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# Effective models for non-perfectly conducting thin coaxial cables

Geoffrey Beck<sup>1,\*</sup>, Sébastien Imperiale<sup>2</sup>, Patrick Joly<sup>1</sup>

<sup>1</sup>POEMS (CNRS-INRIA-ENSTA Paristech), Palaiseau, France

<sup>2</sup>Inria & LMS, Ecole polytechnique, CNRS, Université Paris-Saclay, France

\*Email: geoffrey.beck.poems@gmail.com

## Abstract

Continuing past work on the modelling of coaxial cables, we investigate the question of the modeling of non-perfectly conducting thin coaxial cables. Starting from 3D Maxwell's equations, we derive, by asymptotic analysis with respect to the (small) transverse dimension of the cable, a simplified effective 1D model. This model involves a fractional time derivatives that accounts for the so-called skin effects in highly conducting regions.

**Keywords:** Maxwell's equations, Coaxial cables, Asymptotic analysis

## Statement of the problem

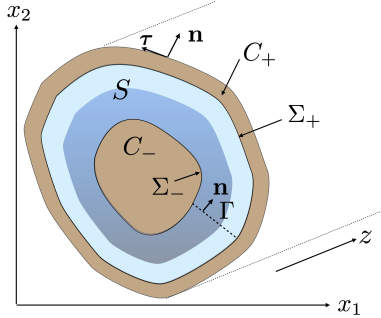


Figure 1: Section the coaxial cable.  $\Sigma_+$  and  $\Sigma_-$  are the outer and inner boundary of  $S$ .

Denoting  $\delta > 0$  a small parameter, we consider a family of (thin) domains  $\Omega^\delta = \mathcal{G}_\delta(\Omega)$  where

$$\mathcal{G}_\delta : (x_1, x_2, z) \longrightarrow (\delta x_1, \delta x_2, z).$$

and  $\Omega$  is the disjoint union of a conducting domain  $\Omega_c$  and a dielectric one  $\Omega_d$ ,

$$\Omega_c = C \times \mathbb{R}, \quad \Omega_d = S \times \mathbb{R},$$

where  $C = C^+ \cup C^-$ ,  $C^+$  corresponding to the outer metallic shield and  $C^-$  to the inner metallic wire and  $S$  is non-simply connected, see Figure 1. Accordingly, we have, with obvious notation

$$\Omega^\delta = \Omega_d^\delta \cup \Omega_c^\delta.$$

We are interested in the solution  $(E^\delta, H^\delta)$  of 3D Maxwell's equation in  $\Omega^\delta$  :

$$\begin{cases} \varepsilon^\delta \partial_t E^\delta + \sigma^\delta E^\delta - \mathbf{curl} H^\delta = \mathbf{j}^\delta, \\ \mu^\delta \partial_t H^\delta + \mathbf{curl} E^\delta = \mathbf{0}, \end{cases} \quad (1)$$

with zero initial data. In  $\Omega_c^\delta$ ,  $(\varepsilon^\delta, \mu^\delta)$  are constant equal to  $(\varepsilon_c, \mu_c)$  and  $\mathbf{j}^\delta = \mathbf{0}$ . In  $\Omega_d^\delta$ ,  $(\varepsilon^\delta, \mu^\delta)$  do not depend on  $z$  and are obtained by a scaling in the transverse variable  $\mathbf{x}_T = (x_1, x_2)$  of fixed distributions in the reference domain  $\Omega_d$ , for instance

$$\varepsilon^\delta(\mathbf{x}_T, z) = \varepsilon(\mathbf{x}_T/\delta).$$

The source term  $\mathbf{j}^\delta$  is defined similarly, moreover it is compactly supported, it has no longitudinal component and is divergence free. The conductivity is weak in the dielectric  $\Omega_d^\delta$ , but very high in  $\Omega_c^\delta$ . More precisely

$$\sigma^\delta(\mathbf{x}_T, z) = \begin{cases} \delta^{-4} \sigma_c & \text{in } \Omega_c^\delta, \\ \delta \sigma(\mathbf{x}_T/\delta), & \text{in } \Omega_d^\delta. \end{cases} \quad (2)$$

Note that the  $O(\delta^{-4})$  magnitude of  $\sigma^\delta$  in  $\Omega_c^\delta$  gives rise to a skin depth in  $O(\delta^2)$ , small with respect to  $\delta$ .

Our approach consists in obtaining the formal behaviour of the solution for small  $\delta$ . To do so, we propose two distinct asymptotic expansions of the solution  $\Omega_d^\delta$  and  $\Omega_c^\delta$  that we match using transmission conditions. We present below our main results.

## Electromagnetic field in the dielectric

We introduce the following notations.

- $\nabla$  for the 2D transverse gradient in  $\mathbf{x}_T$ , identified to a 3D vector with third component 0,
- $S_\Gamma := S \setminus \Gamma$  where  $\Gamma$  is a cut that makes  $S_\Gamma$  simply connected (see Figure 1),
- $[\cdot]_\Gamma$  for the jump across  $\Gamma$  in the direction  $\mathbf{n}$ ,
- $\tilde{\nabla}$  is the 2D transverse gradient in  $S_\Gamma$ ,
- $\partial_{\mathbf{n}}$  is the normal derivative, and  $\partial_\tau \psi = \tilde{\nabla} \psi \cdot \boldsymbol{\tau}$  the tangential derivative.

We obtain that, for small  $\delta$  and all  $\mathbf{x}_T \in S^\delta$ ,

$$\begin{aligned} E^\delta(\mathbf{x}_T, z, t) &\sim V^\delta(z, t) \nabla \varphi_e(\mathbf{x}_T/\delta) \\ &+ \delta \left( \int_0^t V^\delta(z, s) ds \right) \nabla \varphi_r(\mathbf{x}_T/\delta), \\ &+ \delta \partial_z V^\delta(z, t) (\varphi_e - \varphi_m)(\mathbf{x}_T/\delta) \mathbf{e}_z, \end{aligned}$$

$$\begin{aligned} H^\delta(\mathbf{x}_T, z, t) &\sim I^\delta(z, t) \nabla \psi_m(\mathbf{x}_T/\delta), \\ &+ \delta \left( \int_0^t \partial_t^{\frac{1}{2}} I^\delta(z, s) ds \right) \nabla \psi_r(\mathbf{x}_T/\delta), \\ &+ \delta \partial_z I^\delta(z, t) (\psi_e - \psi_m)(\mathbf{x}_T/\delta) \mathbf{e}_z, \end{aligned}$$

where  $\mathbf{e}_z = (0, 0, 1)^t$ . Moreover:

- i) The potential  $\varphi_e \in H^1(S)$  satisfies,  
 $\operatorname{div} \varepsilon \nabla \varphi_e = 0 (S)$ ,  $\varphi_e = 0 (\Sigma_+)$ ,  $\varphi_e = 1 (\Sigma_-)$ ,  
and the same for  $\varphi_m$  with  $\mu^{-1}$  instead of  $\varepsilon$ .  
ii) The potential  $\psi_m \in H^1(S_\Gamma)$  satisfies

$$\operatorname{div} \mu \nabla \psi_m = 0 (S_\Gamma), \quad \partial_n \psi_m = 0 (\partial S),$$

and  $[\psi_m]_\Gamma = 1$ ,  $[\partial_n \psi_m]_\Gamma = 0$ . The same holds for  $\psi_e$  with  $\varepsilon^{-1}$  instead of  $\mu$ . Moreover

$$\int_S \mu \psi_e = \int_S \mu \psi_m = 0.$$

- iii) The function  $\varphi_r \in H_0^1(S)$  is the solution of

$$\operatorname{div} \varepsilon \nabla \varphi_r = -\operatorname{div} \sigma \nabla \varphi_e.$$

- iv) The function  $\psi_r \in H^1(S)$  satisfies

$$\operatorname{div} \mu \nabla \psi_r = 0 (S), \quad \mu \partial_n \psi_r = -\sqrt{\frac{\mu_c}{\sigma_c}} \partial_\tau^2 \psi_m (\partial S).$$

- v) The electric potential  $V^\delta(z, t)$  and current  $I^\delta(z, t)$  are 1D unknowns governed by generalized telegrapher's equations:

$$\begin{cases} C \partial_t V^\delta + \delta G V^\delta + \partial_z I^\delta = j, \\ L \partial_t I^\delta + \delta R \partial_t^{\frac{1}{2}} I^\delta + \partial_z V^\delta = 0, \end{cases} \quad (3)$$

where  $j(z, t)$  is an effective source term,

$$j(z, t) = \int_S \mathbf{j}(\mathbf{x}_T, z, t) \cdot \nabla \varphi_e(\mathbf{x}_T), \quad (4)$$

and  $\partial_t^{\frac{1}{2}}$  is the square root derivative in the sense of Caputo

$$\partial_t^{\frac{1}{2}} u(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial_\tau u(\tau)}{\sqrt{t-\tau}} d\tau.$$

As in [1], the capacity  $C$ , inductance  $L$  and conductance  $G$  are given by:

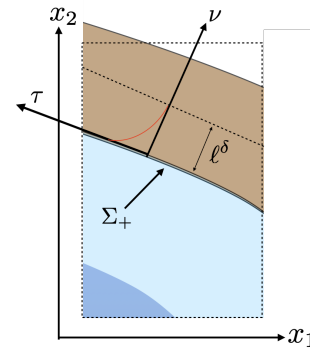
$$C = \int_S \varepsilon |\nabla \varphi_e|^2, \quad L = \int_S \mu |\widetilde{\nabla} \psi_m|^2, \quad G = \int_S \sigma |\nabla \varphi_e|^2.$$

Moreover, we obtain an explicit expression for the resistance  $R$ , which takes into account skin effects:

$$R = \int_{\partial S} \sqrt{\frac{\mu_c}{\sigma_c}} |\partial_\tau \psi_m|^2. \quad (5)$$

This generalizes formulas of the literature (see [2], chapter 13) already derived in very simple cases.

### Electric field in the outer conductor



In the rescaled conducting domain  $C_+$  the electromagnetic fields are described using tangential and normal coordinates  $(\tau, \nu)$ . The penetration depth  $\ell^\delta$  of the fields is in  $O(\delta)$ .

$L^+$  being the length of  $\Sigma_+$ , one shows that there exists a 3D field

$$E^+ : [0, L^+] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$$

such that in  $C_+^\delta \times \mathbb{R}$  and for small  $\delta$ ,

$$E^\delta(\mathbf{x}_T(\tau, \nu), z, t) \sim \delta^2 E^+(\tau/\delta, \nu/\delta^2, z, t),$$

(and a similar property holds for the magnetic field). The important fact is that the component  $E_z^+$  is solution of the 1D heat equation

$$\mu_c \sigma_c \partial_t E_z^+ - \partial_\nu^2 E_z^+ = 0, \quad (6)$$

and thus satisfies, at the boundary  $\nu = 0$ :

$$\partial_\nu E_z^+ + \sqrt{\mu_c \sigma_c} \partial_t^{\frac{1}{2}} E_z^+ = 0.$$

The above equation is used when writing transmission conditions across  $\Sigma_+^\delta \times \mathbb{R}$ .

This explains the appearance of  $\partial_t^{\frac{1}{2}}$  in the effective model (3).

### References

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