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To cite this version:

Bassam Fayad. Lyapunov unstable elliptic equilibria. 2019. hal-02414963

HAL Id: hal-02414963

https://hal.science/hal-02414963v1

Preprint submitted on 16 Dec 2019

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LYAPUNOV UNSTABLE ELLIPTIC EQUILIBRIA

BASSAM FAYAD

ABSTRACT. We introduce a new diffusion mechanism from the neighborhood of elliptic equilibria for Hamiltonian flows in three or more degrees of freedom. Using this mechanism, we obtain the first examples of real analytic Hamiltonians that have a Lyapunov unstable non-resonant elliptic equilibrium. In four or more degrees of freedom, we obtain examples of unstable elliptic equilibria with arbitrary chosen frequency vectors whose coordinates are not all of the same sign. Moreover, the Birkhoff normal form at the origin is divergent in all our examples.

In addition, it is possible to insure a transversality condition at the equilibria and the diffusion coexists therefore with the stability in a probabilistic sense (or KAM stability) of the equilibrium.

Introduction

An equilibrium $(p, q) \in \mathbb{R}^{2d}$ of an autonomous Hamiltonian flow is said to be Lyapunov stable or topologically stable if all nearby orbits remain close to 0 for all forward time.

The topological stability of the equilibria of Hamiltonian flows is one of the oldest problems in mathematical physics. The important contributions to the understanding of this problem, dating back to the 18th century, form a fundamental part of the foundation and of the evolution of the theory of dynamical systems and celestial mechanics up to our days.

The goal of this note is to give examples of real analytic Hamiltonians that have a Lyapunov unstable non-resonant elliptic equilibrium.

A C^2 function $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ such that $DH(0) = 0$ defines a Hamiltonian vector field $X_H(x, y) = (\partial_y H(x, y), -\partial_x H(x, y))$ whose flow ϕ_H^t preserves the origin.

Naturally, to study the stability of the equilibrium at the origin, one has first to investigate the stability of the linearized system at the origin. By symplectic symmetry, the eigenvalues of the linearized system come by pairs $\pm\lambda$, $\lambda \in \mathbb{C}$. It follows that if the linearized system has an eigenvalue with a non zero real part, it also has an eigenvalue with positive real part and this implies instability of the origin for the linearized system as well as for the non-linear flow.

When all the eigenvalues of the linearized system are on the imaginary axis the stability question is more intricate. In the non-degenerate case where the eigenvalues are simple, we say that the origin is an *elliptic* equilibrium. The linear system is then symplectically conjugated to a direct product of planar rotations. The arguments of the eigenvalues are called the frequencies of the equilibrium since they

correspond to angles of rotation of the linearized system. We focus our attention on Hamiltonians $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ of the form

$$(*) \qquad H(x,y) = H_\omega(x,y) + \mathcal{O}^3(x,y),$$
$$H_\omega(x,y) = \sum_{j=1}^{3} \omega_j I_j, \quad I_j = \frac{1}{2}(x_j^2 + y_j^2).$$

where $\omega \in \mathbb{R}^d$ has rationally independent coordinates. The elliptic equilibrium at the origin of the flow of X_H is then said to be *non-resonant*.

The phenomenon of averaging out of the non-integrable part of the nonlinearity effects at a non-resonant frequency is responsible for the long time effective stability around the equilibrium : the points near the equilibrium remain in its neighborhood during a time that is greater than any negative power of their distance to the equilibrium. This can be formally studied and proved using the Birkhoff Normal Forms (BNF) at the equilibrium, that introduce action-angle coordinates in which the system is integrable up to arbitrary high degree in its Taylor series (see Section 1 for some reminders about the BNF, and [Bi66] or [SM71], for example, for more details). Moreover, it was proven in [MG95, BFN15] that a typical elliptic fixed point is doubly exponentially stable in the sense that a neighboring point of the equilibrium remains close to it for an interval of time which is doubly exponentially large with respect to some power of the inverse of the distance to the equilibrium point.

In addition to the long time effective stability of non-resonant equilibria, KAM theory (after Kolmogorov Arnold and Moser), asserts that a non-resonant elliptic fixed point is in general accumulated by quasi-periodic invariant Lagrangian tori whose relative measurable density tends to one in small neighborhoods of the fixed point. This can be viewed as stability in a probabilistic sense, and is usually coined *KAM stability*. In classical KAM theory, KAM stability is established when the BNF has a non-degenerate Hessian. Further development of the theory allowed to relax the non degeneracy condition and [EFK13] proved KAM-stability of a non-resonant elliptic fixed point under the non-degeneracy condition of the BNF (see Section 1).

Despite the long time effective stability, and despite the genericity of KAM-stability, Arnold conjectured that apart from two cases, the case of a sign-definite quadratic part of the Hamiltonian, and generically for $d = 2$, an elliptic equilibrium point of a generic real analytic Hamiltonian system is Lyapunov unstable [Arn94, Section 1.8].

Although a rich literature in the direction of proving this conjecture exist in the C^∞ smoothness (we mention [KMV04] below, but to give a list of contributions would exceed the scope of this introduction), the conjecture is still wide open in the real analytic category. For instance, not a single example of real analytic Hamiltonians was known that has an unstable non-resonant elliptic equilibrium.

Our goal is to give the first such examples in three or more degrees of freedom. The question in two degrees of freedom remains open.

THEOREM A. – *There exists a non-resonant $\omega \in \mathbb{R}^3$ and a real entire Hamiltonian $H : \mathbb{R}^6 \to \mathbb{R}$, such that the origin is a Lyapunov unstable elliptic equilibrium with frequency ω of the Hamiltonian flow Φ_H^t of H.*

For any $\omega \in \mathbb{R}^d, d \geq 4$, such that not all its coordinates are of the same sign, there exists a real entire Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$ such that the origin is a Lyapunov unstable elliptic equilibrium with frequency ω of the Hamiltonian flow Φ_H^t of H.

Moreover, in all our examples, for non-resonant frequencies ω, the Birkhoff normal form at the origin is divergent.

Finally, for non-resonant frequencies ω, it is possible to choose the Hamiltonians H such that the origin is KAM stable.

As explained earlier, Lyapunov instability of an elliptic fixed point of frequency vector ω is only possible when not all the coordinates of ω are of the same sign. However, if one only seeks the divergence of the Birkhoff normal form, then it is possible to modify the construction to have examples for any non-resonant frequency in $\mathbb{R}^d$, $d \geq 4$.

THEOREM B. – *For any non-resonant $\omega \in \mathbb{R}^d, d \geq 4$, there exists a real entire Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$ such that the origin is an elliptic equilibrium with frequency ω of the Hamiltonian flow Φ_H^t, and such that the Birkhoff normal form at the origin is divergent.*

Note that we do not obtain the existence of an orbit that accumulates on the origin. Based on a different diffusion mechanism, [FMS17] gives examples of smooth symplectic diffeomorphisms of $\mathbb{R}^6$ having a non-resonant elliptic fixed point that attracts an orbit.

Detailed statements with an explicit definition of the Hamiltonians that prove Theorem A, will be given in Section 2. We explain in Section 2.3, the required modification to prove Theorem B.

Note that, due to the result of Perez-Marco of [PM03], the existence for any non-resonant $\omega \in \mathbb{R}^d$ of just one example of a real analytic Hamiltonian with divergent BNF, implies that divergence of the BNF is typical for this frequency. Denote by $\mathcal{H}_\omega$, the set of analytic Hamiltonians having an elliptic fixed point of frequency ω at the origin. As a consequence of Theorem B and of [PM03, Theorem 1] we get

COROLLARY. – *For any non-resonant $\omega \in \mathbb{R}^d, d \geq 4$, the generic Hamiltonian in $\mathcal{H}_\omega$ has a divergent BNF at the origin.*

More precisely, all Hamiltonians in any complex (resp. real) affine finite-dimensional subspace V of $\mathcal{H}_\omega$ have a divergent BNF except for an exceptional pluripolar set.

This answers, for any frequency vector in $\mathbb{R}^d, d \geq 4$, the question of Eliasson on the typical behavior of the BNF (see for example [E89, E90, EFK15] and the discussion around this question in [PM03]). What was known up to now, was the generic divergence of the normalization, proved by Siegel in 1954 [Si54] in $\mathcal{H}_\omega$ for any fixed ω.

Examples of analytic Hamiltonians with non-resonant elliptic fixed points and divergent BNF were constructed by Gong [Go12] on $\mathbb{R}^{2d}$ for arbitrary $d \geq 2$, but only for some class of Liouville frequency vectors.

The generic divergence of the BNF was announced by Krikorian for symplectomorphisms of the plane with a Diophantine elliptic fixed point at the origin. The method of Krikorian is completely different from ours and does not rely on the dichotomy proved by Perez-Marco. He proves more than just the generic divergence of the BNF. Indeed, he proves that the convergence of the BNF, combined with torsion (a generic condition), implies the existence of a larger measure set of

invariant curves in small neighborhoods of the origin than what actually holds for a generic symplectomorphism.

In contrast, our proof does not yield any result in dimension lower than 6 for flows. In dimension 6, the generic divergence of the BNF will hold generically in $\mathcal{H}_\omega$, but only for super Liouville frequencies ω such as the ones we will use in our construction of Lyapunov unstable fixed points (see condition ($\mathcal{L}$) of Section 2, although a less restrictive Liouville condition is sufficient for the divergence of the BNF, as seen in the last line of the proof of Theorem 2).

In the C^∞ category, examples of unstable elliptic equilibria can be obtained *via* the successive conjugation method, the Anosov-Katok method. They can be obtained in two degrees of freedom or for $\mathbb{R}^2$ symplectomorphisms, provided the frequency at the elliptic equilibrium is not Diophantine ([AK66, FS05, FS17]). In three or more degrees of freedom, smooth examples with Diophantine frequencies can be obtained through a more sophisticated version of the successive conjugation method (see [EFK15, FS17]). The Anosov-Katok examples are infinitely tangent to the rotation of frequency ω at the fixed point and as such are very different in nature from our construction. In particular, KAM stability is in general excluded in these constructions.

Again in the C^∞ class but in the non-degenerate case, R. Douady gave examples in [Dou88] of Lyapunov unstable elliptic points for symplectic diffeomorphisms on $\mathbb{R}^{2n}$ for any $n \geq 2$. Douady's examples can have any chosen Birkhoff Normal Form at the origin provided its Hessian at the fixed point is non-degenerate. Douady's examples are modeled on the Arnold diffusion mechanism through chains of heteroclinic intersections between lower dimensional partially hyperbolic invariant tori that accumulate toward the origin. The construction consists of a countable number of compactly supported perturbations of a completely integrable flow, and as such was carried out only in the C^∞ category.

In [KMV04], the authors admit Mather's proof of Arnold diffusion for a cusp residual set of nearly integrable convex Hamiltonian systems in 2.5 degrees of freedom, and deduce from it that generically, a convex resonant totally elliptic point of a symplectic map in 4 dimensions is Lyapunov unstable, and in fact has orbits that converge to the fixed point.

A third diffusion mechanism, closely related to Arnold diffusion mechanism, is Herman's synchronized diffusion, and is due to Herman, Marco and Sauzin [MS02]. It is based on the following coupling of two twist maps of the annulus (the second one being integrable with linear twist): at exactly one point p of a well chosen periodic orbit of period q on the first twist map, the coupling consists of pushing the orbits in the second annulus up on some fixed vertical Δ by an amount that sends an invariant curve whose rotation number is a multiple of $1/q$ to another one having the same property. The dynamics of the coupled maps on the line $\{p\} \times \Delta$ will thus drift at a linear speed.

The diffusion mechanism that underlies our constructions is inspired by all these three mechanisms described above but is quite different from each. In 3 degrees of freedom, we start with a product of rotators of frequencies $\omega_1, \omega_2, \omega_3$, where $\omega_1\omega_2 < 0$ and then couple this integrable Hamiltonian with a resonant diffusive Hamiltonian acting in the 4-dimensional space (x_1, y_1, x_2, y_2) and that almost commutes with the rotators, provided $\bar{\omega} = (\omega_1, \omega_2)$ is very well approached by

resonant vectors. We use the third action, $I_3 = x_3^2 + y_3^2$, that is invariant by the whole flow as a coupling parameter.

If only one coupling term is added in this way to the original Hamiltonian, there appears an orbit that diffuse from a small (but not arbitrarily small) neighborhood of the origin towards infinity.

To get diffusion from arbitrary small neighborhoods of the origin, one has to add successive couplings that commute with increasingly better resonant approximations of $\bar{\omega}$. The use of the third action as a coupling parameter, allows to isolate the effect of each successive coupling from the other ones. Indeed, to isolate the effect of each individual coupling from all the successive couplings is easy because these terms can be chosen to be extremely small compared to it. On the other hand, if we look at adequately small values of the third action, the effect of the prior coupling terms is tamed out due to Birkhoff averaging (we refer to Section 4.1 for a more precise description of the diffusion mechanism).

In the case of 4 degrees of freedom (or more) we can take the frequency vector of the equilibrium to be arbitrary, provided all the coordinates are not of the same sign, assuming for definiteness $\omega_1\omega_2 < 0$. Following an idea introduced in [EFK15] (see also [FS17]) we can use the action of the fourth degree of freedom as a parameter, that merely changes the frequency ω_1 into $\omega_1 + I_4$. For a sequence $I_{4,n} \to 0$ the vector $(\omega_1 + I_{4,n}, \omega_2)$ will be resonant which allows to adopt the three degrees of freedom diffusion strategy.

1. Birkhoff Normal Forms.

For H as in $(*)$, ω non-resonant, for all $N \geq 1$, there exists an exact symplectic transformation $\Phi_N = \mathrm{Id} + O^2(x, y)$, and a polynomial B_N of degree N in the variables $I_1, \ldots, I_d$, such that

$$H \circ \Phi_N(x, y) = B_N(I) + O_{2N+1}(x, y).$$

We say that $f \in O^l(x, y)$ when $\partial_z f(0) = 0$ for any multi-index z on the x_i and y_i of size less or equal to l.

There also exists a formal exact symplectic transformation $\Phi_\infty = \mathrm{Id} + O^2(x, y)$, i.e. defined *via* a generating function that is a formal power series in x and y such that

$$H \circ \Phi_\infty(x, y) = B_\infty(I)$$

where B_∞ is a uniquely defined formal power series of the action variables I_j, called the Birkhoff Normal Form (BNF) at the origin.

For more details on the Birkhoff Normal Form at a Diophantine, and more generally at any non-resonant elliptic equilibrium, one can consult for example [SM71].

Divergent Birkhoff Normal Forms. When the radius of convergence of the formal power series $B_\infty(\cdot)$ is 0, we say that the BNF diverges.

Non-degenrate Birkhoff Normal Forms. Following [Rüs01], we say that B_∞ is *Rüssmann non-degenerate* or simply *non-degenerate* if there does not exist any vector γ such that for every I in some neighborhood of 0

$$\langle \nabla B_\infty(I), \gamma \rangle = 0.$$

In [EFK15] the following was proven

THEOREM 1 ([EFK15]). – *Let $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ be a real analytic function of the form $(*)$ and assume that ω is non-resonant. If the BNF of H at the origin is non-degenerate, then in any neighborhood of $0 \in \mathbb{R}^{2d}$ the set of real analytic KAM-tori for X_H is of positive Lebesgue measure and density one at 0.*

Real analytic KAM-tori are invariant Lagrangian tori on which the flow generated by H is real analytically conjugated to a minimal translation flow on the torus $\mathbb{R}^d/\mathbb{Z}^d$.

Calculating the BNF : Resonant and non-resonant terms.

To simplify the computations, we prefer to use the complex variables

$$\xi_j = \frac{1}{\sqrt{2}}(x_j + iy_j), \quad \eta_j = \frac{1}{\sqrt{2}}(x_j - iy_j)$$

Notice in particular that the actions become $I_j = \xi_j \eta_j$, making it very simple to detect the monomials $\xi_{u_1} \dots \xi_{u_k} \eta_{v_1} \dots \eta_{v_{k'}}$ that only depend on the actions, since these are exactly the *resonant* monomials for which $k = k'$ and $\{u_1, \dots, u_k\} = \{v_1, \dots, v_k\}$.

Note also that in these variables H_ω as in $(*)$ reads as $\sum \omega_j \xi_j \eta_j$. We easily verify that, in these variables, the Poisson bracket is given by

$$\{F, G\} = i \sum_j \frac{\partial F}{\partial \xi_j} \frac{\partial G}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j} \frac{\partial G}{\partial \xi_j},$$

while the Hamiltonian equations are given by

$$\begin{cases} \dot{\xi}_j = -i\partial_{\eta_j} H(\xi, \eta) \\ \dot{\eta}_j = i\partial_{\xi_j} H(\xi, \eta) \end{cases} \tag{1}$$

We will say that a function F defined in the variables ξ and η is real when $F(\xi, \bar{\xi})$ is real, which means that in the original variables (x, y), F is real valued.

Finally observe that for a Hamiltonian as in $(*)$, and since ω is non-resonant, it is easy to eliminate a non-resonant term $\xi_{u_1} \dots \xi_{u_k} \eta_{v_1} \dots \eta_{v_{k'}}$ by conjugacy. Indeed, if we take

$$\chi = i \frac{1}{\omega_{u_1} + \ldots + \omega_{u_k} - \omega_{v_1} - \ldots - \omega_{v_{k'}}} \xi_{u_1} \dots \xi_{u_k} \eta_{v_1} \dots \eta_{v_{k'}}$$

we get for the time one map Φ^1_χ of the Hamiltonian flow of χ, also called the Lie transform associated to χ,

$$H \circ \Phi^1_\chi = H + \{H_\omega, \chi\} + \{H - H_\omega, \chi\} + \frac{1}{2!}\{\{H, \chi\}, \chi\} + \dots$$

and observe that $\{H_\omega, \chi\} = -\xi_{u_1} \dots \xi_{u_k} \eta_{v_1} \dots \eta_{v_{k'}}$, while the other terms that appear due to the composition by Φ^1_χ are of higher degree than $k + k'$. The reduction to the BNF is thus done progressively by eliminating non-resonant monomials of higher and higher degree.

2. Explicit Hamiltonians with Lyapunov unstable elliptic equilibria

In this section, we give the explicit constructions that yield Theorem A.

Starting from 4 degrees of freedom and above it is possible to give examples with arbitrary frequency vectors, in particular Diophantine. Recall that ω is said to be Diophantine if there exists $\gamma, \tau > 0$ such that $|\langle k, \omega\rangle| \geq \gamma |k|^{-\tau}$, for all $k \in \mathbb{Z}^d - \{0\}$, with $\langle\cdot\rangle$ being the canonical scalar product and $|\cdot|$ its associated norm.

In his ICM talk of 1998 [He98], Herman conjectured that a real analytic elliptic equilibrium with a Diophantine frequency vector must be accumulated by a set of positive measure of KAM tori. This conjecture is still open. However, our examples can be chosen such that the Birkhoff Normal Form is non-degenerate, which implies KAM-stability as established in [EFK13] (see Theorem 1 above).

In all the sequel, we denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^{2d}$, indifferently on the value of d that will be clear from the context. We also denote indifferently B_n the Euclidean ball of radius n in $\mathbb{R}^{2d}$ for any value of d. For $k \in \mathbb{N}$, we denote by $\|H\|_{C^k(B_R)}$ the C^k norm of H on the ball B_R.

2.1. Lyapunov unstable elliptic equilibrium in three degress of freedom.

We suppose $\omega \in \mathbb{R}^3$ is such that there exists a sequence $\{(k_n, l_n)\} \in \mathbb{N}^* \times \mathbb{N}^*$ satisfying[1]

$$(\mathcal{L}) \qquad 0 < |k_n\omega_1 + l_n\omega_2| < e^{-e^{n^4(k_n+l_n)}}.$$

The set of vectors satisfying $(\mathcal{L})$ is clearly a G_δ-dense set, since resonant vectors form a dense set in $\mathbb{R}^2$. Up to extracting we can also assume that

$$(\mathcal{N}R) \quad k_n \geq \max_{0<k+l\leq k_{n-1}+l_{n-1}} e^{\frac{1}{|k\omega_1+l\omega_2|}}.$$

For $n \in \mathbb{N}$ we define on $\mathbb{R}^4$ the following real polynomial Hamiltonians

$$(2) \qquad F_n(x_1, x_2, y_1, y_2) = \xi_1^{k_n}\xi_2^{l_n} + \eta_1^{k_n}\eta_2^{l_n},$$

We finally define a real entire Hamiltonian on $\mathbb{R}^6$

$$H(x, y) = H_\omega(x, y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2)$$

THEOREM 2. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow Φ_H^t of H. More precisely, for every $n \geq 1$, there exists $z_n \in \mathbb{R}^6$, such that $|z_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi_H^{\tau_n}(z_n)| \geq n$.*

Moreover, the Birkhoff normal form of H at the origin is divergent.

We can modify the definitions of the Hamiltonians H_ω and H on $\mathbb{R}^6$ as follows

$$\widetilde{H}_\omega(x, y) = (\omega_1 + I_3^3)I_1 + (\omega_2 + I_3^4)I_2 + \omega_3 I_3,$$
$$\widetilde{H}(x, y) = \widetilde{H}_\omega(x, y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2).$$

We can assume that $k_0+l_0 > 10$, hence $\widetilde{H}_\omega$ gives the BNF of $\widetilde{H}$ at the origin up to order 5 in the action variables. But $\nabla\widetilde{H}_\omega(I) = (\omega_1 + I_3^3, \omega_2 + I_3^4, \omega_3 + 3I_3^2 I_1 + 4I_3^3 I_2)$

[1]The requirement of double exponential approximations is not uncommon in instability results in real analytic and holomorphic dynamics as is the case for example in [PM97].

is clearly non-degenerate, and this implies that the BNF of $\widetilde{H}$ is non-degenerate. We then have the following

THEOREM 3. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi^t_{\widetilde{H}}$ of $\widetilde{H}$. Moreover, the Birkhoff normal form of $\widetilde{H}$ at the origin is non-degenerate, hence the equilibrium is KAM-stable.*

2.2. Lyapunov unstable elliptic equilibrium in four degrees of freedom. In 4 degrees of freedom (or more), our method yields unstable elliptic equilibria for any frequency vector, provided its coordinates are not all of the same sign. Suppose for instance that $\omega = (\omega_1, \dots, \omega_4)$ is such that $\omega_1\omega_2 < 0$.

We assume (ω_1, ω_2) non-resonant (the resonant case follows from Corollary 1 below). By Dirichlet principle, there exists a sequence $(k_n, l_n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$|k_n\omega_1 + l_n\omega_2| < \frac{1}{k_n^2}.$$

WLOG, we assume that $k_n\omega_1 + l_n\omega_2 < 0$. Then, for $I_{4,n} = -(k_n\omega_1 + l_n\omega_2)/k_n \in (0, \frac{1}{k_n^3})$, it holds

$$(\mathcal{R}) \qquad k_n(\omega_1 + I_{4,n}) + l_n\omega_2 = 0.$$

Since (ω_1, ω_2) is non-resonant, we can, up to extracting, additionally ask that for all $(k, l) \in \mathbb{N}^2 \setminus \{0, 0\}$ such that $k + l \leq k_{n-1} + l_{n-1}$, we have $k(\omega_1 + I_{4,n}) + l\omega_2 \neq 0$ and

$$(\mathcal{N}R) \quad k_n \geq \max_{0<k+l\leq k_{n-1}+l_{n-1}} e^{\frac{1}{|k(\omega_1+I_{4,n})+l\omega_2|}}, \quad k_n \geq e^{e^{e^{n^4(k_{n-1}+l_{n-1})}}}.$$

We define the following real entire Hamiltonians on $\mathbb{R}^8$

$$H_\omega(x, y) = (\omega_1 + I_4)I_1 + \sum_{j=2}^{4} \omega_j I_j,$$
$$H(x, y) = H_\omega(x, y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2)$$

THEOREM 4. – *The origin is a Lyapunov unstable equilibrium for the Hamiltonian flow of H. More precisely, for every $n \geq 1$, there exists $z_n \in \mathbb{R}^8$, such that $|z_n| \leq \frac{1}{n}$, and $\tau_n \geq 0$ such that $|\Phi^{\tau_n}_H(z_n)| \geq n$.*

Moreover, the Birkhoff normal form of H at the origin is divergent.

We can modify the definition of the Hamiltonian on $\mathbb{R}^8$ as follows

$$\widetilde{H}_\omega(x, y) = (\omega_1 + I_4)I_1 + (\omega_2 + I_4^2)I_2 + (\omega_3 + I_4^3)I_3 + \omega_4 I_4,$$
$$\widetilde{H}(x, y) = \widetilde{H}_\omega(x, y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2),$$

where $(k_n, l_n) \in \mathbb{N}^* \times \mathbb{N}^*$ are chosen so that $(\mathcal{R})$and $(\mathcal{N}R)$ hold with ω_2 replaced by $(\omega_2 + I_{4,n}^2)$.

Here also, it is clear that $\nabla\widetilde{H}_\omega(I)$ is non-degenerate. We have the following.

THEOREM 5. – *The origin is a Lyapunov unstable equilibrium of the Hamiltonian flow $\Phi^t_{\widetilde{H}}$ of $\widetilde{H}$. Moreover, the Birkhoff normal form of $\widetilde{H}$ at the origin is non-degenerate, hence the equilibrium is KAM-stable.*

2.3. Examples with divergent Birkhoff Normal Forms for any non-resonant frequency $\omega \in \mathbb{R}^d$, $d \geq 4$.

Theorem 4 covers the case of frequency vectors ω where not all the coordinates are of the same sign. Suppose now that ω is non-resonant and that the coordinates of ω are all of the same sign, say positive. By Dirichlet principle, there exists a sequence $(k_n, l_n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that

$$|k_n\omega_1 - l_n\omega_2| < \frac{1}{k_n^2}.$$

WLOG, we assume that $k_n\omega_1 - l_n\omega_2 < 0$.

For $n \in \mathbb{N}$ we introduce the real polynomial Hamiltonians on $\mathbb{R}^4$

$$D_n(x_1, x_2, y_1, y_2) = \xi_1^{k_n}\eta_2^{l_n} + \eta_1^{k_n}\xi_2^{l_n}.$$

We finally define the following real entire Hamiltonian on $\mathbb{R}^8$

$$H(x, y) = H_\omega(x, y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 D_n(x_1, x_2, y_1, y_2)$$

The origin is an elliptic fixed point for the Hamiltonian flow of H with frequency vector ω.

THEOREM 6. – *The Birkhoff normal form of H at the origin is divergent.*

3. Resonant unstable elliptic fixed points on $\mathbb{R}^4$

In case ω is resonant, it is known that instabilities are more likely to happen. Algebraic examples were known since long time ago [LC1901, Ch26] (see [MS02, §31]). Our construction is actually based on the existence in two degrees of freedom, for resonant frequencies, of polynomial Hamiltonians that have invariant lines that go through the origin such that any point on such a line converges to the origin for negative times and goes to infinity in finite time in the future.

Recall indeed the definition of the following real Hamiltonians for $k, l \in \mathbb{N}^* \times \mathbb{N}^*, k + l > 2$

$$F_{k,l}(x_1, x_2, y_1, y_2) = \xi_1^k\xi_2^l + \eta_1^k\eta_2^l.$$

We have

PROPOSITION 1. – *For any $n \in \mathbb{N}^*$, there exist $t_n \in [0, n^{k+l-2}]$ such that $\Phi^{t_n}_{F_{k,l}}(B_{\frac{1}{2n}}) \bigcap B^c_{2n} \neq \emptyset$.*

If ω_1 and ω_2 are such that $k\omega_1 + l\omega_2 = 0$, then the Hamiltonian flow of $\xi_1^k\xi_2^l + \eta_1^k\eta_2^l$ commutes with that of $\omega_1 I_1 + \omega_2 I_2$. Hence we get the following consequence:

COROLLARY 1. – *If ω_1 and ω_2 are such that $k\omega_1 + l\omega_2 = 0$ for some $k, l \geq 1$ and $k + l > 2$, then for any $a \in \mathbb{R}^*$, the flow of $H(x_1, x_2, y_1, y_2) = \omega_1 I_1 + \omega_2 I_2 + a(\xi_1^k\xi_2^l + \eta_1^k\eta_2^l)$ has an elliptic fixed point with frequency (ω_1, ω_2) that is Lyapunov unstable.*

Proof of Proposition 1. We let $u = \sqrt{l/k}$, $\alpha = k + l - 1$. We assume $\alpha \geq 2$. WLOG, we suppose that $u \geq 1$.

Pick and fix $\nu, \nu' \in (0,1)$ such that

$$-\frac{1}{4} + k\nu + l\nu' = 1.$$

Define a subset of $\mathbb{R}^4$,

$$\Delta := \left\{ (x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : (\xi_1, \xi_2) = \left(re^{i2\pi\nu}, ure^{i2\pi\nu'} \right), r \in \mathbb{R} \right\}.$$

The Hamiltonian equations of H give

$$\dot{\xi}_1 = -ik\eta_1^{k-1}\eta_2^l, \quad \dot{\xi}_2 = -il\eta_1^k\eta_2^{l-1},$$

For $(x_1, x_2, y_1, y_2) \in \Delta$, we get

$$\dot{\xi}_1 = ku^l r^\alpha e^{i2\pi\nu}, \quad \dot{\xi}_2 = ku^{l+1} r^\alpha e^{i2\pi\nu'},$$

which shows that Δ is invariant by the flow $\Phi^t_{F_n}$, and moreover, that the restriction of the vector field on Δ is given by

$$\dot{r} = ku^l r^\alpha.$$

Hence if we start with $r_0 = \frac{1}{2n}$ we see that

$$r(t)^{\alpha-1} = \frac{1}{(2n)^{\alpha-1} - (\alpha-1)ku^l t}.$$

Define then t_n such that $r(t_n) = 2n+1$. Note that $0 \leq t_n \leq T_n := (2n)^{\alpha-1}/(ku^l(\alpha-1)) < (2n)^{\alpha-1}$ since T_n is an explosion time of $r(t)$ with the initial condition $r_0 = \frac{1}{2n}$. □

4. Proofs of the main Theorems.

4.1. Description of the diffusion mechanism.

We first describe the proof of Theorem 2, that is, diffusion in 3 degrees of freedom near a close to resonant elliptic equilibrium.

We want to exhibit diffusive orbits for the flow of
$H_\omega(x,y) + \sum_{n\in\mathbb{N}} e^{-n(k_n+l_n)} I_3 F_n(x_1, x_2, y_1, y_2)$

- From Corollary 1, we know that if $k_n\bar{\omega}_1 + l_n\bar{\omega}_2 = 0$ then the flow of $\bar{\omega}_1 I_1 + \bar{\omega}_2 I_2 + F_n(x_1, x_2, y_1, y_2)$ is unstable
- Due to $(\mathcal{L})$, an approximation argument (section 4.2) will show that, for fixed $I_3 = \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}$, the flow of $H_\omega(x,y) + e^{-n(k_n+l_n)}\mathbf{I}F_n(x_1, x_2, y_1, y_2)$, has a point satisfying $I_1, I_2 \sim 1/n$, $I_3 = \mathbf{I}$, that escapes at time of order $\mathbf{I}^{-1}$.
- The terms $F_l, l > n$ are too small and do not disrupt the diffusion at this time scale
- The terms $F_l, l < n$ average out to an I_3^2 term that contributes with $\mathcal{O}(\mathbf{I}^2)$ magnitude at this level of I_3 and do not distrupt the diffusion at this time scale comparable to $\mathbf{I}^{-1}$.

In four degrees of freedom, we replace the almost resonance condition on ω by the use of the fourth action variable that is also invariant along the flow. Indeed, when we fix the value of this variable to $I_{4,n}$ such that $k_n(\omega_1 + I_{4,n}) + l_n\omega_2 = 0$, we find our Hamiltonian exactly in the form to which we can apply the Corollary 1.

The variable I_3 plays then the same role as in the precedent case, of isolating the effect of a single F_n in the diffusion, for various values of $I_3 \to 0$.

With a bit more scrutiny in the terms of order I_3^2 of the Birkhoff normal form, we can easily see that these forms diverge in our examples.

4.2. Approximation by resonant systems and diffusive orbits.

LEMMA 1. – *Suppose $F, G \in C^2(\mathbb{R}^{2d}, \mathbb{R})$, $\omega \in \mathbb{R}^d$, $A, r, R, a, T > 0$ such that $r \leq R$, $a^2 A T e^{daAT} \leq 1/4$, and*

- $H(x,y) = \sum_{j=1}^{d} \omega_j I_j + aF(x,y)$
- $h(x,y) = \sum_{j=1}^{d} \omega_j I_j + aF(x,y) + a^2 G(x,y)$
- $\|F\|_{C^2(B_{R+1})} \leq A, \quad \|G\|_{C^2(B_{R+1})} \leq A$
- *For all* $s \in [0,T] : \Phi_H^s(B_r) \subset B_R$

Then, for all $s \in [0,T]$ and for all $z \in B_r$:

$$|\Phi_H^s(z) - \Phi_h^s(z)| \leq a^2 A T e^{daAT}$$

Proof. We use the complex coordinates $\xi(s)$ and $\eta(s)$ for the solution $\Phi_H^s(z)$, and $u(s)$ and $v(s)$ for the solution $\Phi_h^s(z)$. Define the matrices $U_j = \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix}$, and introduce the variables $\zeta_j(s) = e^{isU_j} \begin{pmatrix} \xi_j(s) \\ \eta_j(s) \end{pmatrix}$, and $w_j(s) = e^{isU_j} \begin{pmatrix} u_j(s) \\ v_j(s) \end{pmatrix}$. Since e^{isU_j} is a unitary matrix, it suffices to control $|X(s)|$, for $X(s)$ the $2d$ dimensional vector whose entries are given by the coordinates of $\zeta_j(s) - w_j(s)$, $j = 1, \ldots, d$. The Hamiltonian equations and the bounds on F and G then yield, as long as $\Phi_H^s(z)$, and $\Phi_h^s(z)$ are in B_{R+1}

$$|\dot{X}(s)| \leq daA|X(s)| + a^2 A, \quad X(0) = 0$$

and Gronwall's inequality then implies that

$$|X(s)| \leq a^2 A s e^{daAs}$$

which allows to conclude due to the condition $a^2 A T e^{daAT} \leq 1/4$, which also allows to make sure that $\Phi_h^s(B_r) \subset B_{R+1}$ for $s \in [0,T]$. □

COROLLARY 2. – *Let $a \in (e^{-2e^{n^3(k_n+l_n)}}, e^{-e^{n^3(k_n+l_n)}})$. Let $H \in C^2(\mathbb{R}^{2d}, \mathbb{R})$ be such that*

$$H(x,y) = H_\omega(x,y) + aF_n(x_1, x_2, y_1, y_2) + a^2 G_n(x,y)$$

with $\|G_n\|_{C^2(B_{2n})} \leq e^{4n(k_n+l_n)}$.

If $(\mathcal{L})$ holds, there exist $t_n \in [0, (2n)^{k_n+l_n-2}]$ and $z_n \in \mathbb{R}^{2d}$ such that $|z_n| = \frac{1}{2n}$ and $|\Phi_H^{\frac{t_n}{a}}(z_n)| \geq n$.

Proof. From ($\mathcal{L}$), there exists ω_1' such that $|\omega_1' - \omega_1| < e^{-e^{n^4(k_n+l_n)}}$ and $|k_n\omega_1' + l_n\omega_2| = 0$. Then, $\{\omega_1'\xi_1\eta_1 + \omega_2\xi_2\eta_2, F_n\} = 0$. Hence if we define $\omega' = (\omega_1', \omega_2, \dots, \omega_d)$ and

$$H'(x,y) = H_{\omega'}(x,y) + aF_n(x_1,x_2,y_1,y_2),$$

we get that

$$|\Phi_{H'}^{\frac{t}{a}}(z)| = \left|\Phi_{\omega_1' I_1 + \omega_2 I_2}^{\frac{t}{a}}\left(\Phi_{aF_n}^{\frac{t}{a}}(z)\right)\right| = |\Phi_{aF_n}^{\frac{t}{a}}(z)| = |\Phi_{F_n}^{t}(z)|.$$

Hence, by Proposition 1, there exists $t_n \in [0, (2n)^{k_n+l_n-2}]$ and $z_n \in \mathbb{R}^{2d}$ such that $|z_n| \le \frac{1}{2n}$, $|\Phi_{H'}^{\frac{t_n}{a}}(z_n)| = n+1$ and $\Phi_{H'}^{\frac{t}{a}}(B_{\frac{1}{n}}) \subset B_{n+1}$ for every $t \le t_n$.

Now since $|\omega_1' - \omega_1| < e^{-e^{n^4(k_n+l_n)}} \le a^2$, we have that

$$H(x,y) = H_{\omega'}(x,y) + aF_n(x_1,x_2,y_1,y_2) + a^2 G_n'(x,y)$$

with $\|G_n'\|_{C^2(B_{2n})} \le e^{4n(k_n+l_n)} + 1$. Note also that $\|F_n\|_{C^2(B_{2n})} \le e^{n(k_n+l_n)}$.

Let $A = e^{4n(k_n+l_n)} + 1$. Observe that for $T = \frac{t_n}{a}$ we have $a^2 A T e^{daAT} = aAt_n e^{dAt_n} \le \frac{1}{4}$. We can thus apply Lemma 1, with $r = \frac{1}{2n}$, $R = n+1$, and deduce that for all $s \in [0, t_n]$ and for all $z \in B_{\frac{1}{2n}}$:

$$\left|\Phi_{H}^{\frac{s}{a}}(z) - \Phi_{H'}^{\frac{s}{a}}(z)\right| \le aAt_n e^{dAt_n} \le \frac{1}{4}$$

and the conclusion of the corollary thus holds if we apply the latter inequality to $z = z_n$ and $s = t_n$. □

4.3. Proof of Theorem 2. We fix $n \in \mathbb{N}$ and want to show that there exists $z_n \in \mathbb{R}^6$, such that $|z_n| \le \frac{1}{n}$, and $\tau_n \ge 0$ such that $|\Phi_H^{\tau_n}(z_n)| \ge n$.

Note that for any value $\mathbf{I} \in \mathbb{R}_+$, the set $\{(x,y) \in \mathbb{R}^6 : I_3 = \xi_3\eta_3 = \mathbf{I}\}$ is invariant under all the flows we consider in this construction.

We restrict from here on our attention to

$$I_3 \le \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}. \tag{3}$$

For $r > 0$, we then denote $\hat{B}(r) := \{(x_1,x_2,y_1,y_2,x_3,y_3) : (x_1,x_2,y_1,y_2) \in B(r), I_3 \le \mathbf{I}\}$.

Denote $a_j = e^{-j(k_j+l_j)}$. Let $b_j = \frac{e^{-j(k_j+l_j)}}{k_j\omega_1 + l_j\omega_2}$ and define the Hamiltonians on $\mathbb{R}^6$

$$\chi_j = -ib_j I_3 E_j, \quad E_j = \xi_1^{k_j}\xi_2^{l_j} - \eta_1^{k_j}\eta_2^{l_j}.$$

that satisfy

$$\{H_\omega, \chi_j\} = -a_j I_3 F_j. \tag{4}$$

Let $\widehat{\chi}_{n-1} = \sum_{j \le n-1} \chi_j$. Since $k_n \ge e^{\frac{1}{|k_j\omega_1 + l_j\omega_2|}}$ for any $j \le n-1$, we have for any fixed $k \in \mathbb{N}$, and for sufficiently large n

$$\|\widehat{\chi}_{n-1}\|_{C^k(\hat{B}_{2n})} \le k_n^2 \mathbf{I}. \tag{5}$$

Hence, (3) implies that if $z \in \hat{B}(2n)$

$$\left|\Phi_{\widehat{\chi}_{n-1}}^{1}(z) - z\right| \le e^{-n(k_n+l_n)}. \tag{6}$$

Next, we conjugate the flow of H with the time one map of $\widehat{\chi}_{n-1}$. We have

$$H \circ \Phi^1_{\widehat{\chi}_{n-1}} = H + \{H, \widehat{\chi}_{n-1}\} + \frac{1}{2!}\{\{H, \widehat{\chi}_{n-1}\}, \widehat{\chi}_{n-1}\} + \dots$$

Due to (4) we have

$$\{H, \widehat{\chi}_{n-1}\} = -\sum_{j \le n-1} I_3 a_j F_j - I_3^2\{\sum_{j \ge 1} a_j F_j, \sum_{j \le n-1} i b_j E_j\}$$

Hence

$$\begin{aligned} H \circ \Phi^1_{\widehat{\chi}_{n-1}} &= H_\omega + I_3 a_n F_n + \sum_{j \ge n+1} I_3 a_j F_j \\ &\quad - \frac{1}{2} I_3^2\{\sum_{j \le n-1} a_j F_j, \sum_{j \le n-1} i b_j E_j\} - I_3^2\{\sum_{j \ge n} a_j F_j, \sum_{j \le n-1} i b_j E_j\} + \mathcal{R} \end{aligned} \tag{7}$$

where

$$\mathcal{R} = \frac{1}{2}\{\{H - H_\omega, \widehat{\chi}_{n-1}\}, \widehat{\chi}_{n-1}\} + \frac{1}{3!}\{\{\{H, \widehat{\chi}_{n-1}\}, \widehat{\chi}_{n-1}\}, \widehat{\chi}_{n-1}\} + \dots$$

Note that due to (3) and (5), we have that $\mathcal{R}$ is a real analytic Hamiltonian that is of order 3 in I_3 and satisfies

$$\|\mathcal{R}\|_{C^2(\hat{B}(2n))} \le \mathbf{I}^{\frac{5}{2}}. \tag{8}$$

Also, the same bound on the b_j's used for (5) implies that

$$\|\{\sum_{j \le n-1} a_j F_j, \sum_{j \le n-1} i b_j E_j\}\|_{C^2(\hat{B}(2n))} + \|\{\sum_{j \ge n} a_j F_j, \sum_{j \le n-1} i b_j E_j\}\|_{C^2(\hat{B}(2n))} \le k_n^2 \tag{9}$$

Since $k_{n+1} \ge \mathbf{I}^{-1}$ we have that

$$\|\sum_{j \ge n+1} a_j F_j\|_{C^2(\hat{B}(2n))} \le \mathbf{I} \tag{10}$$

Since I_3 is invariant by the Hamiltonian flow of all the functions we are considering, we now fix $I_3 = \mathbf{I}$ and consider the flow of $H \circ \Phi^1_{\widehat{\chi}_{n-1}}$ in restriction to the (x_1, x_2, y_1, y_2) variables. Introduce $a := \mathbf{I} a_n$. We then have from (7), (8), (9) and (10),

$$H \circ \Phi^1_{\widehat{\chi}_{n-1}} = H_\omega + a F_n(x_1, x_2, y_1, y_2) + a^2 G(x, y) \tag{11}$$

with $\|G\|_{C^2(\hat{B}_{2n})} \le a_n^{-2}(1 + k_n^2 + \mathbf{I}^{1/2}) \le e^{3n(k_n + l_n)}$.

We thus apply Corollary 2 and get that there exist $t_n < (2n)^{k_n + l_n - 2}$ and $w_n \in \mathbb{R}^4$ such that $|w_n| = \frac{1}{2n}$ and $|\Phi^{\frac{t_n}{a}}_{H \circ \Phi^1_{\widehat{\chi}_{n-1}}}(w_n)| \ge n$. Now, for z_n we pick any $(x_3, y_3) \in \mathbb{R}^2$ such that $I_3 = \mathbf{I}$ and let $z_n = \Phi^{-1}_{\widehat{\chi}_{n-1}}(w_n, x_3, y_3)$. Thus, $|\Phi^{\frac{t_n}{a}}_H(z_n)| \ge n$, while (6) implies that $|z_n| \le \frac{1}{n}$. This finishes the proof of Lyapunov instability.

We now prove the divergence of the BNF of H. We have from (7) that

$$H \circ \Phi^1_{\widehat{\chi}_{n-1}} = H_\omega - \sum_{j \le n-1} \frac{a_j^2}{k_j \omega_1 + l_j \omega_2} I_3^2 \left(k_j^2 I_1^{k_j - 1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j - 1}\right) + I + II + III \tag{12}$$

with

$$\begin{aligned} I &= I_3 \sum_{j\geq n} a_j F_j(\xi_1,\xi_2,\eta_1,\eta_2) \\ II &= I_3^2 \sum_{\{u_1,u_2\}\neq\{v_1,v_2\}} c_{u_1,u_2,v_1,v_2}\xi_1^{u_1}\xi_2^{u_2}\eta_1^{v_1}\eta_2^{v_2} \\ III &= I_3^3 W(x,y) \end{aligned}$$

where W is a real analytic Hamiltonian. Define

$$\begin{aligned} \mathcal{A}_n := \big\{(u_1,u_2,v_1,v_2)\in\mathbb{N}^4 : \{u_1,u_2\}\neq\{v_1,v_2\} \\ \text{and } u_1+u_2<k_{n-1}+l_{n-1}, v_1+v_2<k_{n-1}+l_{n-1}\big\} \end{aligned}$$

and

$$\psi = I_3^2 \sum_{\mathcal{A}_n} \frac{-ic_{u_1,u_2,v_1,v_2}}{(u_1-v_1)\omega_1+(u_2-v_2)\omega_2}\xi_1^{u_1}\xi_2^{u_2}\eta_1^{v_1}\eta_2^{v_2} \tag{13}$$

and observe that since

$$\{H_\omega,\psi\} = -I_3^2 \sum_{\mathcal{A}_n} c_{u_1,u_2,v_1,v_2}\xi_1^{u_1}\xi_2^{u_2}\eta_1^{v_1}\eta_2^{v_2}$$

then (12) gives

$$\begin{aligned} H\circ\Phi^1_{\widehat{\chi}_{n-1}}\circ\Phi^1_\psi = H_\omega - \sum_{j\leq n-1}\frac{a_j^2}{k_j\omega_1+l_j\omega_2} I_3^2\left(k_j^2 I_1^{k_j-1}I_2^{l_j}+l_j^2 I_1^{k_j}I_2^{l_j-1}\right) \\ + I' + II' + III' \end{aligned} \tag{14}$$

where I', II', III' are real analytic Hamiltonians around the origin of the form

$$\begin{aligned} I' &= I = I_3 \sum_{j\geq n} a_j F_j(\xi_1,\xi_2,\eta_1,\eta_2) \\ II' &= I_3^2 \sum_{\{u_1,v_1\}\neq\{u_2,v_2\}\cap\mathcal{A}_n^c} c_{u_1,u_2,v_1,v_2}\xi_1^{u_1}\xi_2^{u_2}\eta_1^{v_1}\eta_2^{v_2} \\ III' &= I_3^3 W'(x,y) \end{aligned}$$

where W' is a real analytic Hamiltonian. Hence, the terms in III' do not contribute to the $O^2(I_3)$ part of the BNF of H at 0.

Since the order of the $(\xi_1,\xi_2,\eta_1,\eta_2)$-terms in I and II' are higher than $k_{n-1}+l_{n-1}$, and since (14) holds for an arbitrary n, we conclude that the $O^2(I_3)$ part of the BNF of H is given by

$$\sum_{j=1}^{\infty} -\frac{e^{-2j(k_j+l_j)}}{k_j\omega_1+l_j\omega_2}\left(k_j^2 I_1^{k_j-1}I_2^{l_j}+l_j^2 I_1^{k_j}I_2^{l_j-1}\right) I_3^2 \tag{15}$$

which is clearly divergent since $|k_j\omega_1+l_j\omega_2| < e^{-e^{j^4(k_j+l_j)}}$ by our Liouville hypothesis ($\mathcal{L}$). Observe that, in fact, the super Liouville condition $|k_j\omega_1+l_j\omega_2| < e^{-j^2(k_j+l_j)}$ is sufficient for the divergence of the BNF. □

4.4. Proof of Theorem 3. We keep the same definitions of χ_j and $\widehat{\chi}_j$ as in the above proof and observe that, since since $H - \widetilde{H} = I_3^3(I_1 + I_3 I_2)$, it still holds that for $a = a_n \mathbf{I}$

$$\widetilde{H} \circ \Phi^1_{\widehat{\chi}_{n-1}} = H_\omega + aF_n(x_1, x_2, y_1, y_2) + a^2 \tilde{G}(x, y)$$

with $\|\tilde{G}\|_{C^2(\hat{B}_{2n})} \leq e^{3n(k_n+l_n)}$. The rest of the proof of the topological instability of the origin is the same as that of Theorem 2. □

4.5. Proof of Theorem 4. Note that for any value of $(\mathbf{I}, \mathbf{J}) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, the set $\{(x, y) \in \mathbb{R}^8 : I_3 = \xi_3\eta_3 = \mathbf{I}, \xi_4\eta_4 = \mathbf{J}\}$ is invariant under all the flows we consider in this construction.

If we fix now $I_4 = \mathbf{J} := I_{4,n}$ and $I_3 = \mathbf{I} := e^{-e^{n^3(k_n+l_n)}}$, the restriction of the flow of H to the (x_1, x_2, y_1, y_2) space takes the form:

$$H(x, y) = (\omega_1 + \mathbf{J})I_1 + \omega_2 I_2 + \omega_3 \mathbf{I} + \omega_4 \mathbf{J} + \sum_{n\in\mathbb{N}} \mathbf{I} e^{-n(k_n+l_n)} F_n(x_1, x_2, y_1, y_2) \tag{16}$$

which has the same flow as in the proof of Theorem 2 with this difference that ω_1 is replaced by $\omega_1 + \mathbf{J}$. Moreover, the hypothesis $(\mathcal{R})$ and $(\mathcal{N}R)$ of Theorem 4 imply hypotheses $(\mathcal{L})$ and $(\mathcal{N}R)$ of Theorem 2, so the existence of the diffusive orbit follows from the proof of Theorem 2.

As for the BNF, using hypothesis $(\mathcal{N}R)$ we can define $\widehat{\chi}_{n-1}$ and ψ as in (6) and (13), but with $\omega_1 + I_4$ instead of ω_1, and get, in sufficiently small neighborhood of the origin:

$$\begin{aligned} H \circ \Phi^1_{\widehat{\chi}_{n-1}} \circ \Phi^1_\psi = H_\omega - \sum_{j\leq n-1} \frac{a_j^2}{k_j(\omega_1 + I_4) + l_j\omega_2} I_3^2 \left(k_j^2 I_1^{k_j-1} I_2^{l_j} + l_j^2 I_1^{k_j} I_2^{l_j-1}\right) \\ + I' + II' + III' \end{aligned} \tag{17}$$

where I', II', III' are real analytic Hamiltonians around the origin (for this, we restrict to $I_4 \ll 1$) and are of the same form as in Section 4.3. Hence, the terms in III' do not contribute to the $O^2(I_3)$ part of the BNF of H at 0. As in the proof of Theorem 2, we conclude that the $I_3^2 I_1^{k_{n-1}-1} I_2^{l_{n-1}}$ term of the BNF of H is given by

$$-\frac{e^{-2(n-1)(k_{n-1}+l_{n-1})}}{k_{n-1}(\omega_1 + I_4) + l_{n-1}\omega_2} k_{n-1}^2 I_3^2 I_1^{k_{n-1}-1} I_2^{l_{n-1}}$$

which, as a series in I_4 has a radius of convergence smaller than $|k_{n-1}\omega_1 + l_{n-1}\omega_2|/k_{n-1}$, because we assumed that $k_{n-1}\omega_1 - l_{n-1}\omega_2 < 0$. Since this holds for an arbitrary n, and since $|k_{n-1}\omega_1 + l_{n-1}\omega_2|/k_{n-1}$ goes to 0 as $n \to \infty$, we conclude that the BNF of H diverges. □

4.6. Proof of Theorem 5. We just replace ω_2 by $\omega_2 + \mathbf{J}^2$ everywhere in the proof of the topological instability of the fixed point in Theorem 4. □

4.7. Proof of Theorem 6. Following exactly the same lines as in the proof of the divergence of the BNF in Theorem 4, we get that the $I_3^2 I_1^{k_{n-1}-1} I_2^{l_{n-1}}$ term of the BNF of H defined in Section 2.3 is given by

$$-\frac{e^{-2(n-1)(k_{n-1}-l_{n-1})}}{k_{n-1}(\omega_1 + I_4) - l_{n-1}\omega_2} k_{n-1}^2 I_3^2 I_1^{k_{n-1}-1} I_2^{l_{n-1}}$$

which, since we assumed that $k_{n-1}\omega_1 - l_{n-1}\omega_2 < 0$, as a series in I_4 has a radius of convergence smaller than $|k_{n-1}\omega_1 - l_{n-1}\omega_2|/k_{n-1}$. We thus conclude that the BNF of H diverges. □

ACKNOWLEDGMENT. I am thankful to the anonymous referee whose critical reading resulted in considerable improvements of the presentation. I am grateful to H. Eliasson and M. Saprykina for their inputs to this paper. I am also grateful to A. Bounemoura and R. Krikorian for valuable discussions on the topic. Thank you Tolya for always pushing me to work on this!

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