Pips (times) Pods: Dancing towards multiplicative thinking

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*This paper examines the design of an iPad touchscreen application (*TouchTimes*), that provides children with the opportunity for direct mathematical/multiplicative mediation through fingers and gestures. We describe the explorations of a pair of third-grade students with this technology, in which engagement with number in a multiplicative sense draws on a singular interaction between the eyes and both hands. Using a theoretical perspective informed by tool use and by embodiment in mathematical thinking and learning, we seek to gain insight into the affordances of the* TouchTimes *app in the development of multiplicative awareness in young children, with a specific focus on the multimodal nature of their mathematical interactions.*

Keywords: Touchscreen technology, multiplicative reasoning, primary, tangible.

Introduction

Multiplicative reasoning is the ability to work flexibly and efficiently with "the concepts, strategies and representations of multiplication (and division) as they occur in a wide range of contexts" (Siemon, Breed & Virgona, 2005, p. 2), concepts that include direct and indirect proportion. Among other things, such reasoning involves learners viewing situations of comparison in a multiplicative rather than an additive sense. As students progress to larger whole numbers – and then to decimals, fractions, percentages, ratios and proportions – multiplicative reasoning becomes key to a large number of mathematical situations found in upper elementary and middle school (Brown, Küchemann & Hodgen, 2010).

In the primary grades (K–3), however, the action of repeated addition is commonly used as the initial (and, for some, the sole) means for introducing and working with multiplication, and it tends to become firmly entrenched as the dominant perception of multiplicative situations, both for students *and* for primary teachers (Askew, 2018). This becomes problematic when students begin to engage with mathematics that requires a direct capacity to think multiplicatively (e.g. Siemon et al., 2005). Consequently, multiplication may prove a crucial turning point in student learning, possibly a turnstile for mathematical competence. Rather than rely exclusively on repeated addition, approaches need to be developed and implemented in the early grades that highlight the function aspect of multiplicative reasoning that can be so critical to future success with mathematics.

Since Vergnaud (1983) wrote about the conceptual field of multiplicative structures, considerable attention has been given to the comparison of quantities using multiplicative thinking. Extensive research has documented difficulties students have with employing it in middle school and beyond, which Brown, Küchemann & Hodgen (2010) claim has not improved since the 1970s. Limited student experience with different multiplicative situations is proposed as a contributing factor to this significant challenge (e.g. Downton & Sullivan, 2017). Furthermore, Askew (2018) contends that the lack of development of multiplicative reasoning in the primary grades is "a consequence of predominant approaches to teaching multiplication limiting access to opportunities through which thinking functionally can emerge" (p. 1).

Multiplicative reasoning and digital technology

Digital technology is providing new resources and means that show promise in supporting the mathematical learning of young children (e.g. Sedaghatjou & Campbell, 2017). In particular, the multi-touch affordances of *TouchCounts* (hereafter TC) allow children to produce and transform objects directly on an iPad screen using fingers and contact gestures, providing new counting and early arithmetic opportunities that are enabled by the app's visible, audible and tangible design. Sinclair and de Freitas (2014) show how the interaction in TC between fingers and eyes—between the tangible and the visual—enables new ways of thinking about number.

TouchTimes (TT), a novel extension of TC, is a gesture-based, multi-touch environment for multiplication designed to improve children's flexible and relational understanding of multiplication. Multiplicative rather than additive, TT allows children to create the "visual images of composite unit structures in multiplicative situations" (p. 306) that Downton and Sullivan (2017) argue are fundamental to developing multiplicative reasoning directly. Children can produce and transform objects directly on an iPad screen, using new gestural experiences of multiplication that provide direct feedback through both symbolic and visual representations. Our research focus involves exploring the influence of TT as a gesture-based modality for conceptualizing, visualizing, experimenting with and communicating about multiplicative relationships with students in primary grades (2–4).

Brief description of *TouchTimes*

TouchTimes is an iPad application, which when initially opened displays a blank screen that is split in half by a vertical bar (Figure 1a). A user can place and hold her fingers on one side of the screen to create coloured discs, which we call "pips". (In what follows, we presume that the user has chosen the left side of the screen, but the description is also symmetric for the right side of the screen.) Each finger that maintains continuous contact with the left side (LS) produces a different coloured pip (Figure 1b). When the user taps her finger(s) on the right side (RS) of the screen, a unit of coloured discs appears. These units, which we call "pods", are comprised of the coloured pips that correspond to those pips being created by the user's fingers in contact on the LS (Figure 1c). As each tap creates a new pod, *TouchTimes* displays the number sentence that corresponds with the pips and pods created by the user. When a finger is taken off a pod, it remains on the screen, but becomes slightly smaller, so that users can create numerous pods. As long as at least one finger remains in contact with the LS, the pips are maintained within the pods, but when the user removes all fingers, the pods disappear ("multiplying by 0"). Contact with the screen can be made either one finger at a time or several fingers simultaneously. Pods can be dragged into the trash, at the bottom of the screen.

Pimm and Sinclair (2015) note that *TouchCounts* 'takes care of the counting' through both symbolic and auditory means in response to the user's fingered requests. Although done in a different manner, TT 'takes care of the multiplying', both in terms of making sure that the pods on the RS are reflective of the number of pips on the left, and in terms of ensuring that the equation on the screen corresponds to the pips and pods that are displayed in response to the fingered requests of the user.

Downton and Sullivan (2017) argue "that the co-ordination of composite units is the core of multiplication, and that young children's (8 year-olds) concept of multiplication is based on the meaning they give to the composite units they construct" (p. 306). Thinking multiplicatively involves the ability to simultaneously think about units of one and units of more than one. TT embodies a multiplicative model that involves the co-ordination of two quantities similar to Figure 1d below (Boulet, 1998, p. 13). One way to conceptualise this action is to see the LS touches as the number of pips which are then unitised into pods, with the pods then unitised into the product, as in the Davydovian approach (see Boulet, ibid.). In this view, the sentence 3 x 4 = 12 is read as the multiplicand times the multiplier equals the product, which reverses the typical North American approach (3)(4x). Of course, it is also possible to see the pods as being groups of pips, which can be understood in terms of repeated addition. However, the simultaneity of the two-handed touching retains less of the temporal, sequential sense of repeated addition. Part of the research goal is to develop tasks that promote a more multiplicative sense of multiplication.

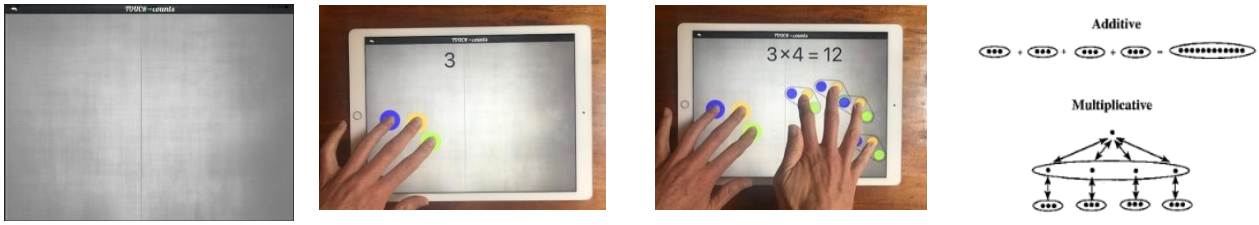


Figure 1: (a) Initial screen of *TT*; (b) Creating pips; (c) Creating pods; (d) Multiplicative model

Theoretical Framing

The theoretical orientation of this study draws principally on theories of embodiment and the relation between bodily movement and mathematical meaning-making (see Nemirovsky et al., 2013; Radford, 2009). We take the monist, ontological position found in inclusive materialism (de Freitas & Sinclair, 2014) on the nature of body and mind, which does not subordinate sensorimotor actions to thinking, but instead recognizes the way in which new ways of moving one's body *are* new ways of thinking. For this reason, we are less interested in studying TT as a manipulative involving acts of moving structures and are more interested in the *structured acts of moving* that TT modulates. Given this orientation, and the gesture-rich design of TT, our focus will include not only verbal explanations but also gesture-based actions, as we are particularly interested in the structured acts of gesturing that arise through the use of the app. The epistemic and communicative nature of gestures has been well documented in the literature (see Sinclair & de Freitas, 2014) and warrants our attention to gestures as particularly relevant structured acts of moving. Since the multitouch environment also enables children to work together, we will also focus on jointly structured acts of moving. Therefore, we will be investigating how a pair of students' interactions with TT prompts new gestures and how these new structured acts of moving are related to multiplicative thinking.

Methods

The data for this paper comes from an exploratory conversation conducted by one of the authors as part of an iterative design experiment aimed at refining the TT prototype and developing appropriate tasks for use with grade two and three children. After the researcher requested a pair of

students who had not yet used TT (three pairs of children had already participated), the two girls, whom we refer to as Jacy and Kyra, were selected by their classroom teacher to explore TT with the researcher. This interaction occurred in an elementary school in a culturally diverse and affluent neighbourhood in British Columbia, where the interviewer worked for approximately 30 minutes with the pair. A video-recording of this interaction was created, and the drawings produced by the girls were kept. The children were initially given time to become familiar with TT through independent exploration prior to any specific requests or questions from the interviewer. Given that this was the pair's first encounter with TT, it provided an opportunity to observe how they made sense of the app and if their interactions with TT would lead to an ability to identify certain multiplicative aspects with distinct handedness. After a period of free exploration that lasted about seven minutes, the interviewer began to use prompts to focus the attention of the children on certain features of their creations on the iPad screen. Using TT, in conjunction with the significant presence of an adult, the initial aim of this research was to understand better the impact of the affordances of TT on young children's multiplicative thinking, with a particular focus on the multimodal and joint nature of the mathematical activity of two girls working together on a single iPad.

Data analysis

In order to account for the multimodal, distributed nature of the phenomenon seen in the video-recordings, following our theoretical tenets, we produced an *orchestral transcription* of ten-second increments involving three separate, but interacting modalities: voices, hands and the iPad screen itself (Figure 2). The top three rows were designated for the voices of each child and the interviewer, thus providing a way to sequence the speaking visually in a manner that would effectively display overlapping voices. Descriptions of what the children's hands were doing on both sides of the iPad screen, were sequenced in time with the voices transcribed in the section above.

As the importance of the iPad screen itself became apparent, additional rows were created to reflect what could be seen on the top, left and right sides of the iPad screen, supplemented by screen shots to illustrate these ideas. The orchestral transcription enabled us easily to identify patterns of structured acts of moving throughout the course of the interaction, by comparing the images.

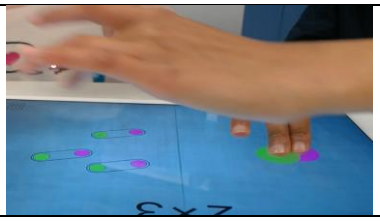
Voices	Jacy	Laughs……	
	Kyra	Wait, now I'll make a three. And then… laughs They're dancing! Laughs	
	Interviewer		
Hands	Left Screen	3-finger simultaneous touch (RH)	3 sequential finger touches
	Right Screen	Pointer finger touch (RH)	Hold two fingers
iPad Screen	Top of Screen	3 3x1=3	2x1, 2x2, 2x3
			
	Left Screen	3 pips	
	Right Screen	1 pod	1, 2, 3 pods

Figure 2: Orchestral transcription

Results

From the orchestral transcript we chose two intervals that illuminate some of the gestures made by the girls during their exploration of TT that we see as being relevant to multiplicative thinking. Prior to presenting these two intervals, however, we describe the pair's initial interactions with TT, which began with the researcher giving the pair permission to "play a little bit". Each girl placed and removed her pointer finger on and off the screen in what appeared to be random motions. After approximately fifteen seconds, Jacy placed her thumb on the screen in addition to her pointer finger, creating two pips. Kyra next touched the screen, creating a pod composed of two pips (Figure 3a). Jacy then began to use both hands. It is shortly thereafter that Jacy said, "How do you get those mini ones?", thus marking a transition from random tapping to more intentional actions on the iPad screen.

Disappearing pips and 'dancing' pods

Approximately five minutes into the exploration, Jacy instructed Kyra to "Wait, just press a lot. Press your whole hand." Kyra responded by placing all five RH fingers simultaneously on the LS and waited while Jacy created pods on the RS one at a time with her index finger. The top of the iPad screen read 5x1, 5x2, 5x3, 5x4, 5x5, 5x6 until Kyra abruptly removed her hand from the screen, causing both girls to laugh. At this point, Jacy began directing the creation of pips through the placement and removal of Kyra's fingers on the LS: when Kyra returned her five fingers to the iPad, Jacy physically removed her partner's pinky and thumb from the screen, causing some confusion. Jacy tried again, telling Kyra to "Wait. Put those [indicating Kyra's RH fingers] and then take up your pinky and thumb." Kyra placed three fingers on the LS, while Jacy created one pod on the RS. This was clearly not what Jacy was after, and she further instructed Kyra to "Press your pinky and thumb away. No wait. Put your pinky and thumb down [Jacy physically presses Kyra's pinky and thumb down (Figure 3b)] and now take them away." Each time Kyra's pinky and thumb touched the screen, the resulting pips changed colour, and the composition of the pod that Jacy was 'holding' changed from three pips to five pips and back to three pips as Kyra alternated rhythmically, making the pod appear to be swinging back and forth (Figure 3c). Jacy created a second pod and at this point Kyra pointed to the pods with her free hand and declared with a laugh that "They're dancing!".

Figure 3: (a) Two fingers; (b) Pinky and thumb down; (c) Pinky and thumb up

Throughout this brief episode, Jacy became interested in exploring how the creation and deletion of pips on the LS affected the shape and colour of the pods on the RS, as demonstrated by both her

comments and her actions. This complex interplay between the girls and TT involved a coordination of two pairs of hands, which resulted in a *joint holding and repetitive-tapping gesture.* The appeal of the 'dancing' pod seemed to draw the girls' attention to the relation between the RS and LS finger touches, which we see as significant in terms of multiplicative thinking. It involved the intentional co-ordination of two quantities: the multiplicand (three or five pips) and the multiplier (the one pod). The girls' joint gestures expressed their attention to the way in which the pods on the RS are directly affected by changes related to the number of pips controlled by the finger actions on the LS.

Counting by fours and fives

After seven minutes of exploration, the interviewer asked Kyra to put four fingers down, which she did with a simultaneous four-finger gesture, creating four pips on the LS. Without prompting, Jacy created a single pod on the RS with her index finger (Figure 4a). When questioned about what she noticed, Jacy replied, "Four times one equals four". She then created additional pods with her thumb while maintaining her index finger on the original pod (Figure 4b). When asked how Jacy could make five with one finger on the RS, Kyra's thumb touched the screen to create five pips, while Jacy continued to add pods on the RS (Figure 4c). After a pause, the interviewer asked, "Are you making a five now?" At this point Jacy, who had been creating additional pods with her thumb, glanced towards the top of the screen which displayed 5x10=50, then 5x11=55. She said, "Fifty-five, sixty…" and with excitement, "Wait! It's counting up by fives! Sixty-five, seventy, seventy-five, eighty, eighty-five, ninety, ninety-five…". The pair were then asked what they could do to count by fours. Kyra removed one of her five fingers from the screen. Jacy said, "Count up by fours?" and physically moved closer to the left side of the screen, causing Kyra to remove her hands. While placing her thumb and all fingers except her pinky on the screen, Jacy explained that you would "Put four fingers down". Meanwhile, Kyra used her index finger on the RS (reaching over Jacy's arm) to create pods (Figure 4d). Jacy then said, "Two, four, six, eight, twelve" then "wait, what?" after she noticed the product displayed on the iPad screen go from eight to twelve. She silently observed the next product and then began to count aloud again: "Twenty, twenty-four, twenty-eight…We're counting by fours."

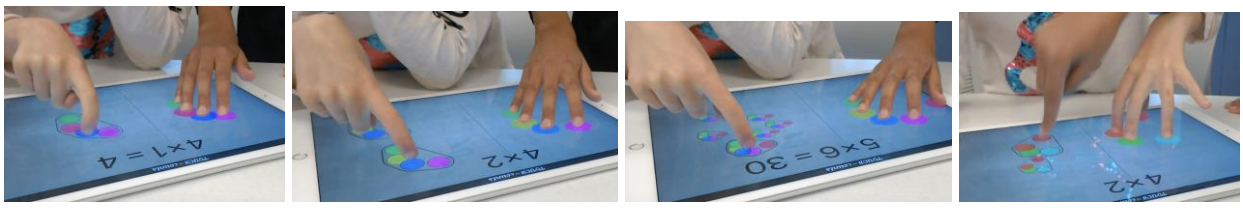


Figure 4: (a) First pod of 4; (b) Second pod of 4; (c) Many pods of 5; (d) Crossover pod making

The girls had transitioned away from single touches and were now making a *simultaneous touch gesture,* as in the four-finger gesture described above. The interviewer's first question was meant to draw attention to the number/colour of pips in the 4-pod they had created, but Jacy described the multiplication statement instead. The interviewer then hoped to elicit the making of a 5-pod in order to prompt the unitising of five pips into one pod. With Kyra's additional finger on the LS, Jacy did indeed begin creating 5-pods, but she did not seem to be aware of the shift from 4-pods until she

noticed the multiplication statements, and that the product was increasing by five with each RS touch. This induced a particular kind of *joint skip-counting gesture* in which one girl was responsible for holding pips while the other girl's sequential touches created pods. The pair then switched roles to skip-count by four, with Jacy holding pips and Kyra sequentially making pods. Unlike the skip-counting that occurs in many classrooms, where children count intransitively by the unit ("four, eight, twelve", etc.), the skip-counting in TT explicitly involves gesturally expressing the unit (the pips) *and* the number of units (the pods).

Jacy's counting self-correction seems to indicate a shift from rote counting to co-ordinating the *joint skip-counting gesture* (tangible expression) with the products (symbolic expression) that she could see on the iPad screen. When asked how Jacy could create a 5-pod, Kyra who was in control of creating the pips, immediately added a finger to the screen and later removed a finger when prompted to count by fours. Kyra was controlling the effects of the multiplier (the pips) on the multiplicand (the pods) with her fingers. Jacy, however, did not attend to the effects of each added pod on the product until after she noticed the product changing in the equation at the top of the iPad screen. Skip-counting in TT using the *joint skip-counting gesture* involves co-ordinating two quantities (instead of one) and is therefore more multiplicative than additive.

Discussion

The intent of the TT design was for learners to notice the relation between the number and colour of pips, and the shape and content of the pods, as this is the basis for the multiplicative operation. Although the girls created numerous pips and pods, it was not until the *joint holding and repetitive-tapping* gesture that they seemed to co-ordinate this relation. Indeed, it appeared to be the shifting pod shape, and then the changing pip colour, that initially drew their attention. The *joint holding and repetitive-tapping* gesture, which arose from manipulating the screen in exploratory ways, became a gesture for expressing the relation between pips and pods. The girls were able to make visible the effect of changing the unit through the tapping on and releasing of pips, which resulted in the changing size, shape and colour of the pods. When there was more than one pod on the screen (as in Figure 4c), the *joint holding and repetitive-tapping* gesture produced the multiplicative effect in which one new tap produced a new pip in *every* pod *simultaneously*.

In the previous section, we outlined how the joint skip-counting gesture can be viewed as more multiplicative than additive. However, we think that a gestural shift from tapping one finger at a time to create new pods, to tapping several fingers simultaneously would provide an even stronger multiplicative effect. We know from prior *TouchCounts* research that sequential tapping was more frequent than simultaneous tapping, during children's initial interactions. Since the girls rarely created several pods simultaneously, we hypothesise that such a gesture involves a more difficult co-ordination and may need to be prompted by a particular task.

Another significant aspect of the actions described throughout this paper involves the joint nature of the girls' interactions. In both of the gestures identified, the pair must co-ordinate their hand movements and there were a few instances during the 30-minute exploration in which one of the girls "took over" TT. We wonder how a situation with two children co-ordinating multiple hands would differ from one in which a single child coordinates only her own hands.

Conclusion

Designed to support the development of multiplicative thinking, TT provides young learners with ways of thinking about multiplication that are not solely dependent upon the use of repeated addition. Two intervals were described in the 30-minute episode reported above, in which the girls created and sustained particular structured ways of moving their hands that we hypothesise as being relevant to the development of multiplicative thinking. The two primary gestures discussed were the *joint holding and repetitive-tapping* and the *joint skip-counting* gestures. The former prompted and enabled the girls to attend to the relation between the number/colour of pips and the pods, and thus the co-ordination of the two quantities. The latter enabled the girls to produce multiples of a number determined by the number of pips. Attention to the symbolic expression on the screen was particularly salient in the second interval. We highlight that the girls primarily engaged in free exploration throughout this experiment, with few prompts from the interviewer. In future explorations, we intend to build upon the insights gathered here to design tasks that can effectively prompt and support similar types of gestures, and to link these gestures to other off-line and TT-based actions.

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