment from 1 Hz up to 5500 Hz. This frequency range considered to get the four complete band gap. The MCS

with 10000 samples is used to get the reference wave characteristics lower and upper bound of wave number.

In FWFEM formulation, stiffness and mass matrix of the two-node rod elements are considered.
The global stiffness and mass matrices formed with 100 elements in the unit cell of periodic rod in the MATLAB environment. The input material and geometric properties are shown in Table 1. It is to be mentioned that the material properties are choosen artibiary for the test case.

Table 1: Geometric parameters and material properties of periodic rod

Geometrical and Material property	**Nominal value**
Rod length (A)	1 m
Rod length (B)	1 m
Radius of rod	0.0644 m
Young's modulus (A)	210 Gpa
Young's modulus (B)	70 Gpa
Mass density (A)	7800 kg/m^3
Mass density (B)	2700 kg/m^3
Loss factor (A) and (B)	0.001

The variation of fuzzy Young's modulus of section A (material 1) and section B (material 2) is considered as furnished in Fig.4.
FWFEM result at α-cut=0 (largest interval) in terms of upper and lower bounds of the wave number is compared with sampling upper and lower bound with 5000 samples. The comparison is shown in Fig.5. The results are in good agreement. The band gap envelops for each value of the fuzzy elasticity in every α-cut can be extracted shown in Fig.6. It is visible that the effect of uncertainty (at the fixed level of uncertainty) on band gap bound is increasing with increasing frequency. The variation bound of the wave number at α-cut=0 is shown in Fig.7 and it can be inferred that the response membership function is not always symmetric about mean/crisp value. The upper bound of wave number is more sensitive to the uncertain elasticity.

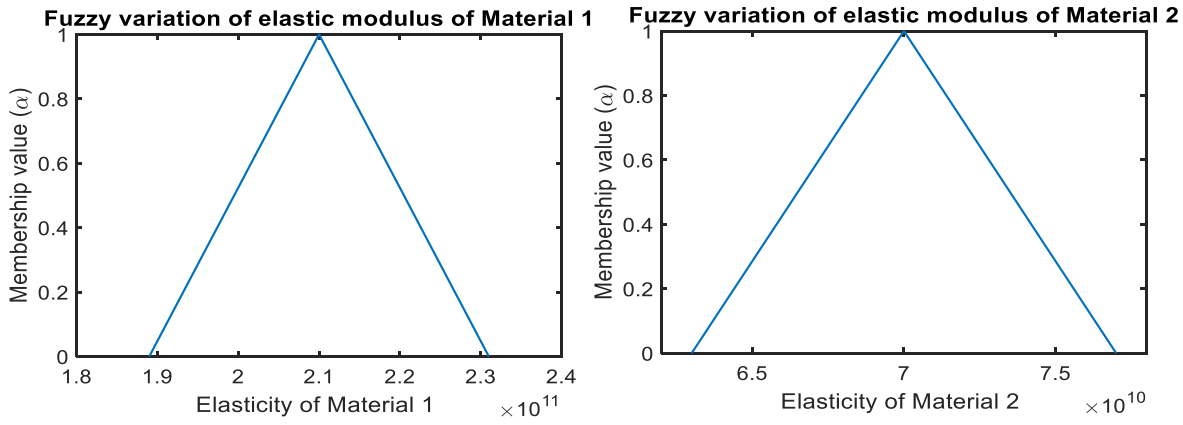


Figure 4: Membership function for the fuzzy parameter

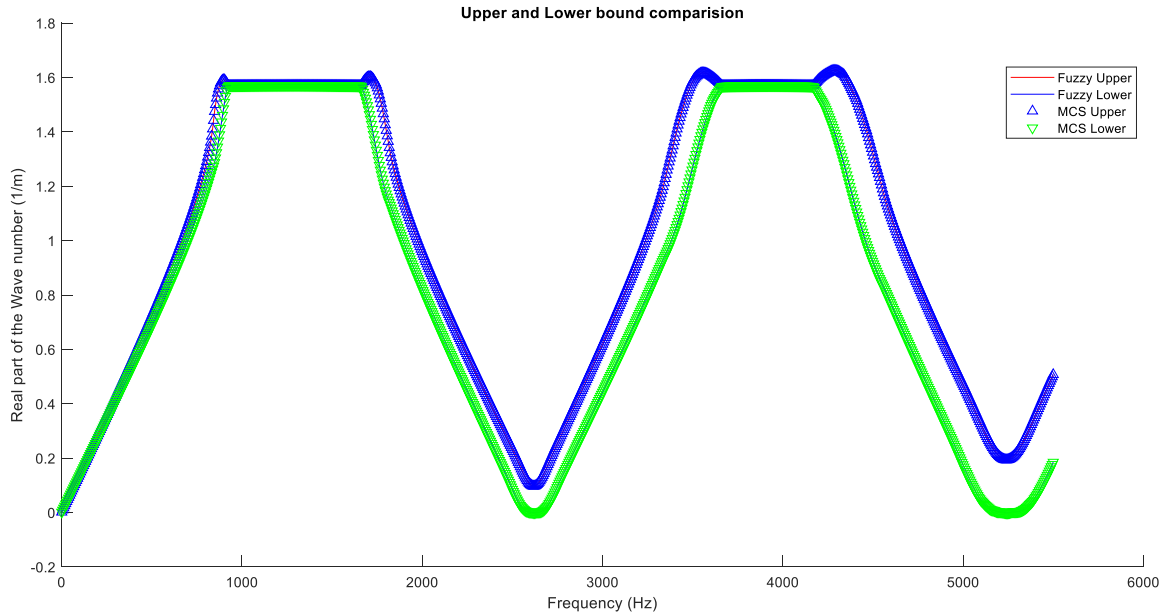


Figure 5: Upper and lower bound comparison

The FRF of a periodic rod of the finite extent of 20 m excited by unit force on the left end is furnished in Fig. 8 indicating the upper and lower bounds comparison at α-cut=0 (maximum bound). In Fig. 9 fuzzy bound of the FRF for the different value of α-cut is plotted. It can be seen that as the value of α-cut increase the fuzzy bound of FRF also decreases and finally at α=1, without any simulation bound as expected, α=1 represent the deterministic value.

3.2 Periodic beam with fuzzy uncertainty

In this subsection, the study of the fuzzy uncertainty effect on the flexural wave in the periodic beam is performed. For numerical simulation, Euler-Bernoulli beam theory is considered. The input material and geometrical properties are listed in Table 2.
For the fuzzy uncertainty propagation, the elasticity of material 1and material 2 is fuzzified using triangular membership function and shown in Fig.4.

A two-node beam element with two degrees of freedom per node is considered. The global matrices (stiffness and mass) formed with 20 elements in the unit cell of the periodic beam in the MATLAB environment with input material and geometric properties given in Table 2.
FWFEM result at α-cut=0 is computed and compared with upper and lower bound obtained with MCS sampling (10000 samples) of the WFEM result and shown in Fig. 10. The frequency range of computation is 1-2000 Hz. The results are in good agreement. The variation bound of the wave number at a different frequency for the 11 α-cut levels is shown in Fig.11. it showed that at the same level of the α-cut upper bound of wave number is more sensitive to the uncertain elasticity.

Table 2: Geometric parameters and material properties of the periodic beam

Geometry/Property	**Value**
Beam length	0.5m
Height of beam	0.003m
Width of beam	0.003m
Young's modulus (A)	210 Gpa
Young's modulus (B)	70 Gpa
Mass density (A)	7800 kg/m^3
Mass density (B)	2700 kg/m^3
Loss factor (A) and (B)	0.01

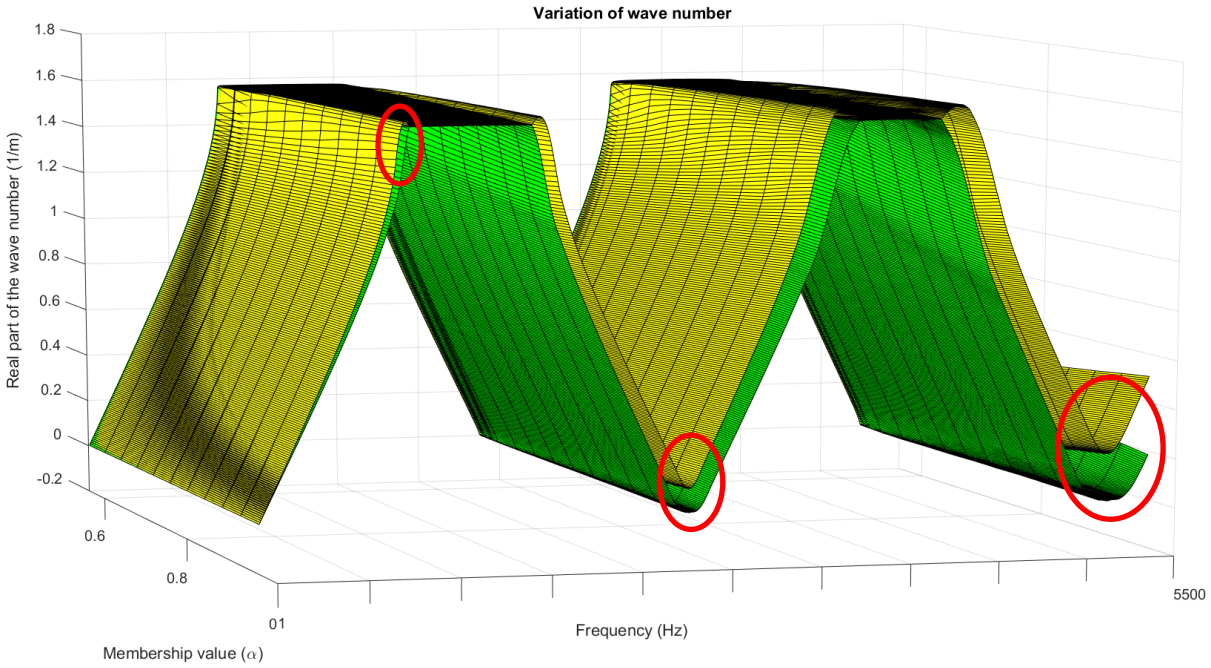


Figure 6: Band gap bound

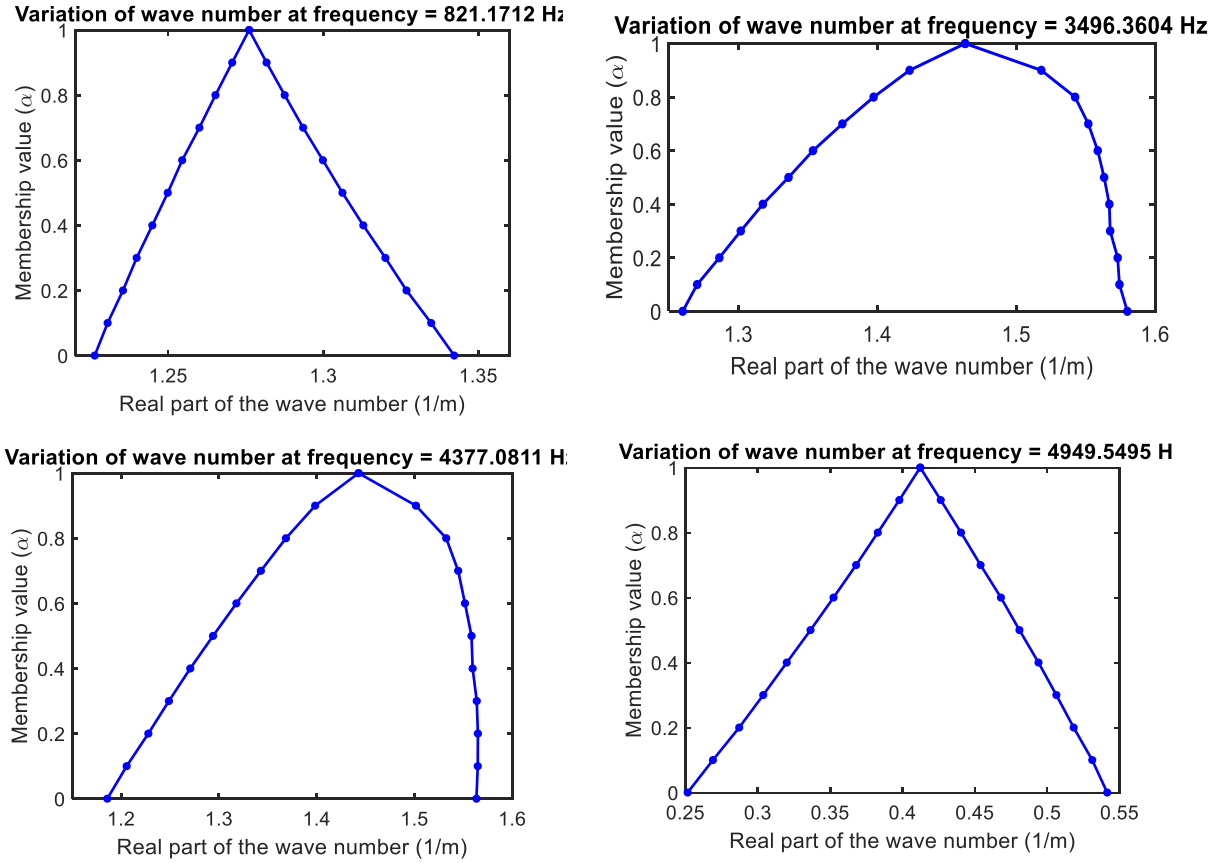


Figure 7: Variation bound of wave number at 11 α-cut level

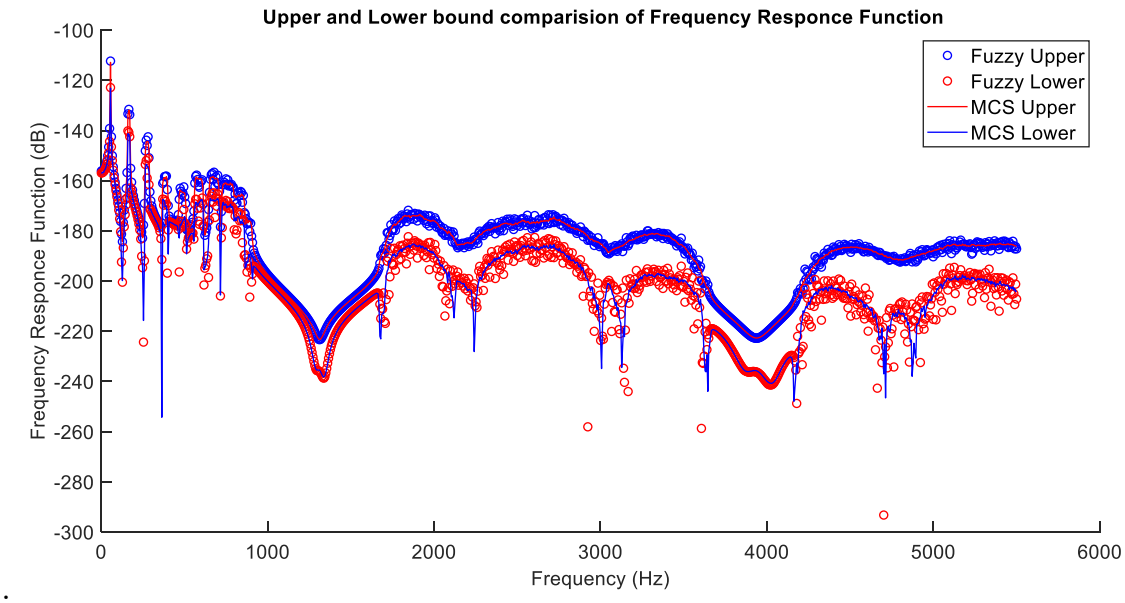


Figure 8: Upper and lower bound comparison of FRF

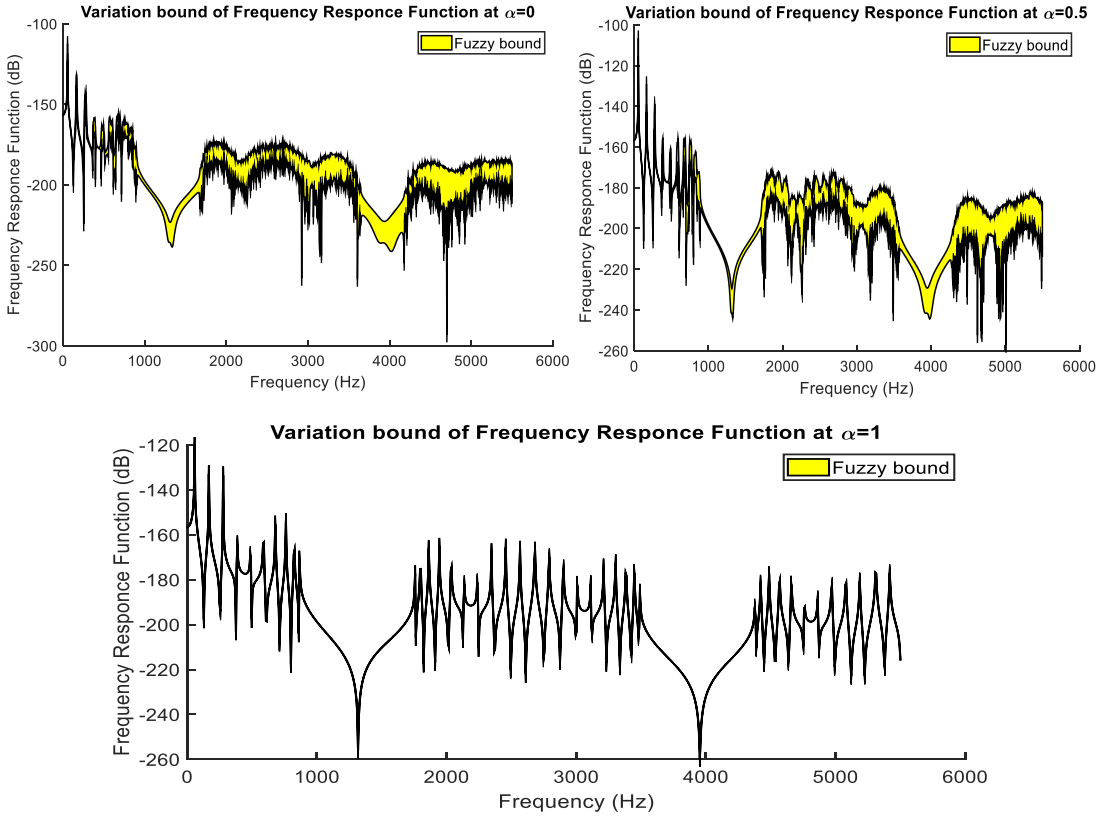


Figure 9: Fuzzy bound of FRF for the different value of α-cut

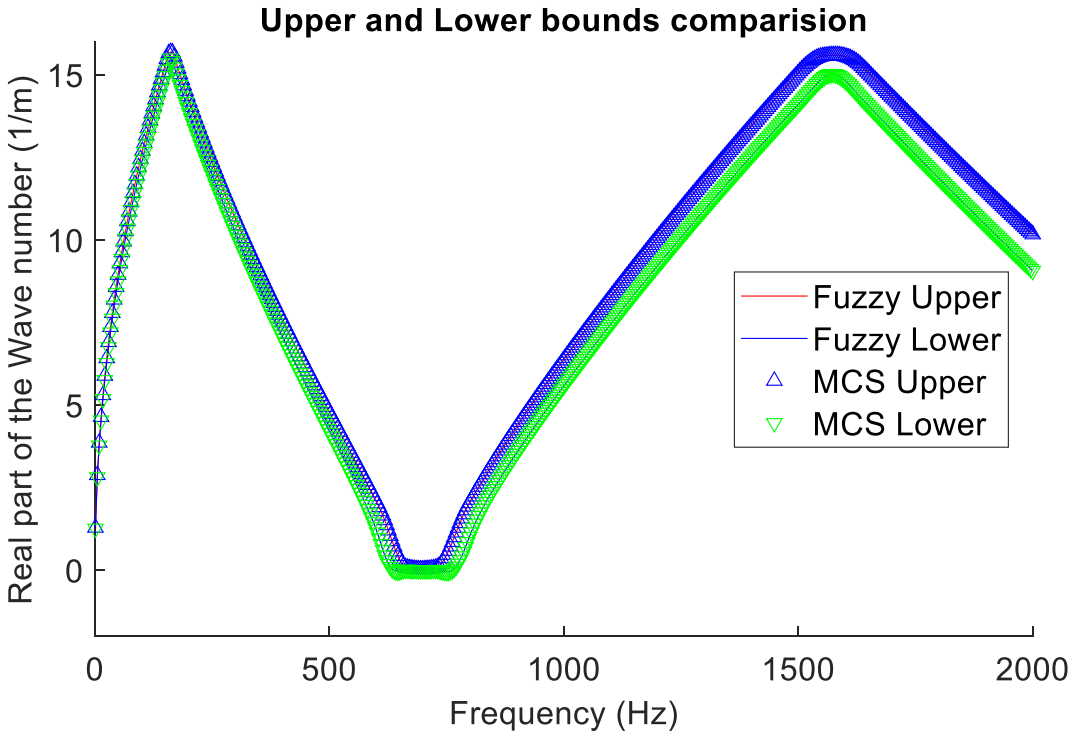


Figure 10: Upper and lower bound comparison

The FRF bound is shown in Fig. 12 representatives the bound at the different α-cut level. As α-cut value increase the fuzzy bound of FRF is expected to decrease. same can be seen from Fig. 13 as the value of membership increase the fuzzy bound is decreasing and lowest at α=0.9.

For the computation cost comparison, elapsed time for the MCS and FWFEM obtained using homemade WFEM code which exploits the resources of the workstation with following characteristics, Intel® Core™i7 7820 HQ CPU@2.90GHz with 32 GB RAM and data storage using solid state drive. The comparison is presented in Table 3.

Table 3: Elapsed time comparison

	WFEM MCS (10000 Samples)	FWFEM(11 α-cut evaluation)
Elapse time (seconds)	3640	234

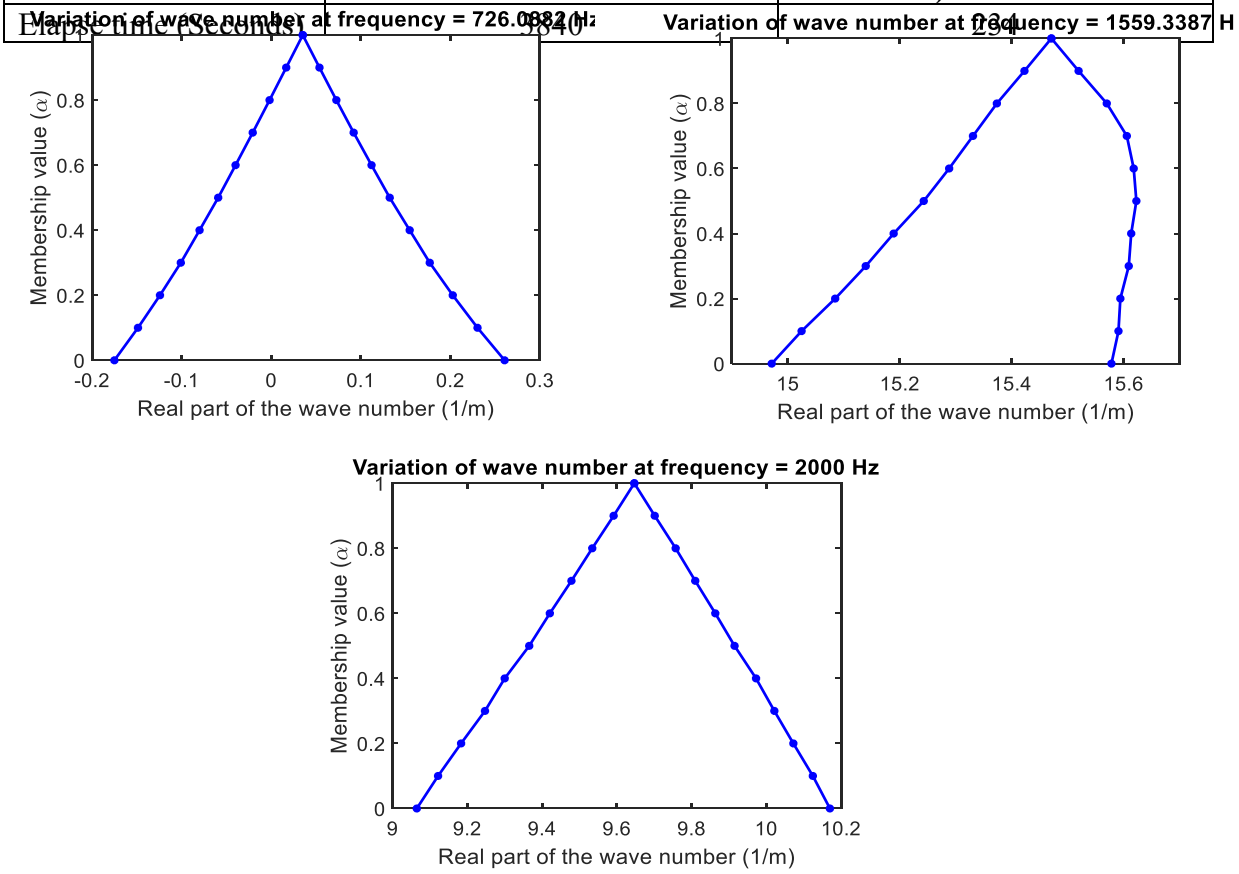


Figure 11: Variation bound of wave number at 11 α-cut level

CONCLUSION

This paper provides a computationally inexpensive possibilistic numerical method for uncertainty propagation in 1D periodic media in conjunction with WFEM. This study proposes a new approach to predict uncertainty bounds of the band gap and FRF of periodic

rod and beam with fuzzy variations. The uncertainty quantification of wave number and frequency response function with fuzzy variables are derived using WFEM.
The approach was presented for very simplified test-cases, but it is found to be more efficient when compared with the conventional MCS approach in terms of computational cost. The present study can be extended for the future investigation for the complex unit cell and a large number of fuzzy variables.

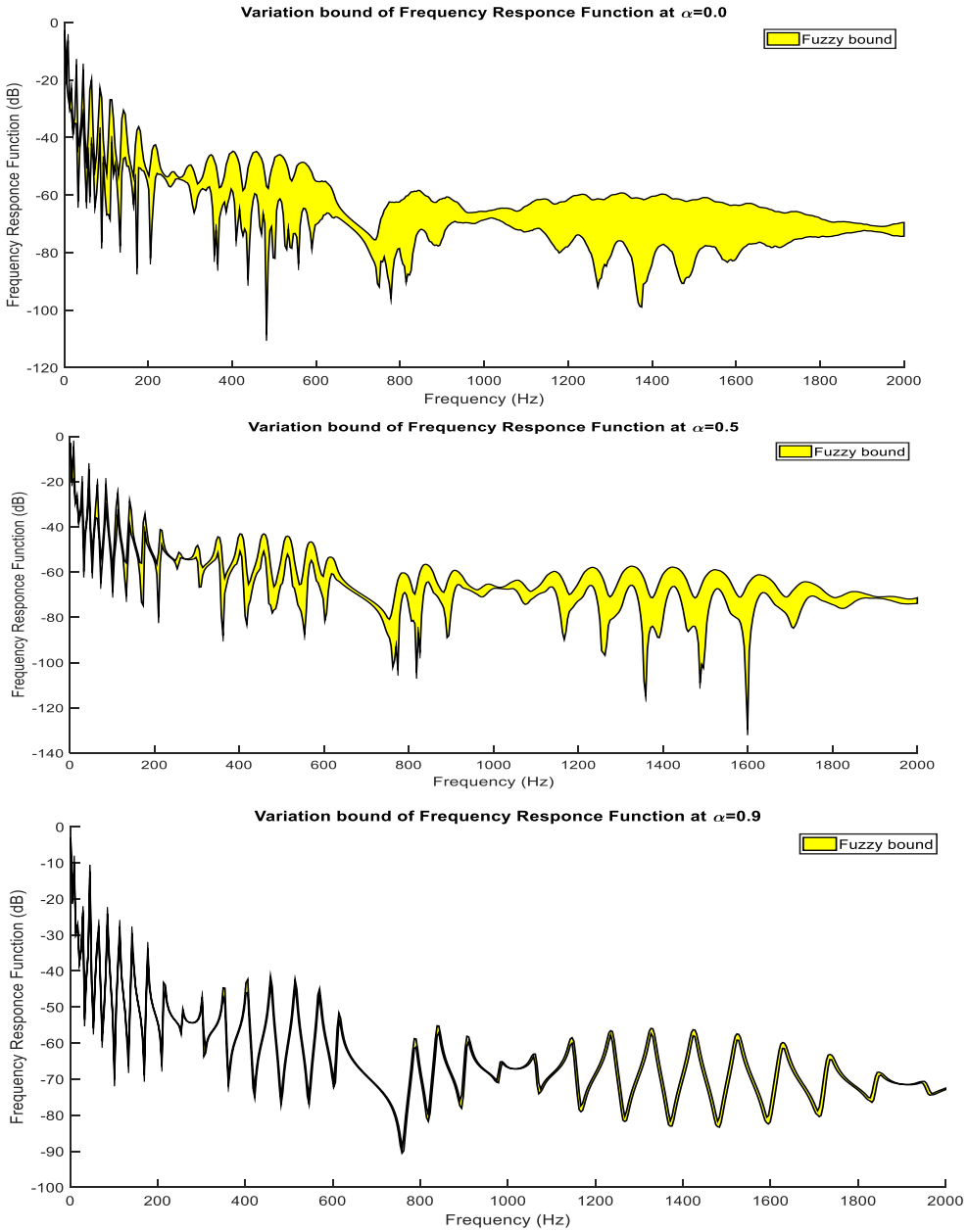


Figure 12: Fuzzy bound of FRF for the different value of alpha-cut

ACKNOWLEDGMENTS

The research of R.P. Singh was funded by an Early Stage Researcher grant within the European Project VIPER Marie Curie European Joint Doctorates (GA 675441).

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