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COMPLETELY MONOTONIC DEGREES OF REMAINDERS OF ASYMPTOTIC EXPANSIONS OF THE DIGAMMA FUNCTION

FENG QI AND MANSOUR MAHMOUD

ABSTRACT. Motivated by several conjectures posed in the paper “F. Qi and A.-Q. Liu, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371; available online at <https://doi.org/10.1016/j.cam.2019.05.001>”, the authors calculate some completely monotonic degrees of remainders of the asymptotic expansion of the logarithm of the gamma function, compute some completely monotonic degrees of remainders of asymptotic expansions of the digamma function, and confirm some conjectures on completely monotonic degrees of remainders of the asymptotic expansion of the logarithm of the gamma function and on completely monotonic degrees of remainders of asymptotic expansions of polygamma functions.

1. PRELIMINARIES

The classical Euler gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the digamma function. As a whole, the derivatives $\psi^{(i)}(x)$ for $i \geq 0$ are called the polygamma functions. For more information on $\Gamma(z)$ and $\psi^{(i)}(x)$, please refer to [1, 2, 6, 9, 19, 22, 29, 37, 43, 44, 45] and closely related references therein.

Recall from [17, Chapter XIII], [38, Chapter 1], and [40, Chapter IV] that, if a function h on an interval I has derivatives of all orders on I and

$$(-1)^n h^{(n)}(t) \geq 0, \quad t \in I, \quad n \in \{0\} \cup \mathbb{N},$$

then we call h a completely monotonic function on I . Theorem 12b in [40, p. 161] states that a necessary and sufficient condition for h to be completely monotonic on the infinite interval $(0, \infty)$ is that

$$h(t) = \int_0^{\infty} e^{-ts} d\sigma(s), \quad s \in (0, \infty), \quad (1.1)$$

where $\sigma(s)$ is non-decreasing and the above integral converges for $s \in (0, \infty)$. In other words, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of a non-negative measure.

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Definition 1.1. Let $h(t)$ be a completely monotonic function on $(0, \infty)$ and let $h(\infty) = \lim_{t \rightarrow \infty} h(t) \geq 0$. If the function $t^\alpha[h(t) - h(\infty)]$ is completely monotonic on $(0, \infty)$ when and only when $0 \leq \alpha \leq r \in \mathbb{R}$, then we say that $h(t)$ is of completely monotonic degree r ; if $t^\alpha[h(t) - h(\infty)]$ is completely monotonic on $(0, \infty)$ for all $\alpha \in \mathbb{R}$, then we say that the completely monotonic degree of $h(t)$ is ∞ .

For convenience, we designed in [7] a notation $\deg_{\text{cm}}^t[h(t)]$ to denote the completely monotonic degree r of $h(t)$ with respect to $t \in (0, \infty)$. Proposition 1.2 in [29] can be modified as that $\deg_{\text{cm}}^x[h(t)] = r > 0$ if and only if

$$h(t) - h(\infty) = \int_0^\infty \left[\frac{1}{\Gamma(r)} \int_0^s (s - \tau)^{r-1} d\mu(\tau) \right] e^{-ts} ds$$

for $0 < t < \infty$, where $\mu(\tau)$ is a family of bounded and non-decreasing measures on $(0, \infty)$. For more information on completely monotonic degree and its properties, please refer to the papers [8, 11, 13, 14, 15, 16, 21, 24, 25, 27, 28, 29, 30, 34, 35, 36] and closely related references therein.

2. MOTIVATIONS AND MAIN RESULTS

In [2, pp. 374–375, Theorem 1] and [9, Theorem 1], the function

$$t^\alpha[\ln t - \psi(t)], \quad \alpha \in \mathbb{R}$$

was proved to be completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$. This means that completely monotonic degree of $\ln t - \psi(t)$ on $(0, \infty)$ is

$$\deg_{\text{cm}}^t[\ln t - \psi(t)] = 1. \quad (2.1)$$

In [37, Theorem 1.7], the function

$$t^2[\psi(t) - \ln t] + \frac{t}{2}$$

was proved to be decreasing and convex on $(0, \infty)$ and, as $t \rightarrow \infty$, to tend to $-\frac{1}{12}$. In [6, Theorem 1], the function

$$t^2[\psi(t) - \ln t] + \frac{t}{2} + \frac{1}{12} \triangleq t^2\phi(t)$$

was verified to be completely monotonic on $(0, \infty)$. In [31], the completely monotonic degree of $\phi(t)$ with respect to $t \in (0, \infty)$ was proved to be

$$\deg_{\text{cm}}^t[\phi(t)] = 2. \quad (2.2)$$

In [2, Theorem 8], [12, Theorem 2], and [42], the functions

$$R_n(t) = (-1)^n \left[\ln \Gamma(t) - \left(t - \frac{1}{2} \right) \ln t + t - \frac{1}{2} \ln(2\pi) - \sum_{k=1}^n \frac{B_{2k}}{(2k-1)2k} \frac{1}{t^{2k-1}} \right]$$

for $n \geq 0$ were proved to be completely monotonic on $(0, \infty)$, where an empty sum is understood to be 0 and the Bernoulli numbers B_n are generated [20, 23, 26] by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi. \quad (2.3)$$

This conclusion implies that the functions $(-1)^m R_n^{(m)}(t)$ for $m, n \geq 0$ are completely monotonic on $(0, \infty)$. See also [10, Section 1.4] and [18, Theorem 3.1]. By the way, we call the function $(-1)^n R_n(t)$ for $n \geq 0$ the remainders of asymptotic formula of $\ln \Gamma(t)$. See [1, p. 257, 6.1.40] and [19, p. 140, 5.11.1]. The completely monotonic degree of the function $R_n(t)$ for $n \geq 0$ with respect to $t \in (0, \infty)$ was proved in [16, Theorem 2.1] to be at least n .

Stimulated by the above results and related ones, the first author of this paper conjectured in [31] that

- (1) the completely monotonic degrees of $R_n(t)$ for $n \geq 0$ with respect to $t \in (0, \infty)$ satisfy

$$\deg_{\text{cm}}^t[R_0(t)] = 0, \quad \deg_{\text{cm}}^t[R_1(t)] = 1, \quad (2.4)$$

and

$$\deg_{\text{cm}}^t[R_n(t)] = 2(n-1), \quad n \geq 2; \quad (2.5)$$

- (2) the completely monotonic degrees of $-R'_n(t)$ for $n \geq 0$ with respect to $t \in (0, \infty)$ satisfy

$$\deg_{\text{cm}}^t[-R'_0(t)] = 1, \quad \deg_{\text{cm}}^t[-R'_1(t)] = 2, \quad (2.6)$$

and

$$\deg_{\text{cm}}^t[-R'_n(t)] = 2n-1, \quad n \geq 2; \quad (2.7)$$

- (3) the completely monotonic degrees of $(-1)^m R_n^{(m)}(t)$ for $m \geq 2$ and $n \geq 0$ with respect to $t \in (0, \infty)$ satisfy

$$\deg_{\text{cm}}^t[(-1)^m R_0^{(m)}(t)] = m-1, \quad \deg_{\text{cm}}^t[(-1)^m R_1^{(m)}(t)] = m, \quad (2.8)$$

and

$$\deg_{\text{cm}}^t[(-1)^m R_n^{(m)}(t)] = m+2(n-1), \quad n \geq 2. \quad (2.9)$$

Motivated by

$$\deg_{\text{cm}}^t[(-1)^2 R_0''(t)] = 2 \quad \text{and} \quad \deg_{\text{cm}}^t[(-1)^2 R_1''(t)] = 3,$$

which can be verified as did in [20], we can correct and modify two conjectures stated in (2.8) as follows: the completely monotonic degrees of $(-1)^m R_n^{(m)}(t)$ for $m \geq 2$ and $n \geq 0$ with respect to $t \in (0, \infty)$ satisfy

$$\deg_{\text{cm}}^t[(-1)^m R_0^{(m)}(t)] = m \quad \text{and} \quad \deg_{\text{cm}}^t[(-1)^m R_1^{(m)}(t)] = m+1. \quad (2.10)$$

In this paper, we will confirm that Qi's conjectures expressed in (2.4), (2.6), and (2.7) are true.

Theorem 2.1. *For $n \geq 0$, completely monotonic degrees of the remainder $R_n(t)$ with respect to $t \in (0, \infty)$ satisfy two equalities in (2.4).*

For $n \geq 0$, completely monotonic degrees of the functions

$$-R'_n(t) = (-1)^{n+1} \left[\psi(t) - \ln t + \frac{1}{2t} + \sum_{k=1}^n \frac{B_{2k}}{2k} \frac{1}{t^{2k}} \right]$$

with respect to $t \in (0, \infty)$ satisfy three equalities in (2.6) and (2.7).

3. PROOF OF THEOREM 2.1

Now we are in a position to prove our main results stated in Theorem 2.1.

3.1. Proofs of equalities in (2.4). It is easy to see that

$$\begin{aligned} R_0(t) &= \ln \Gamma(t) - \left(t - \frac{1}{2} \right) \ln t + t - \frac{1}{2} \ln(2\pi), \\ R_1(t) &= - \left[\ln \Gamma(t) - \left(t - \frac{1}{2} \right) \ln t + t - \frac{1}{2} \ln(2\pi) - \frac{1}{12t} \right], \\ \lim_{t \rightarrow \infty} R_0(t) &= \lim_{t \rightarrow \infty} R_1(t) = 0. \end{aligned}$$

Theorem 2.1 in [16] means that the completely monotonic degree of the remainder $R_n(t)$ for $n \geq 0$ with respect to $t \in (0, \infty)$ is at least n . In particular, we have

$$\deg^t[R_0(t)] \geq 0 \quad \text{and} \quad \deg^t[R_1(t)] \geq 1. \quad (3.1)$$

If $t^\theta R_0(t)$ and $t^\lambda R_1(t)$ were completely monotonic on $(0, \infty)$, then their first derivatives should be non-positive. As a result, we have

$$\theta \leq -\frac{tR_0'(t)}{R_0(t)} \rightarrow 0, \quad t \rightarrow 0^+$$

and

$$\lambda \leq -\frac{tR_1'(t)}{R_1(t)} \rightarrow 1, \quad t \rightarrow 0^+.$$

Consequently, it follows that

$$\deg^t[R_0(t)] \leq 0 \quad \text{and} \quad \deg^t[R_1(t)] \leq 1. \quad (3.2)$$

Combining those inequalities in (3.1) and (3.2) concludes those equalities in (2.4).

3.2. Proofs of equalities in (2.6). It is clear that

$$-R_0'(t) = \ln t - \psi(t) - \frac{1}{2t}$$

and

$$-R_1'(t) = \psi(t) - \ln t + \frac{1}{2t} + \frac{1}{12t^2} = \phi(t).$$

The equation (2.1) together with the limits

$$\lim_{t \rightarrow \infty} t[\ln t - \psi(t)] = \frac{1}{2} \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{t\left[\frac{1}{2t^2} + \frac{1}{t} - \psi'(t)\right]}{\frac{1}{2t} - \ln t + \psi(t)} = 1$$

implies the first conclusion in (2.6).

The equation (2.2) is equivalent to the second conclusion in (2.6).

3.3. Proof of the equality (2.7). Let

$$f_n(t) = (-1)^n \left[\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} t^{2k-1} \right], \quad n \geq 0.$$

Then, by virtue of the formulas

$$\coth t = \frac{1 + e^{-2t}}{1 - e^{-2t}} = \frac{2}{1 - e^{-2t}} - 1,$$

$$\psi(z) = \ln z + \int_0^\infty \left(\frac{1}{t} - \frac{1}{1 - e^{-t}} \right) e^{-tz} dt, \quad \Re(z) > 0$$

in [19, p. 140, 5.9.13], and

$$\frac{1}{z^w} = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1} e^{-zt} dt, \quad \Re(z), \Re(w) > 0$$

in [1, p. 255, 6.1.1], we derive that

$$\begin{aligned} (-1)^n \int_0^\infty f_n(t) e^{-xt} dt &= \int_0^\infty \left[\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} t^{2k-1} \right] e^{-xt} dt \\ &= \int_0^\infty \left[\frac{1}{t} - \frac{1}{1 - e^{-t}} + \frac{1}{2} + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} t^{2k-1} \right] e^{-xt} dt \\ &= \psi(x) - \ln x + \frac{1}{2x} + \sum_{k=1}^{n+1} \frac{B_{2k}}{2k} \frac{1}{x^{2k}}, \quad n \geq 0. \end{aligned}$$

This means that

$$-R'_{n+1}(x) = \int_0^\infty f_n(t) e^{-xt} dt, \quad n \geq 0. \quad (3.3)$$

Using the relation

$$\coth z = \sum_{j=0}^m \frac{2^{2j} B_{2j}}{(2j)!} z^{2j-1} + \sigma_m(z), \quad m \geq 0$$

in [5, Theorem 2.3], where

$$\sigma_m(z) = (-1)^m \frac{2z^{2m+1}}{\pi^{2m}} \sum_{k=1}^{\infty} \frac{1}{k^{2m}(z^2 + \pi^2 k^2)}, \quad m \geq 0,$$

we see that

$$f_n(t) = \frac{(-1)^{n+1}}{2} \sigma_{n+1}\left(\frac{t}{2}\right) > 0, \quad n \geq 0.$$

Combining this with (1.1) and (3.3) reveals that the remainder $-R'_{n+1}(x)$ for $n \geq 0$ is completely monotonic on $(0, \infty)$.

Since the function $\frac{t}{e^t-1} - 1 + \frac{t}{2}$ is even in $t \in \mathbb{R}$, by virtue of the equation (2.3), it follows that

$$\begin{aligned} f_n(t) &= \frac{(-1)^n}{t} \left[1 - \frac{t}{1-e^{-t}} + \frac{t}{2} + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} t^{2k} \right] \\ &= \frac{(-1)^n}{t} \left[-\left(\frac{t}{e^t-1} - 1 + \frac{t}{2} \right) + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} t^{2k} \right] \\ &= (-1)^{n+1} \sum_{k=n+2}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k-1}, \quad t \in \mathbb{R}. \end{aligned}$$

Therefore, we have

$$\lim_{t \rightarrow 0} f_n^{(\ell)}(t) = (-1)^{n+1} \lim_{t \rightarrow 0} \sum_{k=n+2}^{\infty} \frac{B_{2k}}{(2k)!} \langle 2k-1 \rangle_{\ell} t^{2k-\ell-1} = 0$$

for $0 \leq \ell \leq 2n+2$ or $\ell = 2m$ with $m, n \geq 0$, where

$$\langle \alpha \rangle_n = \prod_{k=0}^{n-1} (\alpha - k) = \begin{cases} \alpha(\alpha-1) \cdots (\alpha-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

is called the falling factorial of α . For more information on the falling and rising factorials, please refer to the papers [32, 33]. Since

$$\lim_{t \rightarrow \infty} \frac{d^{\ell}}{dt^{\ell}} \left(\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} \right) = \lim_{t \rightarrow \infty} \frac{d^{\ell}}{dt^{\ell}} \left[\frac{1}{t} - \left(\frac{1}{e^t-1} + \frac{1}{2} \right) \right] = \begin{cases} 0, & \ell \geq 1 \\ -\frac{1}{2}, & \ell = 0 \end{cases}$$

and

$$\lim_{t \rightarrow \infty} (t^m e^{-xt}) = 0, \quad x > 0, \quad m \geq 0,$$

then

$$\begin{aligned} \lim_{t \rightarrow \infty} [f_n^{(\ell)}(t) e^{-xt}] &= (-1)^n \lim_{t \rightarrow \infty} \left[\left(\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} \right)^{(\ell)} e^{-xt} \right. \\ &\quad \left. + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} \langle 2k-1 \rangle_{\ell} t^{2k-\ell-1} e^{-xt} \right] = 0 \end{aligned}$$

for $\ell, n \geq 0$ and $x > 0$. Consequently, integrating by part inductively $2n+1$ times in the equation (3.3) gives

$$-R'_{n+1}(x) = -\frac{1}{x} \int_0^{\infty} f_n(t) d e^{-xt} = -\frac{1}{x} \left[f_n(t) e^{-xt} \Big|_{t=0}^{\infty} - \int_0^{\infty} f'_n(t) e^{-xt} dt \right]$$

$$\begin{aligned}
&= \frac{1}{x} \int_0^\infty f'_n(t) e^{-xt} dt = \dots = \frac{1}{x^{2n+1}} \int_0^\infty f_n^{(2n+1)}(t) e^{-xt} dt \\
&= \frac{1}{x^{2n+1}} \int_0^\infty (-1)^n \left[\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} + \sum_{k=1}^{n+1} \frac{B_{2k}}{(2k)!} t^{2k-1} \right]^{(2n+1)} e^{-xt} dt \\
&= \frac{(-1)^n}{x^{2n+1}} \int_0^\infty K_n(t) e^{-xt} dt,
\end{aligned}$$

where

$$K_n(t) = \frac{d^{2n+1}}{dt^{2n+1}} \left(\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} \right) + \frac{B_{2n+2}}{2n+2}, \quad t > 0, \quad n \geq 0. \quad (3.4)$$

Using the relation

$$\int_0^\infty \frac{x^{2n+1} \cos(ax)}{e^x - 1} dx = (-1)^n \frac{d^{2n+1}}{da^{2n+1}} \left[\frac{\pi}{2} \coth(a\pi) - \frac{1}{2a} \right], \quad a > 0, \quad n \geq 0$$

in [41, p. 2], we arrive at

$$\frac{d^{2n+1}}{dt^{2n+1}} \left(\frac{1}{t} - \frac{1}{2} \coth \frac{t}{2} \right) = (-1)^{n+1} 2 \int_0^\infty \frac{w^{2n+1} \cos(wt)}{e^{2\pi w} - 1} dw \quad (3.5)$$

for $t > 0$ and $n \geq 0$. Using the relation

$$\frac{B_{2m}}{4m} = (-1)^{m-1} \int_0^\infty \frac{w^{2m-1}}{e^{2\pi w} - 1} dw, \quad m \geq 1$$

in [3, p. 29], [4, p. 220], and [39, p. 19], we obtain

$$\frac{B_{2n+2}}{2n+2} = (-1)^n 2 \int_0^\infty \frac{w^{2n+1}}{e^{2\pi w} - 1} dw, \quad n \geq 0. \quad (3.6)$$

Substituting (3.5) and (3.6) into (3.4) reveals

$$K_n(t) = (-1)^n 2 \int_0^\infty \frac{w^{2n+1} [1 - \cos(wt)]}{e^{2\pi w} - 1} dw, \quad t > 0, \quad n \geq 0.$$

Consequently, we have

$$x^{2n+1} [-R'_{n+1}(x)] = 2 \int_0^\infty \left(\int_0^\infty \frac{w^{2n+1} [1 - \cos(wt)]}{e^{2\pi w} - 1} dw \right) e^{-xt} dt, \quad n \geq 0.$$

Hence, the function $x^{2n+1} [-R'_{n+1}(x)]$ is completely monotonic on $(0, \infty)$. This means that

$$\deg_{\text{cm}}^x [-R'_{n+1}(x)] \geq 2n+1, \quad n \geq 0. \quad (3.7)$$

If the function $x^\alpha [-R'_{n+1}(x)]$ were completely monotonic on $(0, \infty)$, then its first derivative would be negative, hence

$$\alpha \leq -\frac{x[-R'_{n+1}(x)]'}{-R'_{n+1}(x)} = -\frac{xR''_{n+1}(x)}{R'_{n+1}(x)}.$$

From

$$\lim_{x \rightarrow 0^+} \left[x^2 \left(\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} \right) \right] = \lim_{x \rightarrow 0^+} \left[x^2 \left(\psi'(x+1) - \frac{1}{x} + \frac{1}{2x^2} \right) \right] = \frac{1}{2}$$

and

$$\lim_{x \rightarrow 0^+} \left[x \left(\psi(x) - \ln x + \frac{1}{2x} \right) \right] = \lim_{x \rightarrow 0^+} \left[x \left(\psi(x+1) - \ln x - \frac{1}{2x} \right) \right] = -\frac{1}{2},$$

it follows that

$$\lim_{x \rightarrow 0^+} \left[-\frac{xR''_{n+1}(x)}{R'_{n+1}(x)} \right] = -\lim_{x \rightarrow 0^+} \frac{x \left[\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} - \sum_{k=1}^{n+1} \frac{B_{2k}}{x^{2k+1}} \right]}{\left[\psi(x) - \ln x + \frac{1}{2x} + \sum_{k=1}^{n+1} \frac{B_{2k}}{2k} \frac{1}{x^{2k}} \right]}$$

$$= - \lim_{x \rightarrow 0^+} \frac{x^2 [\psi'(x) - \frac{1}{x} - \frac{1}{2x^2}] - \sum_{k=1}^{n+1} \frac{B_{2k}}{x^{2k+1}}}{x [\psi(x) - \ln x + \frac{1}{2x}] + \sum_{k=1}^{n+1} \frac{1}{2k} \frac{B_{2k}}{x^{2k-1}}} = 2n + 2.$$

This means that

$$\deg_{\text{cm}}^x[-R'_{n+1}(x)] \leq 2n + 2, \quad n \geq 0. \quad (3.8)$$

Combining (3.7) and (3.8) results in

$$2n + 1 \leq \deg_{\text{cm}}^x[-R'_{n+1}(x)] \leq 2n + 2, \quad n \geq 0. \quad (3.9)$$

As did in the derivation of (3.4), integrating by part $-R'_{n+1}(x)$ inductively $2n+2$ times in (3.3) and utilizing the formula

$$\int_0^\infty \frac{x^{2n} \sin(ax)}{e^x - 1} dx = (-1)^n \frac{d^{2n}}{d a^{2n}} \left[\frac{\pi}{2} \coth(a\pi) - \frac{1}{2a} \right], \quad a > 0, \quad n \geq 0$$

in [41, p. 1] produce

$$x^{2n+2}[-R'_{n+1}(x)] = 2 \int_0^\infty \left[\int_0^\infty \frac{w^{2n+2} \sin(wt)}{e^{2\pi w} - 1} dw \right] e^{-xt} dt, \quad n \geq 0.$$

By (1.1), we see that the function $x^{2n+2}[-R'_{n+1}(x)]$ is not completely monotonic on $(0, \infty)$, that is,

$$\deg_{\text{cm}}^x[-R'_{n+1}(x)] < 2n + 2, \quad n \geq 0. \quad (3.10)$$

Combining (3.10) with (3.9) gives

$$2n + 1 \leq \deg_{\text{cm}}^x[-R'_{n+1}(x)] < 2n + 2, \quad n \geq 0. \quad (3.11)$$

By definition, we can see that, if $x^{2n+1+\epsilon}[-R'_{n+1}(x)]$ for $\epsilon > 0$ and $n \geq 0$ were completely monotonic on $(0, \infty)$, then it should be valid that

$$\begin{aligned} \frac{d^2(x^{2n+1+\epsilon}[-R'_{n+1}(x)])}{d x^2} &= x^{2n-1+\epsilon}[-R'_{n+1}(x)] \left\{ \epsilon^2 + \left[4n + 1 + 2x \frac{[-R'_{n+1}(x)]'}{[-R'_{n+1}(x)]} \right] \epsilon \right. \\ &\quad \left. + \left[x^2 \frac{[-R'_{n+1}(x)]''}{[-R'_{n+1}(x)]} + 2(2n+1)x \frac{[-R'_{n+1}(x)]'}{[-R'_{n+1}(x)]} + 2n(2n+1) \right] \right\} \geq 0 \end{aligned}$$

for $\epsilon > 0$ and $n \geq 0$ on $(0, \infty)$. This means that it is necessary that

$$\begin{aligned} H_n(x) &= \left[4n + 1 + 2x \frac{[-R'_{n+1}(x)]'}{[-R'_{n+1}(x)]} \right]^2 - 4 \left[x^2 \frac{[-R'_{n+1}(x)]''}{[-R'_{n+1}(x)]} \right. \\ &\quad \left. + 2(2n+1)x \frac{[-R'_{n+1}(x)]'}{[-R'_{n+1}(x)]} + 2n(2n+1) \right] \leq 0 \end{aligned}$$

for $\epsilon > 0$ and $n \geq 0$ on $(0, \infty)$. However, straightforward computation shows

$$\lim_{x \rightarrow 0^+} H_n(x) = \lim_{x \rightarrow \infty} H_n(x) = 1.$$

This contradiction demonstrates that

$$\deg^t[-R'_{n+1}(x)] \leq 2n + 1.$$

Combining this inequality with (3.11) yields (2.7).

4. REMARKS

Finally we list several remarks on Qi's conjectures and main results in this paper.

Remark 4.1. From the proof of Theorem 2.1, we can deduce that

$$\lim_{x \rightarrow \infty} R'_n(x) = 0, \quad \lim_{x \rightarrow \infty} [x^{2n-1} R'_n(t)] = 0,$$

$$\lim_{x \rightarrow \infty} [x^{2n+3} R'_n(t)] = (-1)^{n+1} \frac{B_{2n+2}}{2n+2}, \quad \lim_{x \rightarrow 0} [x^{2n+1} R'_n(t)] = (-1)^n \frac{B_{2n}}{2n}$$

for $n \in \mathbb{N}$.

Remark 4.2. The conjectures in (2.5), (2.9) and (2.10) by the first author are still kept open. Theorem 2.1 implies that these open conjectures posed by the first author should be true.

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(Qi) INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO 454010, HENAN, CHINA; COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO 028043, INNER MONGOLIA, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300387, CHINA

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: <https://qifeng618.wordpress.com>

(Mahmoud) MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, MANSOURA UNIVERSITY, MANSOURA 35516, EGYPT

E-mail address: mansour@mans.edu.eg

URL: <https://orcid.org/0000-0002-5918-1913>