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Task characteristics that promote mathematical reasoning among young students: An exploratory case study

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In the paper we examine possible factors that enhance young students’ mathematical reasoning. Particularly, we present the characteristics of seven mathematical tasks, which were chosen based on an a priori analysis, in order to elicit reasoning among a pair of young students. Additionally, we examine the reasoning processes that were manifested during the students’ interactions while solving the tasks. Our results show how particular task characteristics affect reasoning processes.

Keywords: Task design, mathematical reasoning, task characteristics.

Introduction

Algebra constitutes one of the most significant elements of mathematics, and children, even at a small age, have access to mathematical ideas such as mathematisation, connections, argumentation, number sense and mental computation, algebraic reasoning, spatial and geometric thinking, data and probability sense (Perry & Dockett, 2002). This has led to the “Early Algebra” movement, which has been the background of a number of studies (see, e.g., Kieran, Pang, Schifter & Ng, 2016). The main process involved in algebraic thinking – and in all mathematical activities – is undoubtedly generalisation (Mason, 2005), which therefore has been the focus of the majority of studies in early algebra. However, other significant processes are also involved in algebraic thinking: Blanton et al. (2011) (as cited in Kieran et al., 2016, p. 12, our emphasis) refer to “the processes of generalizing, representing, justifying, and reasoning with mathematical structure and relationships”. One of the questions that may then arise is whether it is possible to design and propose mathematical tasks that enhance reasoning among young students. Having this question in mind we designed a case study aiming to examine how particular task characteristics relate to the reasoning processes of two eight-year-old students. Thus, our main research question is what are the characteristics of the tasks which promote mathematical reasoning. Particularly, we aim to examine how specific factors of the tasks are related to reasoning processes.

Algebraic reasoning and task design

Algebraic thinking is defined as “a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways” (Kaput & Blanton, 2005, p. 99). This is in line with “Early Algebra”, which can be characterised by the following shift:

... from a traditional content-centered characterization of algebra to that of the mathematical reasoning processes and representations that would seem appropriate for young children, as well as to the nature of the early algebra activities that might promote the development of these processes and representations (Kieran et al., 2016, p. 5).
The above statement leads us to the importance of engaging young students in activities that promote reasoning. But how should such activities look like? This question might prove misleading, since there are various factors that contribute to a learning situation: besides the task itself, we may refer to the teacher and the students participating and interacting. This list is not exhaustive; one could also consider the wider social and classroom context, the available resources and artefacts, the histories of the students and their interactions, etc.

In our research paper, we focus on the teacher (as a task designer), as well as on the students (their reasoning activities as they manifest during their interactions). The importance of task design in enhancing students’ engagement and promoting mathematical reasoning has been stressed in research focused on reasoning (Francisco & Maher, 2005) as well as assessment (van den Heuvel-Panhuizen, 2005). These studies have provided comprehensive suggestions on task characteristics, including the expected teacher and students’ roles. They can be summarised as follows:

- “the problems must be accessible, inviting and worthwhile to solve” and “the students must have the opportunity to give their own answers in their own words” (van den Heuvel-Panhuizen, 2005, p. 3); moreover, pictures play an important role as context-bearers and they serve multiple functions (van den Heuvel-Panhuizen, 2005);
- “the development of a sense of ownership of the mathematical activity enhances the building of personally meaningful mathematical understanding and students’ confidence in their abilities” and working collaboratively and discussing “helps the students become flexible in problem comprehension and adaptive rather [than] routine experts at solving problems” (Francisco & Maher, 2005, p. 371).

**Context of the study and methodology**

Based on the type of our research questions we have adopted an exploratory case study approach (Yin, 2014). The participants were two eight-year-old female students studying together in the 2nd grade of a public primary school in an urban area in Poland. They both had good marks in mathematics and volunteered to participate in the research. The session took place in one of the students’ house, lasted for two hours and was videotaped, after the parents’ consent was obtained. Both authors of the paper were present in the session and both were familiar to the students, who worked together for solving the tasks. The role of the first author of the paper consisted of the following: challenge pupils, support pupils, evaluate their progress, think mathematically in order to pose challenging and interesting mathematical questions, supply and recall information and promote pupils’ reflection (da Ponte, 2001).

**A priori analysis of the tasks**

Task design was based on the assumptions mentioned in the previous section. Seven tasks were chosen from two sources (Lankiewicz, Sawicka & Swoboda, 2012; Treffers, 2008), based on: their potential to stimulate students’ interest, be solvable, or at least approachable, in more than one way and without the use of tricks, illustrate important mathematical ideas, serve as first steps towards mathematical explorations and be extensible and generalisable (Schoenfeld, 1994). Table 1 below presents all tasks, except Tasks 3 and 5, which will be presented and analysed later.
**Task 1: Chairs**

How you can divide the chairs into 2 sets of 4 chairs in each?

**Task 4: Tiles**

Uncle Ted tiles three terraces. Here are the tiles he uses.

- Help uncle Ted in his work and draw your propositions for the tiling. Check different possibilities. Write under every proposition the number of used tiles.
- Try to tile the following terraces by using the tiles of uncle Ted.
- Count and write how many tiles you used for every terrace. Do you notice any changes? Which ones?

**Task 2: Fish**

How many fish can you colour by three colours so that every fish is different? You have three colours: green, orange and blue.

**Task 6: The parrot Waku-Waku**

This is a talking parrot; his name is Waku-Waku. Unfortunately, he can say only one word: five. You want to show your friends how clever parrot you have. Ask him some questions, for which the answer is “five”.

**Task 7: Caterpillar**

You choose two numbers less than 100. Each successive number is the sum of the previous two. For example:

Try to make a 5-part caterpillar with 100 as the last number.

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**Table 1: Five of the given tasks**

Our a priori analysis of the tasks initiated by identifying the mathematical concepts and processes involved in the tasks’ solutions as anticipated by us and, wherever applicable, by the original task designers. The mathematical fields contained in the tasks correspond to those included in the Polish curriculum for the particular age group. All tasks contained a written description, but in our a priori analysis we also analysed the role of the accompanying images, according to van den Heuvel-Panhuizen’s (2005) pictures’ functions: motivator (M: making the task more attractive), situation describer (S: describing the context of the task), information provider (I: providing necessary information), action indicator (A: an action is elicited that has the potential of a strategy that leads to a solution), model supplier (MS: the picture structuring possibilities that can be used to solve the problem), and solution and solution-strategy communicator (SC: the picture contains the solution and aspects of the applied strategies). The analysis of the tasks’ contexts was based on de Lange’s (1999) assessment framework: zero-order (0): fake context, not considered for the solution (to this category we also inserted mathematical contexts, e.g. Task 5, which will be analysed later); first-order (1): relevant context, needed for the solution; second-order (2): relevant context, mathematisation needed for the solution; third-order (3): the context serves for constructing or reinventing mathematical concepts. Finally, each task was categorised as open or closed. As shown in Table 2, all tasks but one were open.
A posteriori analysis of the tasks

In our a posteriori analysis, we focused on the interactions that took place among the students and the researcher, during the research session. For this analysis we have adopted Lannin, Ellis and Elliot’s (2011) manifestations of essential understandings related to mathematical reasoning: developing conjectures, generalizing to identify commonalities, generalizing by application, conjecturing and generalizing using terms, symbols, and representations, investigating why, justifying based on already-understood ideas, refuting a statement as false, justifying and refuting the validity of arguments and validating justifications. To these we added: reformulating conjectures or justifications, monitoring each other (a form of collaborative reasoning), concluding and explaining (a form of informal justification). After identifying the above understandings, we made a cross-analysis of all tasks, in order to identify commonalities and differences among them.

Results

As we see in Table 3, the reasoning processes which persevered were: developing conjectures and reformulating conjectures or justifications.

Table 3: Tasks and reasoning processes

Next, we present the analyses of Tasks 3 and 5; these tasks are chosen because they demonstrate the range of reasoning processes that we have encountered.
Figure 1: Task 3 and Task 5, containing students’ work

Task 3: Christmas baubles

In Figure 1, on the left, we read: “Design different boxes in which you can put 12 Christmas baubles in such a way that every bauble is in one compartment.” The mathematical concepts come from geometry (shapes, dimensions, area) as well as arithmetic (numbers and multiplication); we anticipated that students will shift from the geometrical representation of the problem to an arithmetic one. The image of the grid can be categorised as a situation describer and a model supplier. The students – signified by A and M, while R stands for the researcher (the first author of the paper) – started by drawing a 4×6 rectangle on the grid and placing their drawn baubles in every second compartment. Initially, it was hard for them to imagine how the baubles can be arranged. Later, they discovered that the box they designed can contain more baubles than they thought, and they justified it. However, they were not able to use this knowledge for the design of boxes of different sizes; they were more focused on the shape, the colours and the sizes of the baubles. In the following excerpts our notes are in brackets and (...) signifies an omitted part of the discussion. The number(s) of the reasoning processes (as they appear in Table 3) are put in brackets at the end of the respective lines.

R: Could we put more baubles to this box? [the box on the right edge of the grid]. For example, could we put one more bauble?

A: I guess not.

R: If I had one more bauble and I would tell you: girls, put that bauble to your box…

M: I would pack it in such a square. (…)

R: (…) then how many could we pack here?


R: How do you know that 24? You didn’t count.

A: Because there are 12 baubles and 12 empty places. [R5]

R: And how do you know that there are 12 empty places?

A: Because here in the 4 squares there are 4 places and in 2 of them there are baubles and those 2 are left. And in the same way you can count to the bottom so it will be 24. [R9]
M: Or 4×6 [showing the dimensions of the rectangle]. (...) [R9]

R: Are those boxes different? [showing the four boxes drawn on the grid]
A: Uhm.
R: By what? [A and M thinking]
A: Here are bigger baubles [the second and third box] and here smaller [the first box].
M: And these are medium ones [the fourth box].

During the whole interaction only two aspects of mathematical reasoning were manifested: justification and reformulating the justification. The students eventually discovered different arrangements of the baubles (in different boxes) but they could not find any relations between them, neither geometric nor arithmetic.

**Task 5: Magic squares**

In Figure 1, on the right, we read: “Complete the empty fields so that the sum of the numbers in rows and columns is 20”; and then: “Try to create a similar table for your friend or somebody from your family”. The main mathematical concept is addition and its properties: identity, commutative and associative. The images of the task are information providers. An extensive analysis of the students’ interactions (Maj-Tatsis & Tatsis, 2015) has led to the following understandings related to mathematical reasoning: developing conjectures, generalizing by application, refuting a statement as false, justifying and refuting the validity of arguments, validating justifications, reformulating conjectures, monitoring each other and explaining. Thus, this task has proved fruitful for mathematical reasoning among our students. In the following excerpt the students have completed working on the first magic square and are discussing on the second. It is interesting that student M started filling the square by number 9 (which means that then one has to put 0 in the third square) which was the extreme case in the first square, and it took them some time to accept it.

M: Here for example 8 [in the centre] and put 1 here [above 8] [R1]
R: Aha (...) What if here we put 7 [in the centre], then what will be here? [above]
M: 2
R: And what if 6?
A & M: [loudly] 3!
R: How do you know it’s 3?
A: Because when it was 7 then it was 2. And if we decrease it more then it will be 3. [R1, R12]
R: What if we put 5?
M: Then 4 (…)
R: What’s the biggest number we can put in the centre? (…)
M: For me 9, because 10+11 would be already 21. [R1, R7]
The differences between this task and the Christmas baubles task lie not only in the extent of reasoning processes, but also in their depth. Although in this task the students were able to articulate, justify and validate mathematical conjectures in order to support their mathematical reasoning, in the baubles task they did not manage to cross the boundaries between arithmetic and geometry; this has resulted in limited opportunities for extensive mathematical reasoning.

Cross-analysis of the tasks

We have seen the greatest extent of reasoning processes in Tasks 5 and 7 (followed by Tasks 2 and 4), while Tasks 1, 3 and 6 proved the poorest concerning the manifested reasoning processes by the students. The common characteristic of Tasks 1 and 6 is that they both required the identification of commonalities and explanations; furthermore, the students’ interactions have shown that there was no explicit need for justification or generalisation. Tasks 5 and 7 were similar concerning the mathematical concepts involved (addition and its properties); additionally, they were both open, but with a finite number of solutions (unlike Task 6). At the same time, the only task with a single solution (Task 2) allowed for a considerable extent of mathematical reasoning; this was probably due to the many ways of solution. Task 3 was the one that demonstrated the influence of context: the students were engaged in a non-mathematical investigation concerning the size and the type of baubles, which in turn minimised the reasoning processes and hindered a thorough investigation of the task. It might be also the influence of the geometrical nature of the task that had created obstacles in its generalisation.

Discussion

The fact that reasoning processes were manifested in all tasks goes in line with relevant studies of the “Early Algebra” movement and strengthens the claim that students at a young age are able to reason mathematically. The variety of our tasks has proved that young students, even with a minimum help, can develop, justify, validate and refute mathematical conjectures.

Our main research question was what are the characteristics of the tasks which promote mathematical reasoning. Firstly, we have not seen any influence of the images that accompanied the tasks; however, we have seen that context might move students away from reasoning (Task 3). Some tasks have proven more fruitful than others concerning the reasoning processes they evoked. It seems that open tasks do not pose any difficulties to the students, when they have a finite number of solutions (Tasks 5, 7). The same is the case with closed tasks that can be solved in many different ways (Task 2). On the contrary, our students experienced difficulties when faced with a task that required a shift from geometrical structures to numerical ones (Task 3). At the same time, they manifested a variety of reasoning processes during another geometrical task (Task 4), which did not require an explicit shift to numerical properties. Based on all the above, although our exploratory study has not led to conclusive results, we believe that it may assist the design of mathematical tasks aiming to promote reasoning among young students.

References


