



## On five papers by Herbert Grötzsch

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# ON FIVE PAPERS BY HERBERT GRÖTZSCH

VINCENT ALBERGE AND ATHANASE PAPADOPOULOS

ABSTRACT. Herbert Grötzsch is the main founder of the theory of quasiconformal mappings. We review five of his papers, written between 1928 and 1932, that show the progress of his work from conformal to quasiconformal geometry. This will give an idea of his motivation for introducing quasiconformal mappings, of the problems he addressed and on the results he obtained concerning these mappings.

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## 1. INTRODUCTION

Herbert Grötzsch is the main founder of the theory of quasiconformal mappings. In a series of papers written between 1928 and 1932, he introduced these mappings as a natural generalization of conformal mappings and he developed their main properties. He saw that several problems in conformal geometry naturally lead to problems on quasiconformal mappings. In this chapter, we review five of his papers that show the progress of his work from conformal to quasiconformal geometry. The five papers are the following:

- (1) “Über einige Extremalprobleme der konformen Abbildung” (On some extremal problems of the conformal mapping), published in 1928 [12];
- (2) “Über einige Extremalprobleme der konformen Abbildung. II” (On some extremal problems of the conformal mapping. II), published in 1928 [13];
- (3) “Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes.” (On the distortion of univalent non-conformal mappings and a related extension of the Picard theorem) [14]; published in 1928.
- (4) “Über die Verzerrung bei nichtkonformen schlichten Abbildungen mehrfach zusammenhängender schlichter Bereiche” (On the distortion of non-conformal schlicht mappings of multiply-connected schlicht regions), published in 1930 [15];
- (5) “Über möglichst konforme Abbildungen von schlichten Bereichen” (On closest-to-conformal mappings of schlicht domains), published in 1932 [16].

Translations of these five papers into English are included in the present volume.

We shall present the main results of these papers, providing introductory remarks, establishing the connections between them, and explaining the background. We shall also indicate some relations with works of Lavrentieff and Teichmüller.

The first two papers are concerned with conformal representations of multiply-connected domains. Grötzsch uses there, for the first time, the so-called *length-area method*, a powerful method which is a direct application of the Schwarz inequality. In the third paper, he introduces the notion of *non-conformal* (“nichtkonformen”) mapping, which he also calls *mapping of bounded infinitesimal distortion* (“Abbildung von beschränkter infinitesimales Verzerrung”). This notion is very close to that of quasiconformal mapping, as it is understood today. The results he proves are all valid for quasiconformal mappings (with the same proofs) and for this reason we shall refer to Grötzsch’s non-conformal mappings as quasiconformal mappings. His aim in this paper is to show that several properties known for conformal mappings remain true in this more general setting. He proves a distortion theorem for the new class of mappings—an analogue of a distortion theorem of Koebe—and a generalization of the big Picard theorem, one of the very classical results on meromorphic functions. The fourth paper is concerned with quasiconformal representations of multiply-connected domains. The results obtained in this paper are generalizations of results known for conformal representations. The fifth paper is concerned with maps that Grötzsch calls *closest-to-conformal* (“möglichst konforme”). These are the mappings that we call today “extremal quasiconformal mappings”.

## 2. CONFORMAL REPRESENTATIONS

The problem of mapping conformally domains of the complex plane  $\mathbb{C}$  onto “canonical” domains was inaugurated by Riemann in his doctoral dissertation [24] (1851), in which he proved that any simply-connected open subset of the plane which is not the whole plane can be conformally mapped in a one-to-one manner onto the unit disc. He also proved that such a mapping is unique up to post-composition by a conformal automorphism of the disc (a Möbius transformation). This is the well-known *Riemann Mapping Theorem*. Riemann proved it using the so-called Dirichlet principle, which he formulated and used extensively in his works on Riemann surfaces. Generalizing the Riemann Mapping Theorem to non-simply-connected planar domains is a broad subject that occupied generations of mathematicians after Riemann. We recall the following two well-known facts:

- (1) Two connected open subsets of the complex plane that have the same connectivity are not necessarily conformally equivalent. For instance, two circular annuli (that is, annuli in the complex plane bounded by two concentric circles) are conformally equivalent if and only if they have the same *module*, that is, if and only if the ratio of the two radii is the same for both annuli.
- (2) There is no natural “canonical” class of domains for multiply-connected domains. Circular annuli may be considered as some kind of “standard” domains for doubly-connected regions, but there are other possibilities.

We now discuss some of these standard multiply-connected domains.

The unit disc slit along an interval of the form  $[0, r]$  ( $r < 1$ ) is a useful object in conformal geometry, and it was employed by Grötzsch in his work on conformal and quasiconformal geometry. It is known in the classical literature on quasiconformal mappings under the name *Grötzsch domain*. Such a domain appears in particular as a solution of an extremal quasiconformal problem described in Theorem 9 below, which is Theorem 3 of Grötzsch’s paper [15]. The same extremality property of this domain is proved in the book by Lehto and Virtanen, *Quasiconformal mappings in the plane*, [23, p. 52]. In the same book, the Grötzsch domain is shown to be the solution of the following extremal problem for conformal mappings: if  $\mu(r)$  denotes the module of this domain, then, the module of any doubly-connected domain separating the points 0 and  $r$  from the circle  $|z| = 1$  is at most equal to  $\mu(r)$ ; see [23, p. 54].

In Ahlfors' book on conformal invariants, the expression *Grötzsch annulus* is used for the complement, in the complex plane, of the closed unit disc and a segment of the form  $[R, \infty]$  for some positive  $R$  [3, p. 72]. It is possible to write explicitly a conformal homeomorphism between the last two domains.

Teichmüller, in his paper *Untersuchungen über konforme und quasikonforme Abbildung* (Investigations on conformal and quasiconformal mappings), translated in the present volume, calls a *Grötzsch extremal region* the complement of the unit disc in the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  cut along a segment of the real axis joining a point  $P > 0$  to the point  $\infty$  [26, §2.1]. He gives estimates for this domain and he uses it in his investigations.

Another “standard model” for doubly-connected domains is the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  slit along two intervals of the form  $[-r_1, 0]$  and  $[r_2, \infty)$  where  $r_1$  and  $r_2$  are positive numbers. This domain is known under the name *Teichmüller extremal domain* [23, p. 52]. Teichmüller uses this domain in his study of extremal properties of topological annuli, in §2 of his paper [26]; see in particular §2.6. This domain satisfies the following extremal property: its module is an upper bound for the module of any doubly-connected domain that separates the pair 0 and  $-r_1 < 0$  from the pair  $r_2 > 0$  and  $\infty$ . Teichmüller proved this characterization in [26, §2.4], using Koebe's one-quarter theorem and Koebe's distortion theorem. His proof is reproduced in Ahlfors' book [3, p. 72]; see also [23, p. 55]. The module of the Teichmüller extremal domain is equal to the quantity  $2\mu\sqrt{\frac{r_1}{r_1+r_2}}$  (where  $\mu$  denotes as above the module of the Grötzsch extremal domain).

Teichmüller, in his paper [26], also works with other “standard” domains, e.g. the circular annulus  $1 < |z| < P_2$  cut along the segment joining  $z = P_1$  to  $z = P_2$ , where  $P_1$  is a point on the real axis satisfying  $1 < P_1 < P_2$  [26, §2.4]. He proves several extremal properties of these domains. He uses them in a geometric proof which simplifies and generalizes a distortion result that Ahlfors obtained in his thesis [1]; see also the discussion in Ahlfors' book [3, p. 76].

Among the other “standard” doubly-connected domains, we mention the *Mori domain*, called so in the monograph [23].

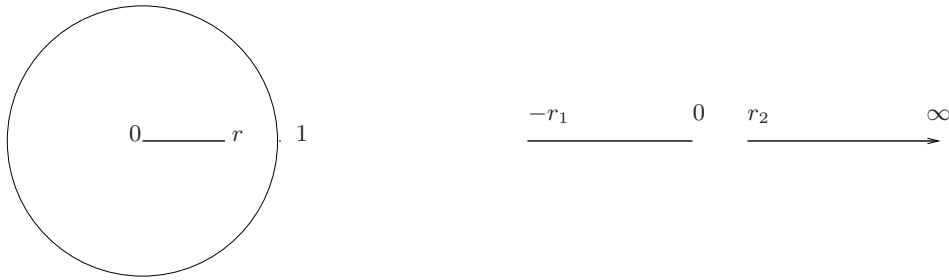


FIGURE 1. A Grötzsch extremal domain (left) and a Teichmüller extremal domain (right), called so in the book of Lehto and Virtanen [23]. In the figure to the right, the doubly-connected domain is the complement in the Riemann sphere of the union of the segment and the ray indicated.

Before Grötzsch, Koebe studied extensively conformal mappings of finitely-connected domains of the plane onto *circle domains*, that is, multiply-connected domains in the plane whose boundary components are all circles (which may be reduced to points), cf. [17, 18]. He proved that every finitely-connected domain in the plane is conformally equivalent to a circle domain. This generalizes the Riemann Mapping Theorem, which says that a simply-connected domain which is not the entire plane

is conformally equivalent to the unit disc. We mention by the way that the question of whether there is a result analogous to Koebe's theorem which is valid for an open subset of the plane having an infinite number of boundary components is open, and is known as the *Kreisnormierungsproblem*, or the *Koebe uniformization conjecture*. Koebe formulated this conjecture in his 1908 paper [19] p. 358. We refer the reader to the recent paper [8] by Bowers in which the author surveys many developments of this conjecture, including the works on circle packings done by Koebe, Thurston and others.

Grötzsch, who was a student of Koebe, was naturally led to study problems connected with conformal representations. In the paper [12], he works with two classes of domains which were already considered by Koebe:

- (1) *Annuli with circular slits*. These are circular annuli from which a certain number ( $\geq 0$ ) of circular arcs (which may be reduced to points) centered at the center of the annulus, have been removed, see Figure 2 (left).
- (2) *Annuli with radial slits*. These are circular annuli from which a certain number ( $\geq 0$ ) of radial arcs (which may be reduced to points) have been removed. A radial arc is a segment in the annulus which, when extended, passes through the center of the annulus, see Figure 2 (right).

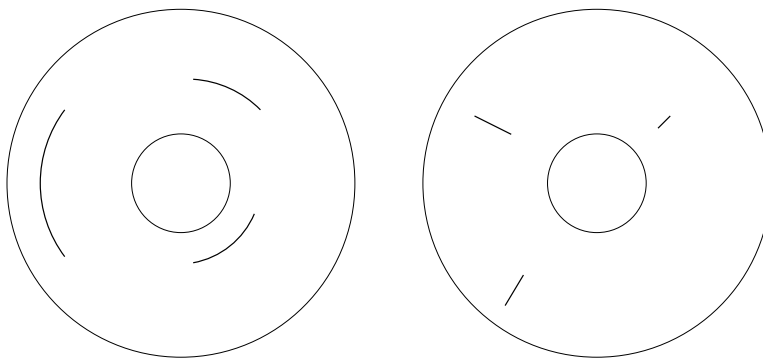


FIGURE 2. An annulus with circular slits (left) and an annulus with radial slits (right)

Koebe, in the paper [12], showed that any finitely-connected domain of the complex plane which is not the whole plane admits a conformal representation (which is essentially unique) onto a domain of one of the two kinds above.

### 3. TWO LEMMAS FROM THE PAPER [12]

Grötzsch's paper *Über einige Extremalprobleme der konformen Abbildung* [12] starts with two lemmas that give a subadditivity result for the modules of (finitely or infinitely many) disjoint subdomains of a circular annulus in the complex plane. In the first lemma, the subdomains are disjoint topological quadrilaterals, each having two opposite sides on the boundary circles of the circular annulus. In the second lemma, the subdomains are disjoint topological annuli that are homotopy-equivalent to the circular annulus.

In each case, the lemma says that the sum of the modules of the subdomains is bounded above by the module of the ambient circular annulus. In particular, if we divide the interior of a circular annulus into two annuli by a simple closed curve homotopic to the boundary components, then the sum of the modules of the two resulting annuli is not greater than the modules of the ambient annulus.

Grötzsch also obtains a characterization of the equality case: equality occurs, in the first case, if and only if the rectangles are obtained by radial slicing of the circular domain, and in the second case, if and only if the decomposition into subdomains is obtained by a slicing of the circular domain by circles concentric to the boundary. Furthermore, in the two equality cases, the subdomains must fill out the ambient circular annulus.

The proofs of these module inequalities are based on the so-called *length-area* method. Regarding this method, Ahlfors writes, in his collected papers edition [5], commenting on his first two published papers on the asymptotic values of entire functions of finite order, which in fact are those of his doctoral dissertation: “[...] The early history of this method is obscure, but I knew it from and was inspired by its application in the well-known textbook of Hurwitz-Courant<sup>1</sup> to the boundary correspondence in conformal mapping.” Talking about Nevanlinna and himself, he adds: “None of us was aware that only months earlier H. Grötzsch had published two important papers on extremal problems in conformal mapping in which the same method is used in a more sophisticated manner [...] The method that Grötzsch and I used is a precursor of the method of extremal length.” In fact, Ahlfors applied this method in his thesis [1] in order to prove what became later known as the *Ahlfors distortion theorem*. Teichmüller, in his papers [26, 27] and [28], uses extensively the length-area method, and he calls it the *Grötzsch–Ahlfors* method.

The module inequalities proved in [12] are considered today as classical results. They were also obtained (with different methods) and widely generalized by Teichmüller in his paper *Untersuchungen über konforme und quasikonforme Abbildung* (Investigations on conformal and quasiconformal mappings) (1938), translated in the present volume (§2.4 and 2.5 of [26]).

#### 4. FOUR THEOREMS FROM THE PAPER [12]

In the paper [12], Grötzsch uses the two lemmas of the previous section in his proof of four theorems which we now state. Theorems 3 and 4 answer questions posed by S. Szegő, see [25].

For  $0 < r < 1$ , we denote by  $\mathcal{K}_r$  the annulus  $r < |z| < 1$  and by  $\overline{\mathcal{K}_r}$  an annulus with circular slits obtained from  $\mathcal{K}_r$  by removing a finite number of circular arcs centered at the origin (Figure 2, left).

**Theorem 1** ([12] §2).— *Let  $R$  be a real number satisfying  $R \geq r$ . If a domain  $\overline{\mathcal{K}_r}$  is mapped conformally and bijectively onto a domain having  $|z| = r$  and  $|z| = R$  as boundary components with the circle  $|z| = r$  sent onto the circle  $|w| = r$  and the circle  $|z| = 1$  sent onto the circle  $|w| = R$ , then  $R \geq 1$ . Furthermore, the case  $R = 1$  holds if and only if the conformal mapping is a rotation.*

Theorem 1 implies in particular that if a circular annulus with circular slits and with boundary circles  $|z| = r$  and  $|z| = 1$  is mapped conformally and bijectively onto a domain in the complex plane that has the same circles  $|z| = r$  and  $|z| = 1$  as boundary components, then the second domain is also an annulus with circular slits and the conformal map is a rotation.

In the next two theorems, a domain  $\mathcal{K}_r$  or  $\overline{\mathcal{K}_r}$  is mapped conformally and bijectively by a mapping  $w = f(z)$  onto a domain called  $\mathcal{B}_r$  (respectively  $\overline{\mathcal{B}_r}$ ) of the extended complex plane  $\mathbb{C} \cup \{\infty\}$  not containing  $\infty$  in its interior, such that the

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<sup>1</sup>Courant and Hurwitz, in their book *Funktionentheorie* (Springer, Berlin, 1922), used the length-area method in their proofs of results of Carathéodory on the boundary values of the Riemann mapping, which the latter has published in then three papers [9, 10, 11].

circle  $|z| = r$  is sent to the circle  $|w| = r$  and such that the image of  $\mathcal{K}_r$  (respectively  $\overline{\mathcal{K}_r}$ ) is contained in the subset  $|w| \geq r$  of  $\mathbb{C} \cup \{\infty\}$ .

The image by  $w$  of the circle  $|z| = 1$  is called the *outer boundary* of  $\mathcal{K}_r$  (respectively  $\overline{\mathcal{K}_r}$ ).

Let  $d$  be the shortest distance from a point on the outer boundary of  $\mathcal{B}_r$  (respectively  $\overline{\mathcal{B}_r}$ ) to the point  $w = 0$ .

**Theorem 2** ([12] §3).— *Consider a conformal mapping of  $\mathcal{K}_r$  sending  $|z| = r$  to  $|w| = r$  and  $|z| = 1$  to a ray whose extension contains the point  $w = 0$  and joining a point  $q$  to  $\infty$ . Then,  $d \geq |q|$ .*

In Theorem 2, the domain represented in Figure 3 appears as an extremal domain. This theorem is also known under the name *Grötzsch module theorem*, and it is usually stated under a form which uses the modulus of an annulus instead of the distance.

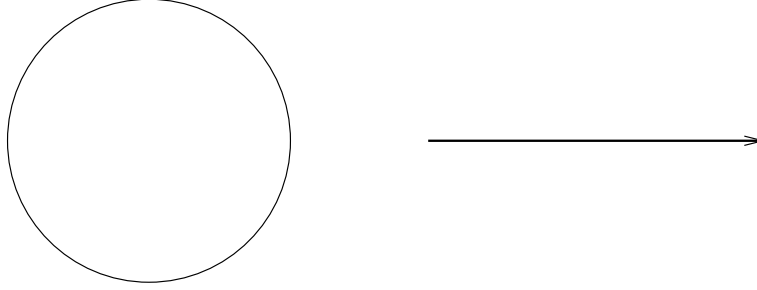


FIGURE 3. An annulus with circular boundary with the other boundary being an infinite ray. This extremal domain appears in the second theorem

In the next theorem, we use the above notation  $\mathcal{B}_r$  and  $\overline{\mathcal{B}_r}$ .

**Theorem 3** ([12] §3).— *Assume there exist  $n$  points on the outer boundary of the image domain  $\mathcal{B}_r$  (respectively  $\overline{\mathcal{B}_r}$ ) realizing the distance of this outer boundary to the point  $w = 0$ , assume that these points are the vertices of a regular  $n$ -gon having  $w = 0$  as center and that there exists a positive constant  $M$  (possibly equal to  $+\infty$ ) such that  $|w(z)| \leq M$  for all  $z$ . Then a map that realizes the shortest distance of the outer boundary of the image domain to the point  $w = 0$  sends  $\mathcal{K}_r$  or  $\overline{\mathcal{K}_r}$  respectively to a domain bounded by  $|z| = r$ ,  $|z| = M$  and which has  $n$  radial slits whose extensions contain 0 and which join the points*

$$\alpha \cdot \rho_{r,M,n} \cdot e^{\frac{2\pi i k}{n}} \quad (r < \rho_{r,M,n} < M, \quad |\alpha| = 1, k = 0, 1, \dots, n-1)$$

*to points on the circle  $|w| = M$ . Furthermore, the mappings satisfying the stated properties are the unique ones that realize the required minimum property.*

The extremal region that appears in Theorem 3 is represented in Figure 4.

In the next theorem,  $\mathcal{K}_0$  and  $\overline{\mathcal{K}_0}$  denote respectively the unit disc in  $\mathbb{C}$  and the unit disc slit along finitely many circular arcs centered at the origin. When we map the domain  $\mathcal{K}_0$  or  $\overline{\mathcal{K}_0}$  by a conformal map  $f$ , we shall say, in analogy with the previous cases, that the image by  $w$  of the circle  $|z| = 1$  is the *outer boundary* of the image  $f(\mathcal{K}_0)$  or  $f(\overline{\mathcal{K}_0})$ .

**Theorem 4** ([12] §4).— *Suppose that  $\mathcal{K}_0$  (respectively  $\overline{\mathcal{K}_0}$ ) is mapped conformally and bijectively by  $w = f(z)$  onto a domain  $\mathcal{B}_0$  (respectively  $\overline{\mathcal{B}_0}$ ) not containing  $\infty$  in its interior satisfying  $f(0) = 0$  and  $|f'(0)| = 1$ . Suppose furthermore that there*

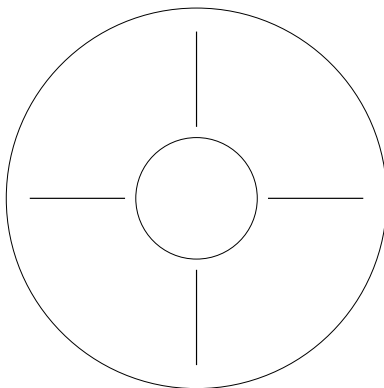


FIGURE 4. An extremal domain for Theorem 3, with  $k = 4$ . The inner slits admit a rotational symmetry

exist  $n$  points on the outer boundary of  $\mathcal{B}_0$  (respectively  $\overline{\mathcal{B}_0}$ ) that are the vertices of a regular  $n$ -gon centered at 0 and whose distance to the point  $w = 0$  is the shortest distance from a point on the outer boundary to that point. Then

$$d \geq \sqrt[n]{\frac{1}{4}},$$

and this extremal value is attained by the mapping  $f(z) = \frac{z}{\sqrt[n]{(1+z^n)^2}}$ . This mapping sends  $\mathcal{B}_0$  to the  $w$ -plane slit along the  $n$  rays emanating from the points  $\sqrt[n]{\frac{1}{4}} \cdot e^{\frac{2\pi i k}{n}}$  ( $k = 0, 1, 2, \dots, n-1$ ) to  $\infty$  (Figure 5.)

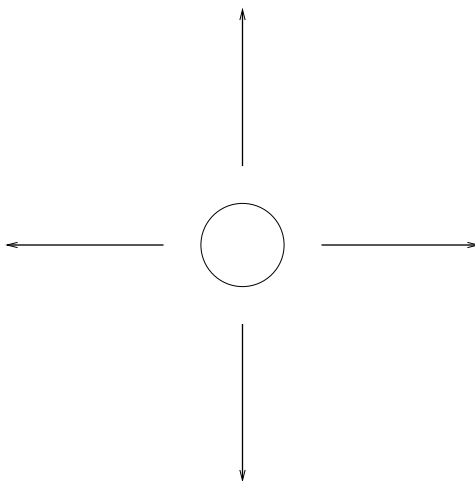


FIGURE 5. An extremal domain for Theorem 4. The inner slits admit a rotational symmetry

**Remarks 4.1.** 1.— An extremal result similar to the one that appears in Theorem 2 was obtained by Lavrentieff in his paper [21] (1934). Lavrentieff attributes the result to Grötzsch, and he uses completely different methods (analysis instead of geometry).

2.— Extremal problems similar to those studied by Grötzsch that we mentioned in this section are analyzed by Teichmüller in his paper [26] (1938), in particular §2.2 to 2.3. Teichmüller attributes the results he obtains to Grötzsch.

3.—

## 5. TWO THEOREMS FROM THE PAPER [13]

The paper (*On Extremal Problems for conformal mappings, II*) [13] by Grötzsch is a sequel to the paper [12] and is concerned with the same subject. The proofs are also based on the lemmas proved in the first paper.

The notation is as follows (we use Grötzsch's notation so that the reader can easily compare with the paper):

The unit disc in the complex plane is denoted by  $\mathfrak{R}_0$ .

For  $r > 0$ ,  $\mathfrak{R}_r$  is the subset of the complex plane defined by  $r \leq |z| < 1$ .

A conformal mapping  $w = f(z)$  defined on  $\mathfrak{R}_0$  is said to be *normalized* if it satisfies  $f(0) = 0$  and  $|f'(0)| = 1$ , and if  $\infty$  is not in the image.

A conformal mapping  $w = f(z)$  defined on  $\mathfrak{R}_r$  is said to be *normalized* if the circle  $|z| = r$  is sent to itself and if  $\infty$  is not in the image.

Grötzsch establishes distortion theorems for the modules  $|f(z)|$  and  $|f'(z)|$  of normalized injective conformal mappings defined on  $\mathfrak{R}_r$  ( $r \geq 0$ ). The formulation of the result uses a conformal mapping  $E_r(z)$  defined as follows:

For  $r > 0$ ,  $E_r(z)$  is the conformal map that maps  $\mathfrak{R}_r$  onto a domain of the  $w$ -plane bounded by the circle  $|w| = r$  and a suitable slit contained in the positive real axis going to infinity and whose extension in the finite direction contains the point  $+r$  on the real line.

$E_r(z)$  is the conformal map that maps  $H_r$  onto the domain of the ....

Cette application est unique!

We set

$$m_r(\rho) = \min_{|z|=\rho} |E_r(z)|$$

and

$$M_r(\rho) = \sup_{|z|=\rho} |E_r(z)|.$$

**Theorem 5.**— We have, for all  $|z| = \rho$ :

$$m_r(\rho) \leq |f(z)| \leq M_r(\rho)$$

with the two inequalities being equalities holding if and only if  $f(z) = \alpha E_r(\beta z)$  with  $|\alpha| = |\beta| = 1$ .

For  $\mathfrak{R}_0$ , a similar statement holds, and in this case an explicit formula for the mapping  $E_0$  and the extremal values  $m_0(\rho)$  and  $M_0(\rho)$  are given. In fact, the resulting map is the so-called *Koebe map*, and the values of  $m_0(\rho)$  and  $M_0(\rho)$  are those that are given by Koebe's distortion map.

In the next theorem, Grötzsch gives estimates on the derivative  $|f'(z)|$ . Here, the notation is as follows:

$$m'_r(\rho) = \min_{|z|=\rho} |E'_r(z)|$$

and

$$M'_r(\rho) = \sup_{|z|=\rho} |E'_r(z)|.$$

The result is then the following:

**Theorem 6.**— We have

$$m'_r(\rho) \leq |f'(z)| \leq M'_r(\rho)$$

for  $|z| = \rho$ , with equality holding only for the functions  $\alpha E_r(\beta z)$  as above.

In the case of  $\mathfrak{R}_0$ , explicit formulae for  $m'_r(\rho)$  and  $M'_r(\rho)$  are given.

These results generalize Koebe's distortion theorems.

## 6. ON THE CONTENT OF GRÖTZSCH'S PAPER [14]

The paper [14] is the first which contains Grötzsch's notion of quasiconformal mappings. This is a mapping with bounded infinitesimal distortion  $Q$ . He introduces such a class of mappings in the first section of the paper and he denotes them by  $\mathfrak{A}_Q$ . As a matter of fact, the term *nichtkonformen* (which we translate by non-conformal) is only used in the title. It was Ahlfors, in [2] who used for the first time the term *quasikonform* ("quasiconformal"); cf. Ahlfors' comments at the beginning of Volume 1 of his *Collected Papers* [5].

For  $Q \geq 1$ , Grötzsch defines in [14] a mapping in the class  $\mathfrak{A}_Q$  to be a bijective mapping of a domain  $G$  of the  $z$ -plane onto a surface which covers the Riemann sphere or a domain in it (Grötzsch uses the term "spread as a Riemann surface over a  $w$ -plane," which is close to the terminology used by Riemann) which satisfies the following three properties:

- (1)  $f$  is continuous on  $G$ ;
- (2) up to a set with at most countably many interior points (called exceptional points) of  $G$ , the mapping  $f$  is a local diffeomorphism which sends an infinitesimal circle to an infinitesimal ellipse whose ratio of minor to major axis is between  $\frac{1}{Q}$  and  $Q$ ;
- (3) In the neighborhood of each exceptional point, the map  $f$  is a finite branched covering.

Such a mapping from a domain of the Riemann sphere onto another domain of the Riemann sphere is not necessarily bijective, but it can be lifted to a bijective map from a domain of the Riemann sphere onto a Riemann surface which is a ramified covering of the sphere. Furthermore, the mapping is not assumed to be sense-preserving. Lehto and Virtanen call these "nicht-conformal maps" of [14] "regular quasiconformal" (see [23, p.17]).

In the second section of the paper, Grötzsch establishes the property known today as the *geometric definition* of quasiconformality, saying that the image of a quadrilateral of module  $M$  by a bijective mapping in  $\mathfrak{A}_Q$  is a quadrilateral whose module  $M'$  satisfies the inequality

$$(1) \quad \frac{1}{Q}M \leq M' \leq QM.$$

The proof that Grötzsch gives of this property is based on the length-area method, which he already used in the papers [12] and [13].

After setting the inequality (1), Grötzsch writes that in the case of rectangles, equality in (1) holds "only for certain immediately determinable extremal affine mappings." This result is now called the *solution of the Grötzsch Problem*; cf. Ahlfors' book [4, p. 8]). The solution was given by Grötzsch in the article [16] which we review below.

Right after the relation (1), Grötzsch deduces the corresponding inequalities for annuli. More precisely, using the logarithm function and then the exponential function, he obtains the fact that if there exists a bijective mapping in the class  $\mathfrak{A}_Q$  from the annulus with inner radius  $r$  and outer radius 1 onto the annulus with inner radius  $\tilde{r}$  and outer radius 1, then

$$(2) \quad r^Q \leq \tilde{r} \leq r^{\frac{1}{Q}}.$$

As in the case of rectangles, Grötzsch says that equalities hold only for certain simple extremal maps which consist of rotations and reflections. By using logarithms, we can deduce from the double inequality (2) the geometrical definition of quasiconformal mappings for doubly-connected domains.

In the same section of the paper, Grötzsch notes that a result of Carathéodory is still valid for bijective  $\mathfrak{A}_Q$  mappings, namely, any bijective  $\mathfrak{A}_Q$  mapping from a simply-connected domain bounded by a Jordan curve onto a domain of the plane can be extended continuously to a homeomorphism.

To conclude this section, Grötzsch states two distortion inequalities that hold for an arbitrary mapping  $f$  in the class  $\mathfrak{A}_Q$  defined on the annulus  $0 \leq r < |z| < 1$ , sending the circle of radius  $r$  onto the circle of radius  $r$  and such that for any  $r \leq |z| < 1$ , we have  $|f(z)| \geq r$ . The first inequality concerns the behaviour of the distance between the circle of radius  $r$  and a point in the image by  $f$  of any circle of radius  $\rho$  with  $r < \rho < 1$ . The second inequality concerns the behaviour of the rate of change of  $f$ .

In the last section, Grötzsch gives an extension of the (big) Picard theorem for mappings in the class  $\mathfrak{A}_Q$ . More precisely, he proves that for a given  $r$ , any  $\mathfrak{A}_Q$  mapping defined on the punctured disc  $0 < |z| < r$  for which 0 is an essential singularity and which omits at most two points in the plane (or three points in the Riemann sphere) is constant. (An essential singularity is a point at which the function does not admit any finite or infinite limit.)

The idea of Grötzsch's proof is quite elementary. Indeed, he takes an  $\mathfrak{A}_Q$  mapping  $f$  defined on the punctured disc  $0 < |z| < r$ , extending continuously to the set  $|z| = r$  with an essential singularity at 0 and which omits three values in the Riemann sphere, that we can assume to be 0, 1 and  $\infty$ . By definition of an  $\mathfrak{A}_Q$  map, we can lift  $f$  to a bijective map  $\tilde{f}$  from  $0 < |z| < r$  onto a Riemann surface  $\mathcal{S}_f$  equipped with a holomorphic map onto a subset of the Riemann sphere such that  $f = \pi \circ \tilde{f}$ . Since  $\mathcal{S}_f$  is doubly-connected, it is biholomorphic by a mapping  $\varphi$  to an annulus  $a < |z| < b$ . Since 0 is an essential singularity, we have  $a > 0$ , because otherwise the map  $\pi \circ \varphi^{-1}$  would be a non-constant holomorphic map from  $0 < |z| < b$  to  $\mathbb{C} \setminus \{0, 1\}$  with an essential singularity at 0, which is impossible by the big Picard theorem (for meromorphic functions). Thus, we have an  $\mathfrak{A}_Q$  mapping  $\varphi \circ \tilde{f}$  from  $0 < |z| < r$  onto  $a < |z| < b$ , which is impossible because of Relation (2) and the superadditivity for the module of an annulus.

In conclusion, Grötzsch in this short paper gives a certain number of results which show that conformal and quasiconformal mappings share several properties.

## 7. SOME COMMENTS ON GRÖTZSCH'S PAPER [14]

1.– Picard's big theorem says that if a holomorphic function has an essential singularity, then in any punctured neighborhood of this essential singularity, the function, takes infinitely often all possible values in the Riemann sphere, with at most two exceptions. Even though the statement that refers to Picard in Grötzsch's paper is not an exact analogue of that theorem, what is proved gives easily such a result. Indeed, if the quasiconformal mapping omits at most two points in the plane, we can lift it to the associated Riemann surface which is necessarily a punctured disc (since otherwise we would have a quasiconformal mapping onto an annulus), and thus, we get a holomorphic function with one essential singularity. Therefore (by Picard's theorem) the quasiconformal mapping takes all values infinitely often with the exception of at most two.

2.– The idea of a class of functions which generalize conformal mappings is also the subject of Lavrentieff’s paper [22] in which this author gives another proof of the big Picard theorem for such maps.

3.– The use of superadditivity in §2 of Grötzsch’s paper, along with the double inequality (2), was used by Teichmüller in [26] in order to prove that quasiconformal mappings preserve the type of a simply connected Riemann surfaces.

4.– Grötzsch continued to investigate similarities between conformal and quasiconformal mappings in his paper [15] which we review next, by solving geometrical extremal problems for quasiconformal mappings, after he studied such problems in the setting of conformal mappings in the papers [12, 13].

## 8. THE PAPER [15]

Grötzsch’s paper *Über die Verzerrung bei nichtkonformen schlichten Abbildungen mehrfach zusammenhängender schlichter Bereiche* [15] published in 1930 is a sequel to the two papers [12] and [13]. Here, instead of working with conformal representations, Grötzsch considers representations of multi-connected domains by quasiconformal mappings, which he calls “nichtkonformen”. The terminology is close to the one he uses in his paper [gr3]. Here, Grötzsch means by this word a mapping defined on a region  $B$  of the  $z$ -plane which can be approximated uniformly, in the neighborhood of each point except for finitely many points in  $B$ —called exceptional points—by an affine mapping, and such that in the neighborhood of each non-exceptional point the ratio  $a/b$  of the great axis to the small axis of the infinitesimal ellipse which is the image of an infinitesimal circle is uniformly bounded by two constants:

$$\frac{1}{Q} \leq a/b \leq Q$$

where  $Q$  is independent of the choice of the non-exceptional point.

As we did for the mappings introduced by Grötzsch in his paper [14], we shall use here the terminology  $\mathfrak{A}_Q$  mapping for such a mapping.

Grötzsch’s results in this paper are based again on the length-area method, which he used extensively in his previous papers in the setting of conformal mappings, together with the two double inequalities (1) and (2) which he proved in his paper [14]. The method, as in the case of conformal mappings, consists in taking appropriate sequences of surface strips.

Grötzsch’s goal is to determine, under some normalization conditions, the mappings in the class  $\mathfrak{A}_Q$  that realize extremal values to certain distortion quantities. He obtains six theorems which we state below. Before stating them, we introduce some notation (which is Grötzsch’s notation).

$B_{n+1}$  is an  $(n+1)$ -connected open region of the complex plane. It is said to be *normalized* if the circle  $|z| = 1$  is one of its boundary components and if it is contained in the interior of that circle.

All the regions  $B_{n+1}$  that we consider are normalized. We say that  $|z| = 1$  is the *outer boundary* of  $B_{n+1}$ , and we let  $R_1, \dots, R_n$  be its remaining boundary components (they may be reduced to points).

Grötzsch gives the solutions of eight extremal theorems. The question, in each case, is to find a mapping in the class  $\mathfrak{A}_Q$  of normalized mappings for which some geometrically defined quantity is extremal.

For each of these problems, any  $\mathfrak{A}_Q$  mapping which is a solution turns out to have constant dilatation  $Q$ .

In the first 4 theorems, a *normalized* mapping  $\mathfrak{A}_Q$  of  $B_{n+1}$  is a bijective mapping from  $B_{n+1}$  onto a normalized region  $\tilde{B}_{n+1}$  in the complex plane sending the outer

boundary  $|z| = 1$  of  $B_{n+1}$  to the outer boundary  $|w| = 1$  of  $\tilde{B}_{n+1}$ . We let  $\tilde{R}_1, \dots, \tilde{R}_n$  be the images of the boundary components  $R_1, \dots, R_n$ .

In the first theorem, the quantity that is maximized is the area of the domain enclosed by  $\tilde{R}_k$ .

**Theorem 7 (Theorem 1 of [15]).**— In the set  $\mathfrak{A}_Q$  of normalized mappings of  $B_{n+1}$ , the area of the domain enclosed by  $\tilde{R}_k$  attains its maximum if and only if

- (1)  $\tilde{R}_k$  is a circle centered at  $w = 0$ ;
- (2) the other  $\tilde{R}_p$  (if they exist) are circular arc slits centered at  $w = 0$ ;
- (3) at each point of the image, the minor axis of all the distortion ellipse (which Grötzsch calls the Tissot indicatrix) is directed radially toward the origin  $w = 0$ ;
- (4) the distortion is everywhere constant and equal to  $Q$ .

Furthermore, any two such extremal mappings differ from each other by at most a rotation around  $w = 0$  and a reflection in a straight line through  $w = 0$ .

The proof of this theorem, like the one of the theorems that follow, is first given in the case of conformal mappings ( $Q = 1$ ), and then for  $\mathfrak{A}_Q$  mappings, using the two distortion results for quadrilaterals and annuli that we recalled.

The remaining seven theorems are of the same sort. Instead of restating them, we indicate, for each one, the quantities that are minimized. In each case, Grötzsch gives a description of the extremal image domain.

**Theorem 8 (Theorem 2 of [15]).**— In the set  $\mathfrak{A}_Q$  of normalized mappings of  $B_{n+1}$ , to find the mappings for which the maximum value of the diameter of  $\tilde{R}_k$  is attained.

In this case, the extremal image domain has a rectilinear slit passing through the origin  $w = 0$  and bisected by that point.

**Theorem 9 (Theorem 3 of [15]).**— In the set  $\mathfrak{A}_Q$  of normalized mappings  $w = f(z)$  of  $B_{n+1}$  and for a given pair  $(z_1, z_2)$  in  $B_{n+1}$ , to find the mappings for which the maximum of the quantity  $|f(z_1) - f(z_2)|$  is attained.

In this case, in the extremal image domain, the images  $f(z_1)$  and  $f(z_2)$  are the endpoints of a rectilinear segment bisected by the origin  $w = 0$ .

**Theorem 10 (Theorem 4 of [15]).**— Assume that  $B_{n+1}$  has  $z = 0$  as interior point. In the set  $\mathfrak{A}_Q$  of normalized mappings  $w = f(z)$  of  $B_{n+1}$  that satisfy  $f(0) = 0$ , and for a given  $z_1$  in  $B_{n+1}$ , to find the mappings for which the maximum of the quantity  $|f(z_1)|$  is attained.

In the remaining 4 theorems, a different normalization of the mappings in  $\mathfrak{A}_Q$  is used. Here, an  $(n + 1)$ -connected region  $B_{n+1}$  is said to be normalized if it is bounded (“from inside”) by a circle  $|z| = r$  and lies entirely outside  $|z| = r$ , and which is bounded (“from outside”) by a linear boundary  $L$ , either separating  $B_{n+1}$  from  $\infty$  (which means the curve is closed) or heading towards to  $\infty$ , and by  $n - 1$  other boundaries  $R_k$  lying between  $|z| = r$  and  $L$ .

Here, a mapping  $\mathfrak{A}_Q$  of  $B_{n+1}$  is called normalized if the image region  $\tilde{B}_{n+1}$  of the  $w$ -plane is also normalized, if the circle  $|z| = r$  is sent to  $|w| = r$ , and if  $L$  is sent to the linear boundary  $\tilde{L}$  joining  $\tilde{B}_{n+1}$  to  $\infty$ . (In particular,  $\infty$  is not an image point).

Let  $\tilde{R}_k$  be the boundary of  $B_{n+1}$  corresponding to  $R_k$ . The next three theorems are stated in the form of problems of which Grötzsch gives the solution.

**Theorem 11 (Theorem 5 of [15]).**— Among the normalized mappings of  $B_{n+1}$ , to find those  $\mathfrak{A}_Q$  mappings for which the distance of the boundary  $\tilde{L}$  from the point  $w = 0$  attains its minimum value.

**Theorem 12 (Theorem 6 of [15]).**— Among the normalized mappings of  $B_{n+1}$ , to find those  $\mathfrak{A}_Q$  mappings for which the distance of the boundary  $\tilde{L}$  from the point  $w = 0$  attains its maximum value.

**Theorem 13 (Theorem 7 of [15]).**— Among the normalized mappings  $w = f(z)$  of  $B_{n+1}$ , and for a given point  $z_1$  in  $B_{n+1}$ , to find those  $\mathfrak{A}_Q$  mappings for which the minimum of the quantity  $|f(z_1)|$  is attained.

**Theorem 14 (Theorem 8 of [15]).**— Among the normalized mappings  $w = f(z)$  of  $B_{n+1}$ , and for a given point  $z_1$  in  $B_{n+1}$ , to find those  $\mathfrak{A}_Q$  mappings for which the maximum of the quantity  $|f(z_1)|$  is attained.

## 9. THE PAPER [16]

The paper *Über möglichst konforme Abbildungen von schlichten Bereichen* [16] by Grötzsch was published in 1932 and is concerned with maps he calls *möglichst konforme* (closest-to-conformal). These are maps that we call today “extremal quasiconformal mappings”. Grötzsch starts by recalling that (by the Riemann Mapping Theorem) one can map conformally and in a one-to-one way an arbitrary simply-connected open subset  $B_1$  of the plane which is not the whole plane onto another such subset  $\tilde{B}_1$ , and that this map is completely determined if one asks that the image of three arbitrarily chosen distinct points on the boundary of  $B_1$  are sent to three arbitrarily chosen distinct points on the boundary of  $\tilde{B}_1$ . He then remarks that if one takes, on the boundary of each domain, four cyclically ordered points instead of three, then, generally speaking, one cannot find a conformal mapping between the two domains that send distinguished points to distinguished points. In this case, one looks for mappings whose deviation from conformality is the smallest. The question is then to introduce an appropriate measure for this deviation from conformality and to investigate the existence of such closest-to-conformal mappings.

The question is also relevant for domains of the plane which are not simply-connected. In the case of multiply-connected domains, the same question appears naturally, with or without distinguished points on the boundary. In the rest of the paper, Grötzsch gives several examples of closest-to-conformal mappings between domains of the plane, some of them simply-connected and others multiply-connected.

Right at the beginning of the investigation, Grötzsch says that one has to allow the maps to have singularities at isolated points in the interior, possibly countably many and converging to points on the boundary. At the non-singular points (which he calls “differential-geometric” and which we call “regular”), the maps are assumed to be differentiable, and the defect in conformality is measured by the ratios of the great axis to the small axis of infinitesimal ellipses which are images of infinitesimal circles, as in the previous papers, [14] and [15]. Likewise, the quantity  $Q \geq 1$  is defined to be the upper bound of such ratios, taken over all the regular points. The measure of deviation of the mapping is taken to be the quantity  $Q - 1$ . Grötzsch notes that this quantity is invariant by (pre- and post-) composition of the mapping with conformal mappings. He then remarks that in a neighborhood of a regular point, infinitesimal squares are sent to infinitesimal parallelograms, and he gives a result on the distortion of sufficiently small parallelograms in the form of a double inequality bounding the quotient of the area over the square of the length of one of the sides in terms of the upper bound  $Q$  of the mapping’s dilatation. More precisely, the result says the following (we use Grötzsch’s notation; formula (1) in his paper):

**Proposition.**— *For every  $\epsilon > 0$  and at every regular point, there exists a neighborhood of this point such that the image of an infinitesimal square is an infinitesimal*

parallelogram satisfying the following: if  $\tilde{s}$  is the length of any one of its sides, and  $\tilde{I}$  its area, then

$$(3) \quad (1 - \varepsilon) \cdot \frac{1}{Q} \cdot \tilde{s}^2 < \tilde{I} < (1 + \varepsilon) \cdot Q \cdot \tilde{s}^2,$$

where  $\varepsilon > 0$  can be arbitrarily small.

After this proposition, Grötzsch gives examples of closest-to-conformal mappings. He calls the problem of finding such mappings “the mapping problem,” a term that suggests the Riemann Mapping Theorem.

#### 10. EXAMPLES OF CLOSEST-TO-CONFORMAL MAPPINGS (EXTREMAL QUASICONFORMAL MAPPINGS)

**Example 1: Rectangles.** The mapping problem is considered between two Euclidean rectangles in the plane, where the distinguished points are the vertices. The rectangles may be assumed to have the same height, and their lengths are denoted by  $p$  and  $\tilde{p}$  respectively, with  $p \leq \tilde{p}$ . Grötzsch proves that we always have the inequality  $Q \geq \frac{\tilde{p}}{p}$ , with equality holding if and only if the map is affine, in which case the value of  $Q$  is equal, at every point, to the ratio of the two axes of the image infinitesimal ellipse.

In his proof, Grötzsch uses the *strip method*, a method that is a form of the length-area method. It is also based on inequalities that relate lengths of curves in a certain family and the area of the domain that the family occupies. For this method, he refers to his paper *Über die Verzerrung bei nichtkonformen schlichten Abbildungen mehrfach zusammenhängender schlichter Bereiche* [15] (Grötzsch calls it so in Example 3, where he uses it again).

**Example 2: Doubly connected domains.** Grötzsch remarks that any annulus can be mapped conformally onto a Euclidean annulus bounded by two circles. By using logarithms, the problem of finding closest-to-conformal mappings between Euclidean annuli is reduced to that of finding closest-to-conformal mappings between Euclidean rectangles, and hence to Example 1. The unique closest-to-conformal mapping is therefore, in this case too, the affine map. Grötzsch notes that in the case where the boundary component of one annulus (and not the other one) is reduced to a point, then we cannot find a closest-to-conformal mapping between the two (the quantity denoted by  $Q$ , that is, the upper bound of the major to the minor axes of infinitesimal ellipses is always infinite).

**Example 3: Simply-connected domains with two distinguished interior points.** Grötzsch considers the problem of finding closest-to-conformal mappings between two simply-connected domains with two distinguished points in the interior. Using the Riemann Mapping Theorem, one may assume that each domain is the open unit disc in the complex plane and that in each such disc, the distinguished points lie on the real axis and are symmetric with respect to the origin. By cutting each of the two discs along the segment joining the two distinguished points, we obtain annuli, and the mapping problem is reduced to that of Example 2. Grötzsch notes that the closest-to-conformal mapping obtained is  $C^1$  at the two distinguished points only in the case where it is conformal.

**Example 4: simply-connected domains with two distinguished points on the boundary and one distinguished point in the interior.** Grötzsch considers the problem of finding closest-to-conformal mappings between two simply-connected domains having two distinguished points on the boundary and one distinguished point in the interior. Like in the Example 3, by composing with a conformal mapping, each of the two domains can be assumed to be the unit disc

in  $\mathbb{C}$ , with the interior distinguished point being the center of the disc and the distinguished points on the boundary being the points  $e^{\pm i\alpha}$  and  $e^{\pm i\beta}$  in the first and the second disc respectively, where  $\alpha$  and  $\beta$  are real numbers satisfying  $0 < \alpha < \pi$  and  $0 < \beta < \pi$ . We may also assume that the mapping sought for (supposing it is orientation-preserving) sends the arc of circle whose extremities are the two points  $e^{\pm i\alpha}$  and containing the point  $-1$  to the arc of circle containing the two points  $e^{\pm i\beta}$  and containing the point  $-1$ . One then cuts the two domains along the segment joining  $0$  to  $-1$ , obtaining in each case a simply-connected domain with four distinguished points on the boundary. We are then reduced to finding the closest-to-conformal map between two simply-connected domains with four boundary points on the boundary, and we proceed as in Example 1. The closest-to-conformal map obtained between the two initial discs is  $C^1$  except at the origin, and it is  $C^1$  there only if it is conformal.

**Remark 10.1.** A few years after Grötzsch's papers appeared, Teichmüller considered the existence and uniqueness problems of closest-to-conformal mappings (which were already called extremal quasiconformal mappings) for arbitrary surfaces: compact or not, orientable or not, with an arbitrary number of distinguished interior or boundary points. See the paper [27] *Extremale quasikonforme Abbildungen und quadratische Differentiale* (1939) and the commentary in [7]. In 1944, he published a paper, *Ein Verschiebungssatz der quasikonformen Abbildung* (A displacement theorem for quasiconformal mapping) [29], in which he considers a new type of existence problem for extremal mappings, in which each point on the boundary is a distinguished point. In other words, he studies the question finding and describing the extremal quasiconformal mapping from the unit disc with the origin as a distinguished point to the unit disc with some distinguished point in the interior which is the identity on the boundary (that is, each point on the unit circle is also considered as a distinguished point). We refer the reader to the paper [6] where the developments of this problem are also discussed.

In the last section of the paper [16], called *Additional remark*, Grötzsch establishes an extension of Schwarz's lemma to the class of differentiable quasiconformal mappings, that is, to the mappings that belong to the class denoted by  $\mathfrak{A}_Q$  in the paper [14] discussed above. The extension is valid for branched coverings of the unit disc.

The proof of this extension involves Riemann surfaces which cover the sphere and it uses the classical Schwarz lemma for conformal mappings.

This result is in the trend of previous results of Grötzsch whose aim is the extension to the class of quasiconformal mappings of results that are known to hold for conformal mappings; see e.g. the extension of Picard's theorem in the paper [14] we reviewed above.

In a biographical note on Grötzsch published in the present volume [20], Reiner Kühnau writes: "It is strange that today many people associate the name Grötzsch only to the ridiculous 'Grötzsch-ring'." We hope that after publishing here the English translations of these five papers of Grötzsch, his work on the theory of quasiconformal mappings will be better known.

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