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► **To cite this version:**

Ö Gültekin, Özgür D. Gürçan. Generalized curvature modified plasma dispersion functions and Dupree renormalization of toroidal ITG. *Plasma Physics and Controlled Fusion*, 2020, 62 (2), pp.025018. 10.1088/1361-6587/ab56a7. hal-02415121

HAL Id: hal-02415121

<https://hal.science/hal-02415121>

Submitted on 30 Nov 2020

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Generalized curvature modified plasma dispersion functions and Dupree renormalization of toroidal ITG

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A new generalization of curvature modified plasma dispersion functions is introduced in order to express Dupree renormalized dispersion relations used in quasi-linear theory. For instance the Dupree renormalized dispersion relation for gyrokinetic, toroidal ion temperature gradient driven (ITG) modes, where the Dupree's diffusion coefficient is assumed to be a low order polynomial of the velocity, can be written entirely using generalized curvature modified plasma dispersion functions: K_{nm} 's. Using those, Dupree's formulation of renormalized quasi-linear theory is revisited for the toroidal ITG mode. The Dupree diffusion coefficient has been obtained as a function of velocity using an iteration scheme, first by assuming that the diffusion coefficient is constant at each v (i.e. applicable for slow dependence), and then substituting the resulting v dependence in the form of complex polynomial coefficients into the K_{nm} 's for verification. The algorithm generally converges rapidly after only a few iterations. Since the quasi-linear calculation relies on an assumed form for the wave-number spectrum, especially around its peak, practical usefulness of the method is to be determined in actual applications. A parameter scan of η_i shows that the form of the diffusion coefficient is better represented by the polynomial form as η_i is increased.

I. INTRODUCTION

A. Background

Curvature driven drift instabilities, driven mainly by inhomogeneities of background profiles, are one of the major sources of turbulence in magnetized fusion devices that lead to anomalous transport, limiting confinement time and therefore our ability to obtain energy from fusion reactions[1]. A proper description of these instabilities under the conditions of a magnetized fusion device requires a kinetic formulation, which takes into account the influence of the large magnetic field[2, 3]. Most linear dispersion relations, relevant for kinetic waves in plasmas can be described using combinations and/or generalizations of the plasma dispersion function[4]. Many generalizations and modifications were proposed in the past[5–8] and efficiencies of different methods of computation compared[9].

A particular such generalization is the introduction of the so called curvature modified plasma dispersion functions[10]. These functions, which appear naturally in the formulation of the local, linear dispersion relation of the toroidal ion temperature gradient driven (ITG) mode, can be used to write the linear dispersion relations of various drift instabilities including ITG or its homologue the electron temperature gradient driven (ETG) mode in a generic form. Since the analytical continuation of these functions have already been incorporated in their definitions, they can be used in order to study damped modes[11] as well as the usual unstable branches.

The primary goal of the study of small scale instabilities in tokamaks is to estimate the resulting transport[12]. Fluctuations that are responsible for example for heat

and particle transport in these devices seem to have the key characteristics of the linear instabilities that drive them[13], especially at those scales whose contribution to the transport is dominant (for example near the peak of the wave-number spectrum[14, 15]). This suggests that instead of fully developed turbulence as in the case of Kolmogorov's theory[16], plasma turbulence is kinetic and relatively weak. This however, does not mean that a weak turbulence theory based on resonant interactions alone [17] solves the problem. Existence of zonal flows [18, 19] and of resonance broadening[20, 21] makes the formulation of plasma response particularly complex.

In this paper we introduce a further generalization of curvature modified plasma dispersion functions, which includes a Dupree diffusion coefficient [22] that has a velocity dependence in the form of a second order polynomial. This allows us to describe the perturbed or renormalized dispersion relations for toroidal drift instabilities using these functions.

The algorithm that we use in this paper is as follows. We start complex frequencies that are obtained as the solutions of the linear dispersion relation $\omega_k = \omega_{rk} + i\gamma_k$. Since the Dupree diffusion coefficient D_D is usually defined in terms of itself via a complicated equation of the sort $D_D \sim \sum_k D_{0k} / (i\omega_{rk} + \gamma_k + D_D)$, we first compute it by taking it as zero on the right hand side. This gives us the first iteration $D_D^{(1)}$. Using $D_D = D_D^{(1)}$ in the perturbed dispersion equation $\varepsilon(\omega, \mathbf{k}, D_D^{(1)}(v)) = 0$, we compute the dominant renormalized complex frequency, which can be dubbed $\omega_k^{(1)}$. Using $D_D^{(1)}$ and $\omega_k^{(1)}$ we compute $D_D^{(2)}$ and so on, until the difference between $(D_D^{(n)} - D_D^{(n-1)}) / D_D^{(n-1)}$ is smaller than a predefined tolerance. The algorithm converges rapidly in only 3-4

iterations in most cases.

The paper is organized as follows. In the remainder of Section I, we review the curvature modified dispersion functions and remind the reader the formulation of the problem of the local linear dispersion relation of toroidal ITG using these functions. In Section II we define the generalized curvature modified plasma dispersion functions, K_{nm} 's, which is the primary novelty of this paper, and discuss their analytical continuation. Section III, reviews the basic idea of Dupree renormalization, explains the numerical implementation and how the Dupree renormalized ITG dispersion relation can be formulated and solved in terms of curvature modified plasma dispersion functions. Section IV is a brief discussion of results and conclusions.

B. Curvature modified dispersion functions

The functions, dubbed $I_{nm}(\zeta_\alpha, \zeta_\beta, b)$, and defined for $\text{Im}[\zeta_\alpha] > 0$ as:

$$I_{nm}(\zeta_\alpha, \zeta_\beta, b) \equiv \frac{2}{\sqrt{\pi}} \int_0^\infty dx_\perp \int_{-\infty}^\infty dx_\parallel \frac{x_\perp^n x_\parallel^m J_0^2(\sqrt{2b}x_\perp) e^{-x^2}}{\left(x_\parallel^2 + \frac{x_\perp^2}{2} + \zeta_\alpha - \zeta_\beta x_\parallel\right)}, \quad (1)$$

can be written as a 1D integral of a combination of plasma dispersion functions as discussed in detail in Ref. [10]:

$$\Delta I_{nm}(\zeta_\alpha, \zeta_\beta, b) = -i\sqrt{\pi} 2^{\frac{(n+3)}{2}} w^{\frac{n}{2}} \int_{-1}^1 d\mu (1-\mu^2)^{\frac{(n-1)}{2}} \left(\mu\sqrt{w} + \frac{\zeta_\beta}{2}\right)^m J_0^2\left(2\sqrt{b(1-\mu^2)}w\right) e^{-2(1-\mu^2)w - \left(\mu\sqrt{w} + \frac{\zeta_\beta}{2}\right)^2} \times \begin{cases} 0 & \zeta_{\alpha i} > 0 \quad \text{or } w_r < 0 \\ \frac{1}{2} & \zeta_{\alpha i} = 0 \quad \text{and } w_r > 0 \\ 1 & \zeta_{\alpha i} < 0 \quad \text{and } w_r > 0 \end{cases} \quad (5)$$

with $w = \frac{\zeta_\beta^2}{4} - \zeta_\alpha$, $\zeta_{\alpha i} = \text{Im}(\zeta_\alpha)$ and $w_r = \text{Re}[w]$.

Derivatives of curvature modified dispersion functions can also be defined as analytical functions and implemented similarly [24].

C. Toroidal ITG

Following the kinetic formulation of toroidal ITG mode[25, 26] starting from the linear gyrokinetic equation[27–29], which can be written in Fourier space for the non-adiabatic part of the distribution function

$$I_{nm}(\zeta_\alpha, \zeta_\beta, b) = \int_0^\infty s^{\frac{n-1}{2}} G_m(z_1(s), z_2(s)) J_0\left(\sqrt{2bs}\right)^2 e^{-s} ds, \quad (\text{Im}[\zeta_\alpha] > 0) \quad (2)$$

using the straightforward multi-variable generalization of the standard plasma dispersion function: $G_m(z_1, z_2, \dots, z_n) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{x^m e^{-x^2}}{\prod_{i=1}^n (x - z_i)} dx$ with

$$z_{1,2}(s) = \frac{1}{2} \left(\zeta_\beta \pm \sqrt{\zeta_\beta^2 - 2(s + 2\zeta_\alpha)} \right). \quad (3)$$

The $G_m(z_1, z_2)$ in (2) can be written in terms of the standard plasma dispersion function as:

$$G_m(z_1, z_2) = \frac{1}{\sqrt{\pi}(z_1 - z_2)} \left[z_1^m Z_0(z_1) - z_2^m Z_0(z_2) + \sum_{k=2}^m (z_1^{k-1} - z_2^{k-1}) \Gamma\left(\frac{m-k+1}{2}\right) \right], \quad (4)$$

which can be implemented using the Weideman method [23].

Furthermore, in order to extend the validity of the definition (2) we must use $I_{nm} = I'_{nm} + \Delta I_{nm}$ [where I'_{nm} is the integral in (2)], where

as:

$$\left(\frac{\partial}{\partial t} + i v_\parallel k_\parallel + i \hat{\omega}_D(v) \right) \delta g_{\mathbf{k}} = \left(\frac{\partial}{\partial t} + i \omega_{*T}(v) \right) \frac{e}{T_i} \delta \Phi_{\mathbf{k}} F_0 J_0 \left(\frac{k_\perp v_\perp}{\Omega_i} \right) \quad (6)$$

Where $\hat{\omega}_D(v) = \frac{\omega_D}{2} \left(\frac{v_\parallel^2}{v_{ti}^2} + \frac{v_\perp^2}{2v_{ti}^2} \right)$, $\omega_{*Ti}(v) \equiv \omega_{*i} \left[1 + \left(\frac{v^2}{2v_{ti}^2} - \frac{3}{2} \right) \eta_i \right]$, $\omega_D = 2 \frac{L_n}{R} \omega_{*i}$, $\omega_{*i} = v_{ti} \rho_i k_y / L_n$, $L_n = -(d \ln n_0 / dr)^{-1}$, $\eta_i = d \ln T_i / d \ln n_0$, R is the major radius of the tokamak, v_{ti} is the ion thermal velocity,

Ω_i is the ion cyclotron frequency and ρ_i is the ion Larmor radius. The gyrokinetic equation is complemented by the quasi-neutrality relation, which is written here for adiabatic electrons for $k_{\parallel} \neq 0$:

$$\frac{e}{T_e} \delta\Phi_k = -\frac{e}{T_i} \delta\Phi_k + \int J_0 \delta g_k d^3v. \quad (7)$$

Taking the Laplace-Fourier transform of (6), solving for $\delta g_{\mathbf{k},\omega}$ and substituting the result into (7), we obtain the following dispersion relation, written in terms of $\varepsilon(\omega, \mathbf{k})$, the plasma dielectric function:

$$\varepsilon(\omega, \mathbf{k}) \equiv 1 + \frac{1}{\tau} - P(\omega, \mathbf{k}) = 0. \quad (8)$$

where

$$P(\omega, \mathbf{k}) = \frac{1}{\sqrt{2\pi}v_{ti}^3} \int \frac{(\omega - \omega_{*Ti}(v)) J_0^2\left(\frac{k_{\perp}v_{\perp}}{\Omega_i}\right) e^{-\frac{v^2}{2v_{ti}^2}}}{\left(\omega - v_{\parallel}|k_{\parallel}| - \omega_{Di}\frac{1}{2}\left(\frac{v_{\parallel}^2}{v_{ti}^2} + \frac{v_{\perp}^2}{2v_{ti}^2}\right)\right)} v_{\perp} dv_{\perp} dv_{\parallel}$$

ormalising k with ρ_i^{-1} and Using $\omega/|k_y| \rightarrow \omega$, and $\omega_D/|k_y| \rightarrow \omega_D$, the dispersion relation (8) can be written as:

$$\varepsilon(\omega, \mathbf{k}) \equiv 1 + \frac{1}{\tau} + \frac{1}{\omega_{Di}} \left(I_{10} \left[\omega + \left(1 - \frac{3}{2} \eta_i \right) \right] + (I_{30} + I_{12}) \eta_i \right) = 0 \quad (9)$$

where $I_{nm} \equiv I_{nm} \left(-\frac{\omega}{\omega_{Di}}, -\frac{\sqrt{2}k_{\parallel}}{\omega_{Di}k_y}, b \right)$ with $b \equiv k_{\perp}^2$.

II. GENERALIZED CURVATURE MODIFIED DISPERSION FUNCTIONS

A further generalization of I_{nm} 's can be proposed in the form of the following integrals:

$$K_{nm}(\zeta_a, \zeta_b, \zeta_c, \zeta_d, b) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} dx_{\perp} \int dx_{\parallel} \frac{x_{\perp}^n x_{\parallel}^m J_0^2(\sqrt{2b}x_{\perp}) e^{-x^2}}{\left(x_{\parallel}^2 + \zeta_a + \zeta_b x_{\parallel} + \zeta_c x_{\perp} + \zeta_d x_{\perp}^2\right)} \quad (10)$$

with complex variables $\zeta_a, \zeta_b, \zeta_c$ and ζ_d , and the real variable b , so that, we can write

$$I_{nm}(\zeta_{\alpha}, \zeta_{\beta}, b) = K_{nm} \left(\zeta_{\alpha}, -\zeta_{\beta}, 0, \frac{1}{2}, b \right).$$

The definition of the I_{nm} 's in terms of the generalized plasma dispersion functions $G_m(z_1, z_2, \dots, z_n)$ as in (2)

can be extended to K_{nm} 's:

$$K_{nm}(\zeta_a, \zeta_b, \zeta_c, \zeta_d, b) = 2 \int_0^{\infty} x_{\perp}^n G_m(z'_1(x_{\perp}), z'_2(x_{\perp})) J_0(\sqrt{2b}x_{\perp})^2 e^{-x_{\perp}^2} dx_{\perp},$$

using

$$z'_{1,2}(x_{\perp}) = \frac{1}{2} \left(-\zeta_b \mp \sqrt{\zeta_b^2 - 4(\zeta_c x_{\perp} + \zeta_d x_{\perp}^2 + \zeta_a)} \right).$$

Note that the function defined in (10) can represent any complex polynomial denominator up to second order in x_{\parallel} and x_{\perp} . The resulting functions can be implemented using (4) to write the G_m 's and computing the remaining one dimensional x_{\perp} integral using a numerical quadrature.

A. Analytical Continuation

The integral in (10) has a singularity in the complex plane when the polynomial in the denominator vanishes. Since the polynomial is a quadratic one that is a function of both x_{\parallel} and x_{\perp} , it vanishes on a curve. If the coefficients were all real, this would be a curve in two dimensions. However since the coefficients are complex, the actual curve lives in 4 dimensional space made of $x_{\parallel r}, x_{\parallel i}, x_{\perp r}$ and $x_{\perp i}$ (i.e. real and imaginary parts of the extended x_{\parallel} and x_{\perp}). However since it remains a "curve", we can still parametrize it.

The analytical continuation of the I_{nm} 's are performed by first scaling the x_{\perp} and x_{\parallel} so that the ellipse becomes a circle, and then switching to polar coordinates so that the singularity becomes a point on the r axis. The residue contribution is then equal to the integral over the angular variable on the circle. This residue is to be added to the original integral when $\text{Im}(\zeta_{\alpha}) < 0$ and $w_r > 0$ as shown in (5).

In the case of the K_{nm} the issue is more complicated. Considering the denominator

$$d(x_{\perp}, x_{\parallel}) = x_{\parallel}^2 + \zeta_b x_{\parallel} + \zeta_c x_{\perp} + \zeta_d x_{\perp}^2 + \zeta_{\alpha}$$

can be written using $\rho = \sqrt{\left(x_{\parallel} + \frac{\zeta_{br}}{2}\right)^2 + (\sqrt{\zeta_{dr}} x_{\perp})^2}$ and $\mu = \left(x_{\parallel} + \frac{\zeta_{br}}{2}\right) / \rho$ as:

$$d(\rho, \mu) = \rho^2 \left(1 + i \frac{\zeta_{di}(1 - \mu^2)}{\zeta_{dr}} \right) + \left(\zeta_c \frac{\sqrt{(1 - \mu^2)}}{\sqrt{\zeta_{dr}}} + i \zeta_{bi} \mu \right) \rho - w$$

where $w = \frac{\zeta_{br}^2}{4} + i \zeta_{bi} \zeta_{br} - \zeta_{\alpha}$. The denominator becomes zero at the two complex roots:

$$\rho_{\pm} = -\frac{\beta}{2\alpha} \pm \frac{1}{2\alpha} \sqrt{4\alpha w + \beta^2} \quad (11)$$

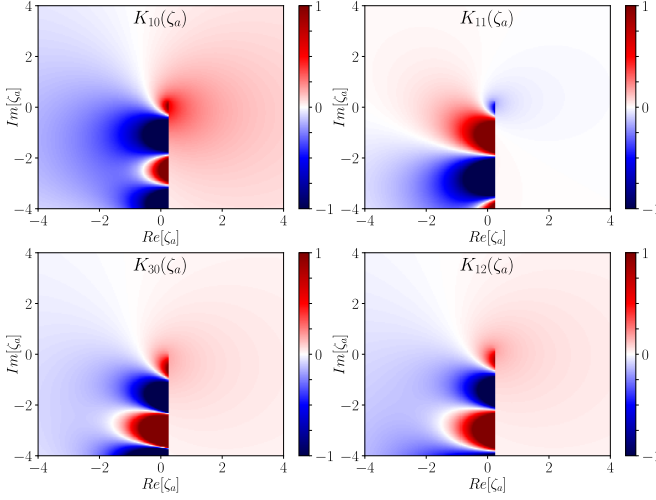


Figure 1. Plots of K_{10} , K_{11} , K_{30} and K_{12} (from top left to bottom right) as functions of the real and imaginary parts of ζ_a , where the other parameters are fixed at $\zeta_b = 1.0 + 0.5i$, $\zeta_c = 0.5 + 0.5i$, $\zeta_d = 0.5 + 0.2i$ and $b = 1.0$.

with $\alpha \equiv \left(1 + i \frac{\zeta_{di}(1-\mu^2)}{\zeta_{dr}}\right)$ and $\beta \equiv \left(\zeta_c \frac{\sqrt{(1-\mu^2)}}{\sqrt{\zeta_{dr}}} + i \zeta_{bi} \mu\right)$.

Note that when we do the transformation from x_\perp, x_\parallel to ρ and μ , the surface element becomes:

$$dx_\perp dx_\parallel = \frac{\rho}{\sqrt{\zeta_{dr}}(1-\mu^2)} d\mu d\rho$$

so that the integral can be written as:

$$K_{nm}(\zeta_a, \zeta_b, \zeta_c, \zeta_d, b) \equiv \frac{2}{\sqrt{\pi}} \int_{-1}^1 d\mu \int_0^\infty \rho d\rho \frac{x_\perp(\rho, \mu)^n x_\parallel(\rho, \mu)^m (J_0^\rho)^2 e^{-x^2(\rho, \mu)}}{\alpha(\rho - \rho_+) (\rho - \rho_-) \sqrt{\zeta_{dr}}(1-\mu^2)}$$

where the shorthand notation $J_0^\rho = J_0(\sqrt{2b}x_\perp(\rho, \mu))$ and $x^2(\rho, \mu) = x_\perp(\rho, \mu)^2 + x_\parallel(\rho, \mu)^2$ has been used. This means that the residue contribution

$$\Delta K_{nm}(\zeta_a, \zeta_b, \zeta_c, \zeta_d, b) \equiv 4\sqrt{\pi}i \int_{-1}^1 \rho_+ d\mu \frac{x_\perp(\rho_+, \mu)^n x_\parallel(\rho_+, \mu)^m (J_0^{\rho_+})^2 e^{-x(\rho_+, \mu)^2}}{\sqrt{4\alpha w + \beta^2} \sqrt{\zeta_{dr}}(1-\mu^2)}$$

should be added to the integral when $\text{Im}[\rho_+(\mu)] > 0$ (and we should add 1/2 times the residue contribution when $\text{Im}[\rho_+(\mu)] = 0$). Note that writing the condition $\text{Im}[\rho_+(\mu)] > 0$ in terms of real and imaginary parts of ζ parameters is complicated enough that we find it more practical to check this condition by computing ρ_+ using (11) numerically.

As a function of the real and imaginary parts of its primary variable ζ_a , the function K_{nm} can be plotted fixing the values of its other variables. The results are shown in figures 1 and 2.

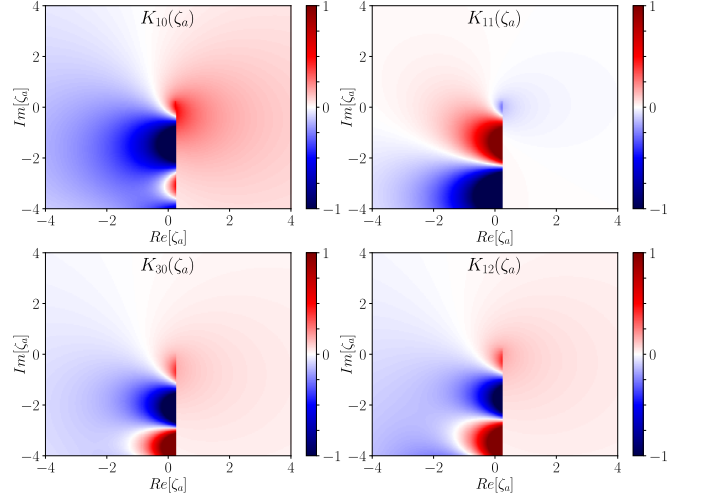


Figure 2. Plots of K_{10} , K_{11} , K_{30} and K_{12} (from top left to bottom right) as functions of the real and imaginary parts of ζ_a , where the other parameters are fixed at the same parameters as figure 1 but with $\zeta_c = 0.5 + 1.0i$.

III. DUPREE RENORMALIZATION

Dupree's renormalized form of the gyrokinetic equation can be written as [21, 30]:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + iv_\parallel k_\parallel + i\hat{\omega}_D(v) + D_D(v) k_\perp^2 \right) \delta g_{\mathbf{k}} \\ & = \left(\frac{\partial}{\partial t} + i\omega_{*T}(v) + D_D(v) k_\perp^2 \right) \frac{e}{T_i} \delta \Phi_{\mathbf{k}} F_0 J_0 \left(\frac{k_\perp v_\perp}{\Omega_i} \right). \end{aligned} \quad (12)$$

Assuming a general complex $D_D(v)$, and short correlation times, we can write the discrete equivalent of Dupree's integral equation as:

$$D_D(v_\perp, v_\parallel) = \sum_{\mathbf{k}} \frac{k^2 J_{0k}^2 |\delta \Phi_{\mathbf{k}}|^2}{i(\omega_{\mathbf{k}}^* - \omega_{Dk}(v) - k_\parallel v_\parallel) + D_D^*(v_\perp, v_\parallel) k_\perp^2} \quad (13)$$

Note that here $\omega_{\mathbf{k}} = \omega_{r\mathbf{k}} + i\gamma_{\mathbf{k}}$ is the solution of the *renormalized* linear dispersion relation due to (12):

$$\begin{aligned} \varepsilon(\omega, \mathbf{k}) & \equiv 1 + \frac{1}{\tau} \\ & - \frac{1}{\sqrt{2\pi} v_{ti}^3} \int \frac{(\omega - \omega_{*T_i}(v) + iD_D(v) k_\perp^2) J_{0k}^2 e^{-\frac{v^2}{2v_{ti}^2}}}{\omega - v_\parallel k_\parallel - \hat{\omega}_{Di}(v) + iD_D(v) k_\perp^2} v_\perp dv_\perp dv_\parallel \end{aligned} \quad (14)$$

which is usually considered to introduce a nonlinear damping on top of the linear solution (i.e. $\omega_{\mathbf{k}} = \omega_{\mathbf{k}, \text{lin}} - i\eta_{\text{eddy}}$). This is exactly the case, if the Dupree diffusion coefficient is a real constant independent of v . However in the case of a complex function of v_\perp and v_\parallel , the issue is more complicated.

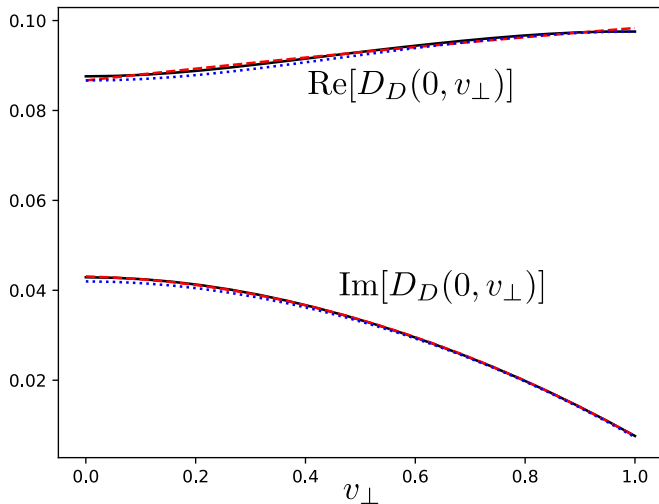


Figure 3. The real and imaginary parts of Dupree diffusion coefficient as a function of v_{\perp} . The solid (black) lines show the $D_D(0, v_{\perp})$ computed using iteration at each v_{\perp} by assuming D_D is constant in the dispersion relation, dashed (red) lines show the polynomial fit. The result that is obtained, when this polynomial fit is used to compute the D_D again via (14) and (13), is shown as the (blue) dotted line.

One approach to this problem is to consider the dependence of D_D on v as being weak. This allows us to solve (13) at each v separately while considering D_D as a constant in solving ω_k from (14). Of course this is only an intermediate step and one should finally consider a v dependent $D_D(v)$ and somehow substitute that into (14) and show that ω_k that one can obtain together with the $D_D(v)$ gives us back the $D_D(v)$ that we used as a requirement of verification of this solution.

Unfortunately for an arbitrary function of v , this is hard to do. However when $D_D(v)$ is computed at each v by assuming it as a constant in (14), the function that is obtained as a function of v_{\parallel} or v_{\perp} is rather close to a low order polynomial in a range of v values. It means that we can fit a polynomial of the form

$$D_D = d_0 + d_1 x_{\parallel} + d_2 x_{\parallel}^2 + d_3 x_{\perp} + d_4 x_{\perp}^2 \quad (15)$$

where $x_{\perp, \parallel} = v_{\perp, \parallel} / \sqrt{2} v_{ti}$, to the form obtained by solving (13) at each v_{\perp} and v_{\parallel} . For simplicity we performed this fit by fixing $v_{\perp} = 0$ and computing $D(v_{\parallel}, 0)$ and fitting and then fixing $v_{\parallel} = 0$ and computing $D_D(0, v_{\perp})$ and fitting.

For a given v_{\perp} , v_{\parallel} we solve (13) by iteration. In practice we start the iteration by setting $D_D = 0$ and computing ω_k from (12) and using this ω_k and $D_D = 0$, we compute the perturbed D_D using (13), which can be called $D_D^{(1)}$ since it is the first iteration. Then using this $D_D^{(1)}$ in (12), we can compute $\omega_k^{(1)}$ and substituting this $\omega_k^{(1)}$ and $D_D^{(1)}$ on the right hand side of (13), we obtain the $D_D^{(2)}$ and so on. We stop the iteration when conver-

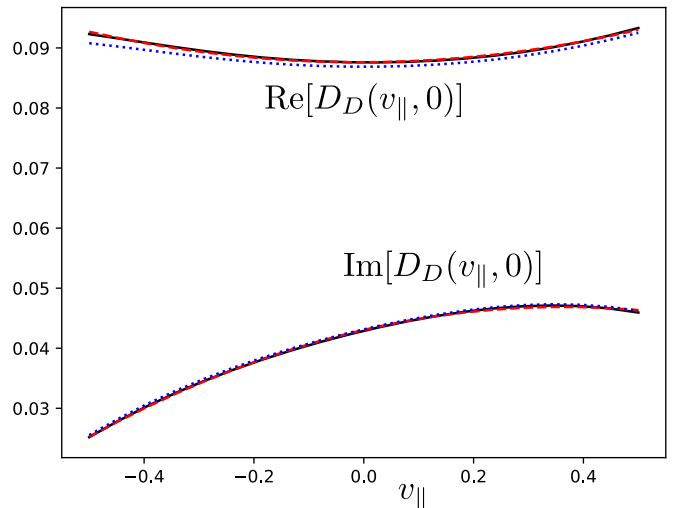


Figure 4. The real and imaginary parts of Dupree diffusion coefficient as a function of v_{\parallel} . The solid (black) lines show the $D_D(v_{\parallel}, 0)$ computed using iteration at each v_{\parallel} by assuming D_D is constant in the dispersion relation, dashed (red) lines show the polynomial fit. The result that is obtained, when this polynomial fit is used to compute the D_D again via (14) and (13), is shown as the (blue) dotted line.

gence, defined by $\left| \frac{D_D^{(n)} - D_D^{(n-1)}}{D_D^{(n-1)}} \right| < \epsilon$ where the tolerance ϵ is taken to be around %1. Fortunately it takes in general about 3 iterations for this algorithm to converge.

One issue, which is a common problem in quasi-linear theory is that, a priori, one can not compute the spectrum $|\delta\Phi_k|^2$ that goes into (13) within the theory itself. Renormalization can be formulated in such a way that we could compute the spectrum using a local balance condition such as $D_D k_{\perp}^2 \sim \gamma_k$. However while maybe the maximum of the spectrum could be determined from a condition of this form, the wave-number spectrum in plasma turbulence is well known to not follow the form implied by $D_D k_{\perp}^2 \sim \gamma_k$ at every scale. In particular this is nonsensical when γ_k becomes negative in some part of the k space. Therefore here we chose to impose the following reasonable form for the k -spectrum:

$$|\delta\Phi_k|^2 = \begin{cases} e^{-(k-k_0)^2/2\sigma^2} & k < k_p \\ \left(e^{-(k_p-k_0)^2/2\sigma^2} k_p^3 \right) k^{-3} & k > k_p \end{cases}$$

where k_0 is the maximum of the wave-number spectrum around which it is taken to be a Gaussian with a width σ which then joins a power law spectrum at the transition wave-number k_p . Obviously the D_D that will be computed will depend on these parameters as well as the plasma parameters R/L_n , η_i etc. Finally we also take $k \rightarrow k_y$ above in order to keep the computation tractable. This form is similar to the one used in quasi-linear models [31], with a peak quasi-linear region matching a power law solution [32, 33] that falls off as k^{-3} partly motivated

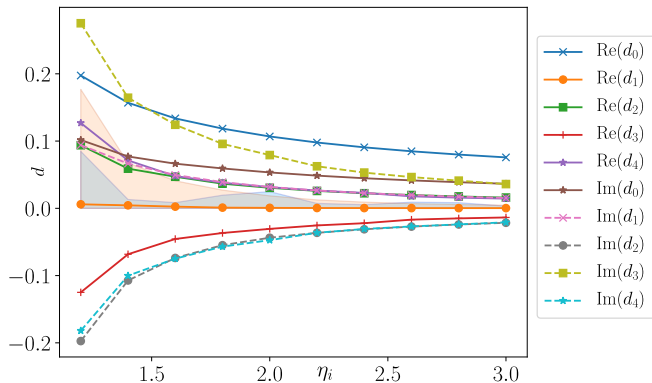


Figure 5. Polynomial coefficients d_i of Dupree diffusion as a function of η_i . The shaded regions represent the mean absolute difference the v dependent form of the diffusion coefficient and the polynomial approximation in (15) (blue shaded region for the dependence on v_\perp and orange shaded region for v_\parallel), multiplied by 10 to make them visible in the plot. Note that both the polynomial coefficients converge and the error is reduced as η_i is increased.

by the forward enstrophy cascade solution in two dimensional turbulence [34], which can be seen as a simple parametrization of the observed wave-number spectrum near the transport range in tokamak plasmas[14, 15, 35]. Note that the exact form of this power law, or its refinement at higher k for example by using a spectrum of the form $k^{-3}/(1+k^2)^2$ [36] makes no practical difference for the computation of D_D .

With all these assumptions, we can finally compute $D_D(v)$, the results are shown in figures 3 and 4. The coefficients of the polynomial (15) are computed in the two fits, with the parameters $\eta_i = 2.5$, $L_n/R = 0.2$, $k_\parallel = 0.01$ and $k_x = 0.1$ as:

$$\begin{aligned} d_0 &= 0.08667974 + 0.04308331i \\ d_3 &= 0.01342785 + 0.00293719i \\ d_4 &= -0.00176120 + 0.03260807i \end{aligned} \quad (16)$$

and

$$\begin{aligned} d_0 &= 0.08760955 + 0.04306338i \\ d_1 &= 0.00034246 + 0.02108074i \\ d_2 &= 0.02101908 + 0.02903672i. \end{aligned} \quad (17)$$

Since the two polynomials are consistent (i.e. d_0 is approximately the same in both cases), we can use these coefficients together in a single two dimensional polynomial form.

A parameter scan, performed by varying η_i shown in figure 5 shows that the polynomial coefficients converge to well defined values as η_i is increased. Also the difference between the polynomial form and the v dependent form of the diffusion coefficient (for example the

difference between the two curves in figures 3 and 4) is observed to decrease with increasing η_i . This suggests that the approximation that is proposed in this paper works better far from marginal stability, which is also the domain of validity of quasilinear theory. This means that a renormalized quasilinear theory based on a polynomial approximation for the Dupree diffusion can be used far from marginal stability as a reasonable approximate method to compute turbulent transport in fusion devices.

A. Solving the renormalized dispersion relation:

Until this point, we talked about solving the renormalized dispersion relation (14). While this is a simple matter of replacing $\omega \rightarrow \omega + ik_\perp^2 D_D$ when D_D is taken to be independent of v , when D_D is taken to be of the form (15), the issue is more complicated. In this case, the dispersion relation have to be rewritten using the K_{nm} 's as follows:

$$\begin{aligned} \varepsilon(\omega, \mathbf{k}) &\equiv 1 + \frac{1}{\tau} + \\ &\frac{1}{\omega_{Di} - id_2 k_\perp^2} \left[\left(\omega - \omega_{*i} \left(1 - \frac{3}{2} \eta_i \right) + ik_\perp^2 d_0 \right) K_{10} \right. \\ &+ ik_\perp^2 d_3 K_{20} + (ik_\perp^2 d_4 - \omega_{*i} \eta_i) K_{30} \\ &\left. + (ik_\perp^2 d_2 - \omega_{*i} \eta_i) K_{12} + ik_\perp^2 d_1 K_{11} \right] = 0 \end{aligned} \quad (18)$$

where $K_{nm} = K_{nm}(\zeta_a, \zeta_b, \zeta_c, b)$ is used as a shortcut notation with

$$\begin{aligned} \zeta_a &= -\frac{\omega + id_0 k_\perp^2}{\omega_{Di} - id_2 k_\perp^2}, \quad \zeta_b = \frac{\sqrt{2} k_\parallel v_{ti} - id_1 k_\perp^2}{\omega_{Di} - id_2 k_\perp^2} \\ \zeta_c &= -\frac{id_3 k_\perp^2}{\omega_{Di} - id_2 k_\perp^2}, \quad \zeta_d = \frac{\frac{1}{2} \omega_{Di} - id_4 k_\perp^2}{\omega_{Di} - id_2 k_\perp^2}. \end{aligned}$$

When the dispersion relation is written using the K_{nm} 's which are already analytic everywhere on the complex plane thanks to the analytical continuation discussed above, $\varepsilon(\omega, \mathbf{k})$ also becomes analytic everywhere also. Note that the form used in (14) is actually not analytical when the imaginary part of ρ_+ as defined in (11) is greater than zero. Using a least square solver, we can then find the zeros of the renormalized dispersion relation and trace these results. The result with combined set of coefficients from (16) and (17) is shown in figure 6.

IV. RESULTS AND CONCLUSIONS

Using a generalization of curvature modified plasma dispersion functions, we were able to implement a linear solver that can solve the dispersion relation with a

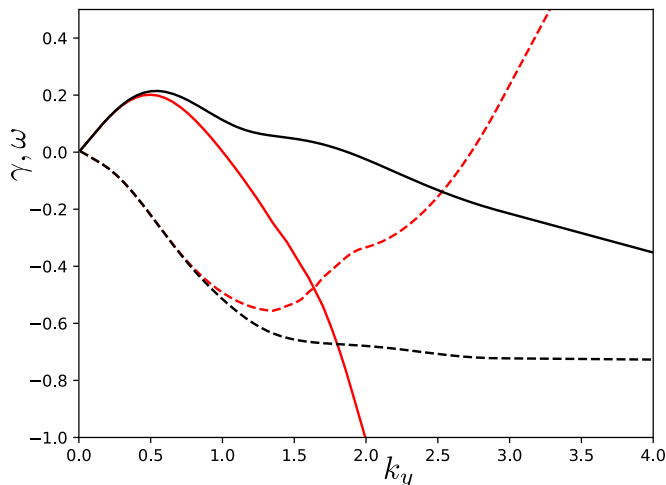


Figure 6. The growth rate (solid lines) and the frequency (dashed lines), with (red) and without (black) renormalization. Note that while the frequency changes sign and becomes positive, since the growth rate becomes strongly negative, it is unlikely that this positive frequency can actually be realized.

Dupree diffusion coefficient in the form of a second order polynomial. The resulting solver was used in an algorithm based on iteration in order to solve the Dupree integral relation and therefore obtain the renormalized Dupree diffusion coefficient together with the renormalized growth rate and frequency for the gyrokinetic, local, electrostatic ITG, with adiabatic electrons.

Since the growth rate is obtained as the solution of the renormalized dispersion relation, it becomes rapidly negative for large k , in particular, due to the effect of Dupree diffusion. Thus, one has to define the generalized curvature modified plasma dispersion functions, the K_{nm} 's together with their analytical continuation. This guarantees that when the dispersion relation is written as in (18) using the K_{nm} 's it is also analytic everywhere in the complex plane.

The Dupree diffusion coefficient is rather important in quasi-linear transport formulation[31], which has been developed into a full transport framework over the years [37], since in this formulation the effective correlation time is argued to be renormalized as compared to the linear one. It is also true that the heat flux in ITG should be proportional to the Dupree diffusion coefficient[30].

The basic computation that is shown here is based on a number of assumptions such as a particular form that one has to assume for the k -spectrum, or the assumption that during the iteration procedure for every v value one can at first assume that D_D is a constant, etc. Nonetheless it proposes a numerically tractable, practical renormalization algorithm based on iteration including curvature effects. The algorithm can be generalized to include k dependence of the eddy damping as well using a closure such as direct interaction approximation, or the

eddy damped quasi-normal Markovian approximation, or rather its realizable variant[21, 38].

A scan of the main parameter η_i of the ITG mode, which represents the strength of the instability drive (and therefore distance to threshold), shows that as the instability drive increases, polynomial coefficients of the Dupree diffusion decrease, together with the difference to the velocity dependent form of the Dupree diffusion. This means that the farther away we are from the stability threshold the better the polynomial approximation becomes, and that the renormalized dispersion relation, therefore a renormalized quasilinear theory based on such a formulation, becomes better justified far from marginal stability.

Finally, we think that further studies, including similar formulation for other important instabilities such as the trapped electron modes or ETG etc. as well as the importance of various parameters in the model are required in order to conclude the robustness and the practical importance of the renormalization perspective in the modeling efforts of fusion plasma turbulence.

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