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PARABOLIC SL_r -OPERS

INDRANIL BISWAS, SORIN DUMITRESCU, AND CHRISTIAN PAULY

ABSTRACT. We define SL_r -opers in the set-up of vector bundles on curves with a parabolic structure over a divisor. Basic properties of these objects are investigated.

1. INTRODUCTION

The notion of oper was introduced by Beilinson and Drinfeld [BD] as an essential ingredient in the geometric Langlands program; they were influenced by earlier work of Drinfeld and Sokolov [DS1], [DS2]. Since then opers have appeared in numerous contexts and by now this notion has been established as an important topic.

Let X be a compact connected Riemann surface and G a semisimple affine algebraic group defined over \mathbb{C} ; fix a Borel subgroup $B \subset G$. A G -oper on X is a holomorphic principal G -bundle E_G over X , together with

- a holomorphic reduction of structure group $E_B \subset E_G$ of E_G to $B \subset G$, and
- a holomorphic connection on E_G with respect to which the reduction E_B satisfies a certain transversality condition. The transversality condition in question is described below for the case of $G = \mathrm{SL}_r(\mathbb{C})$.

In case of $G = \mathrm{SL}_r(\mathbb{C})$, an $\mathrm{SL}_r(\mathbb{C})$ -bundle corresponds to a holomorphic vector bundle E on X of rank r such that $\bigwedge^r E = \mathcal{O}_X$. A B -reduction of it is given by a complete flag

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{r-1} \subset E_r = E$$

of holomorphic subbundles such that $\mathrm{rank}(E_i) = i$. A holomorphic connection D on this filtered bundle defines an $\mathrm{SL}_r(\mathbb{C})$ -oper if

- (1) $D(E_i) \subset E_{i+1} \otimes K_X$ for all i , where K_X is the holomorphic cotangent bundle of X ,
- (2) the corresponding second fundamental form $E_i/E_{i-1} \rightarrow (E_{i+1}/E_i) \otimes K_X$ is an isomorphism for all $1 \leq i \leq r-1$, and
- (3) $\bigwedge^r D = d$, the standard de Rham differentiation on \mathcal{O}_X .

The underlying holomorphic vector bundle of a $\mathrm{SL}_r(\mathbb{C})$ -oper is unique up to tensor product with a finite set of line bundles of order r : the underlying bundle of a $\mathrm{SL}_2(\mathbb{C})$ -oper is the unique nontrivial extension V_0 of $K_X^{-1/2}$ by $K_X^{1/2}$, for some theta characteristic $K_X^{1/2}$ on X . More generally, the underlying bundle of a $\mathrm{SL}_r(\mathbb{C})$ -oper is, up to tensor product with an r -torsion line bundle, the symmetric power $\mathrm{Sym}^{r-1}(V_0)$, where V_0 is the underlying bundle of a $\mathrm{SL}_2(\mathbb{C})$ -oper.

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The isomorphism classes of $\mathrm{SL}_2(\mathbb{C})$ -opers are in bijection with the projective structures on X ; see [Gu] for projective structures on X (see also [GKM]).

Fix finitely many distinct points x_i on X , and consider the effective divisor $S = x_1 + \cdots + x_m$. A quasiparabolic structure, over S , on a holomorphic vector bundle E on X is a decreasing filtration of subspaces of each fiber E_{x_i} (this filtration need not be complete). A parabolic structure on E is a quasiparabolic structure together with weights for the subspaces that are nonnegative real numbers strictly less than 1. For any (decreasing) filtration, these numbers are actually strictly increasing. A vector bundle with a parabolic structure is called a parabolic vector bundle [MS], [MY].

Our aim here is to introduce and study the SL_r -opers in the context of parabolic vector bundles.

The article is organized in the following way. Section 2 presents general facts about jet bundles and differential operators on Riemann surfaces. The SL_2 -opers in the parabolic set-up are defined and studied in Section 4. The main result of that section (Theorem 4.6) says that the space of parabolic SL_2 -opers with fixed parabolic weights is an affine space for the vector space $H^0(X, K_X^2 \otimes \mathcal{O}_X(S))$. The starting point is to actually identify the rank two parabolic bundle underlying parabolic SL_2 -opers. The underlying parabolic bundle for SL_2 -opers is investigated in Section 3. Parabolic SL_r -opers are defined in Section 5. Section 6 focuses on the case where the parabolic weights are rational of special type and r is odd. In this case we show that parabolic SL_r -opers on X are in natural bijection with invariant SL_r -opers on a ramified Galois covering Y over X equipped with an action of the Galois group (see Theorem 6.3). This result uses in an essential way the correspondence studied in [Bi1], [Bo1], [Bo2], and only works for certain rational parabolic weights under the assumption that r is odd. In this case we deduce a natural parametrization of the space of parabolic SL_r -opers on X , with given (special rational) parabolic weights (see Theorem 6.4). In the last section we discuss alternative definitions of SL_r -opers and related questions.

Finally we would like to add that similar constructions have recently been carried out by Y. Wakabayashi ([W], Theorem A).

2. DIFFERENTIAL OPERATORS

Let X be a compact connected Riemann surface. The holomorphic cotangent bundle of X will be denoted by K_X . Let $p_i : X \times X \rightarrow X$ be the projection to the i -th factor, where $i = 1, 2$. Let

$$\Delta = \{(x, x) \mid x \in X\} \subset X \times X$$

be the reduced diagonal divisor. We shall identify Δ with X using the map $x \mapsto (x, x)$.

Take a holomorphic vector bundle V on X . For any integer $k \geq 0$, the k -th jet bundle $J^k(V)$ for V is defined to be the direct image

$$J^k(V) := p_{1*}((p_2^*V)/((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))) \rightarrow X.$$

The line bundle $\mathcal{O}_{X \times X}(-\Delta)|_\Delta$ is identified with K_X by the Poincaré adjunction formula; more precisely, for any holomorphic coordinate function z defined on any open subset $U \subset X$, the section of $\mathcal{O}_{X \times X}(-\Delta)$ over $\Delta \cap (U \times U)$ given by $z \circ p_2 - z \circ p_1$ coincides with the

section dz of $K_X|_U$. This identification between $\mathcal{O}_{X \times X}(-\Delta)|_\Delta$ and K_X produces a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{X \times X}(-(k+2)\Delta) \longrightarrow \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow K_X^{\otimes(k+1)} \longrightarrow 0 \quad (2.1)$$

on $X \times X$, where $K_X^{\otimes(k+1)}$ is supported on Δ . Consider the short exact sequence of sheaves on $X \times X$

$$\begin{aligned} 0 \longrightarrow ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) / ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+2)\Delta)) \longrightarrow \\ (p_2^*V) / ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+2)\Delta)) \longrightarrow (p_2^*V) / ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow 0. \end{aligned} \quad (2.2)$$

Let

$$\begin{aligned} 0 \longrightarrow p_{1*}(((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) / ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+2)\Delta))) \longrightarrow \\ p_{1*}((p_2^*V) / ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+2)\Delta))) \longrightarrow p_{1*}((p_2^*V) / ((p_2^*V) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))) \longrightarrow 0. \end{aligned}$$

the direct image of it on X by the map p_1 ; note that the higher direct images vanish because the supports of the sheaves in the short exact sequence in (2.2) are finite over X . Using (2.1), this exact sequence of direct images becomes the short exact sequence of vector bundles

$$0 \longrightarrow V \otimes K_X^{k+1} \longrightarrow J^{k+1}(V) \xrightarrow{q_V^k} J^k(V) \longrightarrow 0 \quad (2.3)$$

on X .

For any holomorphic vector bundle W on X , any \mathcal{O}_X -linear homomorphism $\delta : V \longrightarrow W$ produces a homomorphism

$$\delta^{(k)} : J^k(V) \longrightarrow J^k(W) \quad (2.4)$$

for every $k \geq 0$.

Consider the homomorphism q_V^0 in (2.3). Let

$$(q_V^0)^{(1)} : J^1(J^1(V)) \longrightarrow J^1(V)$$

be the corresponding homomorphism in (2.4). On the other hand, we have the homomorphism

$$q_{J^1(V)}^0 : J^1(J^1(V)) \longrightarrow J^1(V)$$

by setting $k = 0$ and $J^1(V)$ in place of V in (2.3). Note that the following two compositions

$$J^1(J^1(V)) \xrightarrow{(q_V^0)^{(1)}} J^1(V) \xrightarrow{q_V^0} V$$

and

$$J^1(J^1(V)) \xrightarrow{q_{J^1(V)}^0} J^1(V) \xrightarrow{q_V^0} V$$

coincide. The kernel of the homomorphism $(q_V^0)^{(1)} - q_{J^1(V)}^0$ coincides with $J^2(V)$. Therefore, we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow J^2(V) \xrightarrow{\mathbf{t}} J^1(J^1(V)) \xrightarrow{(q_V^0)^{(1)} - q_{J^1(V)}^0} V \otimes K_X \longrightarrow 0 \quad (2.5)$$

on X .

The sheaf of *holomorphic differential operators* of order k from V to another holomorphic vector bundle W is the sheaf of holomorphic sections of the holomorphic vector bundle

$$W \otimes J^k(V)^* = \text{Hom}(J^k(V), W) =: \text{Diff}^k(V, W). \quad (2.6)$$

Using the inclusion $V \otimes K_X^k \hookrightarrow J^k(V)$ in (2.3), we have a surjective homomorphism of vector bundles

$$\mathrm{Diff}^k(V, W) \longrightarrow \mathrm{Hom}(V \otimes K_X^k, W) = \mathrm{Hom}(V, W) \otimes T_X^{\otimes k}; \quad (2.7)$$

it is known as the *symbol* map.

3. A RANK TWO PARABOLIC BUNDLE

Let X be a compact connected Riemann surface. The genus of X will be denoted by g . Fix a finite nonempty subset

$$S := \{x_1, \dots, x_m\} \subset X, \quad (3.1)$$

(so $m \geq 1$), and also fix a function

$$\mathbf{c} : S \longrightarrow \{t \in \mathbb{R} \mid t > 1\}; \quad (3.2)$$

for notational convenience, the real number $\mathbf{c}(x_i)$ will also be denoted by c_i .

We shall assume that $m = \#S$ is an even integer. Fix a pair

$$(\mathbb{L}, \varphi_0), \quad (3.3)$$

where \mathbb{L} is a holomorphic line bundle on X such that $\mathbb{L}^{\otimes 2}$ is isomorphic to $\mathcal{O}_X(-S)$, and

$$\varphi_0 : \mathbb{L}^{\otimes 2} \longrightarrow \mathcal{O}_X(-S)$$

is a holomorphic isomorphism of line bundles.

Let $(K_X^{1/2}, I_X)$ be a theta characteristic on X ; this means that $K_X^{1/2}$ is a holomorphic line bundle on X of degree $g - 1$, and

$$I_X : (K_X^{1/2})^{\otimes 2} \longrightarrow K_X$$

is a holomorphic isomorphism of holomorphic line bundles. We shall identify $(K_X^{1/2})^{\otimes (2i+j)}$ with $K_X^{\otimes i} \otimes (K_X^{1/2})^{\otimes j}$ using $I_X^{\otimes i} \otimes \mathrm{Id}_{(K_X^{1/2})^{\otimes j}}$. The line bundle $(K_X^{1/2})^{\otimes i}$ will be denoted by $K_X^{i/2}$. For notational convenience, the above isomorphism I_X will be suppressed, and a theta characteristic will be denoted simply by $K_X^{1/2}$. The isomorphism I_X will be used without mentioning it.

Using Serre duality, we have

$$H^1(X, \mathrm{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})) = H^1(X, K_X) = H^0(X, \mathcal{O}_X)^* = \mathbb{C}$$

(the line bundle \mathbb{L} is as in (3.3)). Hence

$$1 \in H^1(X, \mathrm{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})) = \mathbb{C}$$

corresponds to a nontrivial extension

$$0 \longrightarrow K_X^{1/2} \otimes \mathbb{L} \longrightarrow E \xrightarrow{p_0} K_X^{-1/2} \otimes \mathbb{L} \longrightarrow 0. \quad (3.4)$$

So the short exact sequence in (3.4) does not split holomorphically.

The holomorphic vector bundle E constructed in (3.4) will be equipped with a parabolic structure; see [MS], [MY] for parabolic bundles.

The parabolic divisor for the parabolic structure on E is the subset S in (3.1). For any $x_i \in S$, the quasiparabolic filtration of E_{x_i} is

$$0 \subset (K_X^{1/2} \otimes \mathbb{L})_{x_i} \subset E_{x_i}, \quad (3.5)$$

where $(K_X^{1/2} \otimes \mathbb{L})_{x_i}$ is the fiber of the line subbundle $K_X^{1/2} \otimes \mathbb{L} \hookrightarrow E$ in (3.4). The parabolic weight of this line $(K_X^{1/2} \otimes \mathbb{L})_{x_i}$ is $\frac{2c_i-1}{2c_i}$, while the parabolic weight of E_{x_i} is $\frac{1}{2c_i}$, where $c_i := \mathbf{c}(x_i)$ with \mathbf{c} being the function in (3.2); note that $\frac{1}{2c_i} < \frac{2c_i-1}{2c_i}$. The parabolic vector bundle thus obtained will be denoted by E_* .

The parabolic degree of the above parabolic bundle E_* is:

$$\text{par-deg}(E_*) = \text{degree}(E) + \sum_{i=1}^m \left(\frac{2c_i-1}{2c_i} + \frac{1}{2c_i} \right) = \text{degree}(\mathbb{L}^{\otimes 2}) + m = 0. \quad (3.6)$$

A *logarithmic connection* on a holomorphic vector bundle F singular over S is a holomorphic differential operator

$$D : F \longrightarrow F \otimes K_X \otimes \mathcal{O}_X(S)$$

of order one such that

$$D(fs) = f \cdot D(s) + s \otimes df, \quad (3.7)$$

where s is any locally defined holomorphic section of F and f is any locally defined holomorphic function on X . So a logarithmic connection D on F gives a \mathcal{O}_X -linear homomorphism

$$J^1(F) \longrightarrow F \otimes K_X \otimes \mathcal{O}_X(S) \quad (3.8)$$

(see (2.6)). The Leibniz condition in (3.7) implies that the symbol of D is

$$\text{Id}_F \in H^0(X, \text{End}(F) \otimes \mathcal{O}_X(S)).$$

Recall that a logarithmic connection on a Riemann surface is flat (its curvature vanishes identically) because the sheaf of holomorphic two forms on X is the zero sheaf.

Note that for any $x_i \in S$, the fiber $(K_X \otimes \mathcal{O}_X(S))_{x_i}$ of $K_X \otimes \mathcal{O}_X(S)$ over x_i is identified with \mathbb{C} using the Poincaré adjunction formula; more precisely, for any holomorphic coordinate function z defined around x_i , with $z(x_i) = 0$, the evaluation of the local section $\frac{1}{z}dz$ of $K_X \otimes \mathcal{O}_X(S)$ at the point x_i is independent of the choice of the coordinate function z . Consequently, we have the isomorphism

$$\mathbb{C} \longrightarrow (K_X \otimes \mathcal{O}_X(S))_{x_i}, \quad b \longmapsto b \cdot \frac{1}{z}dz \Big|_{z=x_i}.$$

The composition

$$F \xrightarrow{D} F \otimes K_X \otimes \mathcal{O}_X(S) \longrightarrow (F \otimes K_X \otimes \mathcal{O}_X(S))_{x_i} = F_{x_i}$$

(recall that $(K_X \otimes \mathcal{O}_X(S))_{x_i} = \mathbb{C}$) is \mathcal{O}_X -linear, so it produces an endomorphism of the fiber F_{x_i} . This element of $\text{End}(F_{x_i})$ is called the residue of D at x_i , and it is denoted by $\text{Res}(D, x_i)$ (see [De]).

A *connection* on the parabolic bundle E_* is a logarithmic connection D on E , singular over S , such that

- (1) for all $1 \leq i \leq m$, the residue $\text{Res}(D, x_i) \in \text{End}(E_{x_i})$ preserves the line $(K_X^{1/2} \otimes \mathbb{L})_{x_i}$ in (3.5), and $\text{Res}(D, x_i)$ acts on $(K_X^{1/2} \otimes \mathbb{L})_{x_i}$ as multiplication by the parabolic weight $\frac{2c_i-1}{2c_i}$, and
- (2) the endomorphism of the quotient $E_{x_i}/(K_X^{1/2} \otimes \mathbb{L})_{x_i} = (K_X^{-1/2} \otimes \mathbb{L})_{x_i}$ (see (3.4)) induced by $\text{Res}(D, x_i)$ coincides with multiplication by the parabolic weight $\frac{1}{2c_i}$.

(See [BL, Section 2.2].)

Note that for a logarithmic connection D on E defining a connection on E_* , the trace of the residue of D at each point of S is 1.

Remark 3.1. We observe that there exist logarithmic connections D on E with more general residue maps $\text{Res}(D, x_i)$. We need the connection D to be compatible with the parabolic structure.

A connection D on E_* is called *reducible* if there is a holomorphic line subbundle $L' \subset E$ preserved by D , meaning

$$D(L') \subset L' \otimes K_X \otimes \mathcal{O}_X(S). \quad (3.9)$$

A connection on E_* is called *irreducible* if it is not reducible.

Proposition 3.2. *Assume that $\text{genus}(X) = g \geq 1$. Then the following two hold:*

- (1) *The parabolic bundle E_* admits a connection.*
- (2) *Any connection on E_* is irreducible.*

Proof. We shall first prove (2). To prove (2) by contradiction, let D be a connection on E_* which is reducible. Let $L' \subset E$ be a holomorphic line subbundle such that (3.9) holds. Let $L'_* \subset E_*$ be the parabolic line subbundle given by L' equipped with the parabolic structure induced from E_* . Since (3.9) holds, in particular, L'_* admits a connection, we have

$$\text{par-deg}(L'_*) = 0 \quad (3.10)$$

[BL, p. 598, Lemma 4.2], [Oh, p. 16, Theorem 3].

Equip the quotient bundle $K_X^{-1/2} \otimes \mathbb{L}$ in (3.4) with the parabolic structure induced by the parabolic structure of E_* . For this parabolic line bundle we have

$$\text{par-deg}(K_X^{-1/2} \otimes \mathbb{L}) = -g + 1 - \frac{m}{2} + \sum_{i=1}^m \frac{1}{2c_i} < 0 = \text{par-deg}(L'_*) \quad (3.11)$$

(see (3.10)).

Let Φ denote the composition

$$L' \hookrightarrow E \xrightarrow{p_0} K_X^{-1/2} \otimes \mathbb{L}$$

(see (3.4)). This Φ is a homomorphism of parabolic line bundles. Indeed, if the parabolic weight of L'_* at a point $x_i \in S$ is strictly bigger than the parabolic weight of $K_X^{-1/2} \otimes \mathbb{L}$ at x_i , then

$$L'_{x_i} = (K_X^{1/2} \otimes \mathbb{L})_{x_i} \subset E_{x_i},$$

and hence $\Phi(x_i) = 0$. Since Φ is homomorphism of parabolic line bundles, from (3.11) we conclude that $\Phi = 0$; note that there is no nonzero parabolic homomorphism from a parabolic line bundle of higher parabolic degree to a parabolic line bundle of lower parabolic degree. Consequently, L' coincides with the line subbundle $K_X^{1/2} \otimes \mathbb{L} \subset E$ in (3.4).

On the other hand, the parabolic degree of the parabolic line subbundle $K_X^{1/2} \otimes \mathbb{L}$ equipped with the induced parabolic structure is

$$g - 1 - \frac{m}{2} + \sum_{i=1}^m \frac{1 - 2c_i}{2c_i} > 0 = \text{par-deg}(L'_*).$$

Therefore, L' can't coincide with $K_X^{1/2} \otimes \mathbb{L}$. In view of this contradiction we conclude that the connection D on E_* is irreducible.

To prove (1) in the proposition, it suffices to show that the vector bundle E is indecomposable. Indeed, any indecomposable parabolic vector bundle of parabolic degree zero admits a connection [BL, p. 599, Proposition 4.1], and hence from (3.6) it follows that E_* admits a connection if E is indecomposable.

We first assume that $\text{genus}(X) = g > 1$.

To prove that E is indecomposable by contradiction, assume that

$$E = L \oplus M, \tag{3.12}$$

where L and M are holomorphic line bundles on X with $\text{degree}(M) \geq \text{degree}(L)$. Therefore, we have

$$\text{degree}(M) \geq \frac{\text{degree}(L) + \text{degree}(M)}{2} = \frac{\text{degree}(E)}{2} = \text{degree}(K_X^{-1/2} \otimes \mathbb{L}) + g - 1. \tag{3.13}$$

From this it follows that $H^0(X, \text{Hom}(M, K_X^{-1/2} \otimes \mathbb{L})) = 0$, because we have $\text{degree}(M) > \text{degree}(K_X^{-1/2} \otimes \mathbb{L})$. In particular, the composition

$$M \hookrightarrow E \xrightarrow{p_0} K_X^{-1/2} \otimes \mathbb{L} \tag{3.14}$$

vanishes identically (see (3.4) (for p_0) and (3.12)). Consequently, the subbundle M of E in (3.12) coincides with the subbundle $K_X^{1/2} \otimes \mathbb{L}$ in (3.4). This implies that the decomposition in (3.12) gives a holomorphic splitting of the short exact sequence in (3.4). But, as noted earlier, the short exact sequence in (3.4) does not split holomorphically. So the vector bundle E is indecomposable.

Next assume that $g = 1$. We shall show that E is indecomposable in this case as well. To prove this, assume, as before, that we have a holomorphic decomposition as in (3.12). If the composition in (3.14) is the zero homomorphism, then the previous argument shows that E is indecomposable. So assume that the composition in (3.14) is not identically zero. Then we have

$$\text{degree}(M) \leq \text{degree}(K_X^{-1/2} \otimes \mathbb{L}) = \text{degree}(K_X^{1/2} \otimes \mathbb{L}) + g - 1,$$

so from (3.13) it follows immediately that $\text{degree}(M) = \text{degree}(K_X^{-1/2} \otimes \mathbb{L})$. This implies that the composition in (3.14) is actually an isomorphism. Hence M is a direct summand of the subbundle $K_X^{1/2} \otimes \mathbb{L} \subset E$ in (3.4), and the short exact sequence in (3.4) splits

holomorphically. Since the short exact sequence in (3.4) does not split holomorphically, we once again conclude that E is indecomposable. \square

Remark 3.3. Proposition 3.2(1) is not valid for $g = 0$. To construct an example, take $m = 2$, $c_1 = 3$ and $c_2 = 2$. Any holomorphic vector bundle on \mathbb{CP}^1 holomorphically splits into a direct sum of holomorphic line bundles [Gr, p. 122, Théorème 1.1]. Using the fact that the exact sequence in (3.4) does not split holomorphically, it is straightforward to check that the vector bundle E is isomorphic to $\mathbb{L} \oplus \mathbb{L}$ with $\text{degree}(\mathbb{L}) = \text{degree}(E)/2 = -1$. Indeed, if $E = L \oplus M$, with $\text{degree}(M) > \text{degree}(L)$, then

$$\text{degree}(K_X^{1/2} \otimes \mathbb{L}) = -2 < \text{degree}(M) \geq \text{degree}(K_X^{-1/2} \otimes \mathbb{L}) = 0,$$

and hence M projects isomorphically to the quotient $K_X^{-1/2} \otimes \mathbb{L}$ in (3.4), making M a direct summand of the line subbundle $K_X^{1/2} \otimes \mathbb{L}$, and thus implying that (3.4) splits holomorphically. Therefore, we have $E = \mathbb{L} \oplus \mathbb{L}$.

Take a direct summand $\mathbb{L}' = \mathbb{L}$ of E such that $\mathbb{L}'_{x_1} \subset E_{x_1}$ coincides with $(K_X^{1/2} \otimes \mathbb{L})_{x_1}$ in (3.4). Note that the two subbundles \mathbb{L}' and $K_X^{1/2} \otimes \mathbb{L}$ of E are distinct because $\text{degree}(\mathbb{L}') \neq \text{degree}(K_X^{1/2} \otimes \mathbb{L})$. Consider the short exact sequence on X

$$0 \longrightarrow \mathbb{L}' \oplus (K_X^{1/2} \otimes \mathbb{L}) \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

Since $\text{degree}(E) - (\text{degree}(\mathbb{L}') + \text{degree}(K_X^{1/2} \otimes \mathbb{L})) = 1$, it follows that Q is of degree one and hence it is supported on x_1 ; note that x_1 is contained in the support of Q by the condition on \mathbb{L}' that \mathbb{L}'_{x_1} coincides with $(K_X^{1/2} \otimes \mathbb{L})_{x_1}$. In particular, the fiber $\mathbb{L}'_{x_2} \subset E_{x_2}$ does not coincide with $(K_X^{1/2} \otimes \mathbb{L})_{x_2}$. Therefore, the parabolic degree of this line subbundle $\mathbb{L}' \subset E$, equipped with the induced parabolic structure, is

$$-1 + \frac{5}{6} + \frac{1}{4} = \frac{1}{12} \neq 0.$$

This parabolic line subbundle $\mathbb{L}' \subset E$ has a parabolic direct summand given by the copy of \mathbb{L} whose fiber over x_2 coincides with $(K_X^{1/2} \otimes \mathbb{L})_{x_2}$. Hence the parabolic bundle E_* does not admit a connection [BL, p. 601, Corollary 5.1].

Henceforth, we shall always assume that $g \geq 1$.

Let

$$\eta : K_X = \text{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L}) \longrightarrow \text{End}(E) \quad (3.15)$$

be the homomorphism that sends any $w \in \text{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})_x$ to the composition

$$E_x \xrightarrow{p_0(x)} (K_X^{-1/2} \otimes \mathbb{L})_x \xrightarrow{w} (K_X^{1/2} \otimes \mathbb{L})_x \hookrightarrow E_x$$

(see (3.4)). It is easy to check that $\eta(K_X)$ is a holomorphic line subbundle of $\text{End}(E)$. Using the injective map

$$H^0(X, K_X) \longrightarrow H^0(X, \text{End}(E)), \quad \alpha \longmapsto \eta(\alpha),$$

$H^0(X, K_X)$ will be considered as a subspace of $H^0(X, \text{End}(E))$. This subspace

$$H^0(X, K_X) \subset H^0(X, \text{End}(E)) \quad (3.16)$$

evidently consists of only nilpotent endomorphisms of E .

Lemma 3.4. *Any endomorphism T of the holomorphic vector bundle E is of the form*

$$T = c \cdot \text{Id}_E + \alpha,$$

where $c \in \mathbb{C}$ and $\alpha \in H^0(X, K_X)$. Hence any holomorphic automorphism of E is of the form $c \cdot \text{Id}_E + \alpha$ with $c \in \mathbb{C} \setminus \{0\}$ and $\alpha \in H^0(X, K_X)$.

Proof. In the proof of Proposition 3.2(2) we saw that the vector bundle E is indecomposable. Therefore, any element $T \in H^0(X, \text{End}(E))$ is of the form $c \cdot \text{Id}_E + \alpha'$, where $c \in \mathbb{C}$ and α' is nilpotent [At2, p. 201, Proposition 15] (see also [At1]). Take a nonzero nilpotent endomorphism

$$\alpha' \in H^0(X, \text{End}(E)),$$

so $\alpha' \circ \alpha' = 0$. Let $F \subset E$ be the holomorphic line subbundle generated by $\text{kernel}(\alpha')$; so F is the inverse image, in E , of the torsion part of $E/\text{kernel}(\alpha')$. We note that the image of α' also generates F . To prove the lemma it suffices to show that the composition

$$F \hookrightarrow E \xrightarrow{p_0} K_X^{-1/2} \otimes \mathbb{L} \quad (3.17)$$

vanishes identically, where p_0 is the projection in (3.4).

To prove by contradiction that the composition in (3.17) vanishes identically, assume that the composition in (3.17) does not vanish identically. Then we have the short exact sequence of sheaves

$$0 \longrightarrow F \oplus K_X^{1/2} \otimes \mathbb{L} \longrightarrow E \longrightarrow Q \longrightarrow 0 \quad (3.18)$$

on X which is constructed using the inclusions of F and $K_X^{1/2} \otimes \mathbb{L}$ in E . Note that Q in (3.18) is a torsion sheaf. So we have

$$\text{degree}(F) + \text{degree}(K_X^{1/2} \otimes \mathbb{L}) + \text{degree}(Q) = \text{degree}(E) = -m. \quad (3.19)$$

Since F contains a nonzero quotient of E , namely $\alpha'(E)$, as a subsheaf, it can be deduced that

$$\text{degree}(F) \geq \text{degree}(K_X^{-1/2} \otimes \mathbb{L}); \quad (3.20)$$

indeed, if $\text{degree}(F) < \text{degree}(K_X^{-1/2} \otimes \mathbb{L})$, then none of the two line bundles $K_X^{1/2} \otimes \mathbb{L}$ and $K_X^{-1/2} \otimes \mathbb{L}$ has a nonzero homomorphism to F , in which case F can't contain a nonzero quotient of E .

From (3.20) we have

$$\text{degree}(F) + \text{degree}(K_X^{1/2} \otimes \mathbb{L}) \geq \text{degree}(K_X^{-1/2} \otimes \mathbb{L}) + \text{degree}(K_X^{1/2} \otimes \mathbb{L}) = \text{degree}(E).$$

Hence from (3.19) it now follows that $\text{degree}(Q) = 0$. So we have $Q = 0$, because Q is a torsion sheaf of degree zero. This implies that F gives a holomorphic splitting of the short exact sequence in (3.4).

Since (3.4) does not split holomorphically, we conclude that the composition in (3.17) vanishes identically. As noted before, the lemma follows from this. \square

Corollary 3.5. *Any endomorphism of the holomorphic vector bundle E preserves the parabolic structure of E_* .*

Proof. This follows immediately from Lemma 3.4. \square

Let

$$D : E \longrightarrow E \otimes K_X \otimes \mathcal{O}_X(S)$$

be a connection on E_* . Consider the line subbundle $K_X^{1/2} \otimes \mathbb{L}$ of E in (3.4). The composition homomorphism

$$K_X^{1/2} \otimes \mathbb{L} \hookrightarrow E \xrightarrow{D} E \otimes (K_X \otimes \mathcal{O}_X(S)) \xrightarrow{p_0 \otimes \text{Id}} K_X^{-1/2} \otimes \mathbb{L} \otimes (K_X \otimes \mathcal{O}_X(S)) = K_X^{1/2} \otimes \mathbb{L}^{-1}, \quad (3.21)$$

where p_0 is the projection in (3.4), is known as the second fundamental form of the line subbundle $K_X^{1/2} \otimes \mathbb{L}$ for the logarithmic connection D ; here $\mathbb{L} \otimes \mathcal{O}_X(S)$ is identified with \mathbb{L}^{-1} using φ_0 in (3.3). From (3.7) it follows immediately that the composition homomorphism in (3.21) is \mathcal{O}_X -linear. Let

$$\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L}) \in H^0(X, \text{Hom}(K_X^{1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L}^{-1})) = H^0(X, \mathbb{L}^{-2}) = H^0(X, \mathcal{O}_X(S)) \quad (3.22)$$

be the second fundamental form of $K_X^{1/2} \otimes \mathbb{L}$ for the logarithmic connection D .

We recall from the definition of a connection on E_* that the residue of D at any $x_i \in S$ preserves the line $(K_X^{1/2} \otimes \mathbb{L})_{x_i} \subset E_{x_i}$. From this it follows immediately that the section $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L})$ in (3.22) vanishes at all $x_i \in S$. Consequently, we have

$$\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L}) \in H^0(X, \mathcal{O}_X) \subset H^0(X, \mathcal{O}_X(S)). \quad (3.23)$$

Lemma 3.6. *The holomorphic function $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L})$ in (3.23) does not vanish identically.*

Proof. Assume that $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L}) = 0$. This implies that the connection D preserves the line subbundle $K_X^{1/2} \otimes \mathbb{L}$ inducing a logarithmic connection on it. Let D' be the logarithmic connection on $K_X^{1/2} \otimes \mathbb{L}$ induced by D . We recall the general formula relating the residue of a logarithmic connection with the degree of the vector bundle:

$$\text{degree}(K_X^{1/2} \otimes \mathbb{L}) + \sum_{i=1}^m \text{trace}(\text{Res}(D', x_i)) = 0, \quad (3.24)$$

where $\text{Res}(D', x_i)$ is the residue of the logarithmic connection D' at x_i [Oh, p. 16, Theorem 3]. Since $\text{degree}(K_X^{1/2} \otimes \mathbb{L}) = g - 1 - \frac{m}{2}$ and $\text{Res}(D', x_i) = \frac{2c_i - 1}{2c_i}$, it follows that

$$\text{degree}(K_X^{1/2} \otimes \mathbb{L}) + \sum_{i=1}^m \text{trace}(\text{Res}(D', x_i)) > 0;$$

recall that $g \geq 1$ and $m \geq 1$. Since this contradicts (3.24), we conclude that $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L}) \neq 0$. \square

The following is an immediate consequence of Lemma 3.6.

Corollary 3.7. *Let D be a connection on E_* . The section $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L})$ in (3.23) is given by a nonzero constant function on X . In other words, $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L})$, considered as a section of \mathcal{O}_X , does not vanish anywhere on X , and $\mathbb{S}_D(K_X^{1/2} \otimes \mathbb{L})$, considered as a section of $\mathcal{O}_X(S)$, vanishes exactly on S .*

4. PARABOLIC SL_2 -OPERS

Recall that a logarithmic connection on E induces a logarithmic connection on $\bigwedge^2 E$. Note also that $\det E = \bigwedge^2 E = \mathbb{L}^{\otimes 2} = \mathcal{O}_X(-S)$. The de Rham differential $f \mapsto df$ produces a logarithmic connection on $\mathcal{O}_X(-S)$. The residue of this logarithmic connection on $\mathcal{O}_X(-S)$ at every point $x_i \in S$ is 1.

Two connections on E_* are called *equivalent* if they are conjugate by a holomorphic automorphism of the parabolic vector bundle E_* . From Corollary 3.5 we know that any holomorphic automorphism of E is an automorphism of the parabolic vector bundle E_* . Consequently, two connections on E_* are equivalent if they are conjugate by a holomorphic automorphism of E . Note that if a connection D on E_* has the property that the logarithmic connection on $\bigwedge^2 E$ induced by D coincides with the tautological logarithmic connection on $\mathcal{O}_X(-S)$ given by the de Rham differential, then any connection on E_* equivalent to D also has this property. Indeed, if D' and D'' are two equivalent connections on E_* differing by a holomorphic automorphism T of E , then the two logarithmic connections on $\det E = \mathcal{O}_X(-S)$ induced by D' and D'' differ by the automorphism of $\det E$ induced by T . On the other hand, the automorphisms of the holomorphic line bundle $\det E$ are constant scalar multiplications, and a constant scalar multiplication preserves any logarithmic connection.

Definition 4.1. A *parabolic SL_2 -oper* is an equivalence class of connections D on E_* such that the logarithmic connection on $\bigwedge^2 E$ induced by D coincides with the tautological logarithmic connection on $\det E = \mathcal{O}_X(-S)$ given by the de Rham differential.

Remark 4.2. Note that our definition of SL_2 -oper (in the case $S = \emptyset$) is slightly more restrictive than the original one (see [BD] section 2.8), as we choose a theta-characteristic $K_X^{1/2}$, which completely determines the underlying vector bundle.

Lemma 4.3. *The space of parabolic SL_2 -opers on X is nonempty.*

Proof. Let D be a connection on E_* , which exists by Proposition 3.2(1). Let D' be the logarithmic connection on $\bigwedge^2 E = \mathcal{O}_X(-S)$ induced by D . For any $x_i \in S$, since the eigenvalues of $\mathrm{Res}(D, x_i)$ are $\frac{2c_i-1}{2c_i}$ and $\frac{1}{2c_i}$, we conclude that $\mathrm{Res}(D', x_i) = \mathrm{trace}(\mathrm{Res}(D, x_i)) = 1$. In particular, $\mathrm{Res}(D', x_i)$ coincides with the residue at x_i of the tautological logarithmic connection D_0 on $\mathcal{O}_X(-S)$ given by the de Rham differential. So we have

$$D' = D_0 + \beta,$$

where $\beta \in H^0(X, K_X)$. Now it is straight-forward to check that the logarithmic connection $D - \frac{\beta}{2}$ on E defines a parabolic SL_2 -oper. \square

Lemma 4.4. *Let D be a connection on E_* defining a parabolic SL_2 -oper on X . Then D produces a holomorphic isomorphism of E with the jet bundle $J^1(K_X^{-1/2} \otimes \mathbb{L})$. This isomorphism takes the subbundle $K_X^{1/2} \otimes \mathbb{L} \subset E$ in (3.4) to the subbundle*

$$(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X = K_X^{1/2} \otimes \mathbb{L} \subset J^1(K_X^{-1/2} \otimes \mathbb{L})$$

in (2.3).

Proof. Consider the \mathcal{O}_X -linear homomorphism

$$D' : J^1(E) \longrightarrow E \otimes K_X \otimes \mathcal{O}_X(S)$$

given by D (see (3.8)). We shall determine $\text{kernel}(D') \subset J^1(E)$. It can be shown that on the complement $X \setminus S$, we have $\text{kernel}(D')|_{X \setminus S} = E|_{X \setminus S}$. Indeed, D is a holomorphic connection on $E|_{X \setminus S}$, and, as mentioned before, any holomorphic connection on a Riemann surface is flat. The above map $E|_{X \setminus S} \longrightarrow \text{kernel}(D')|_{X \setminus S}$ sends any $v \in E_x$ to the element of $J^1(E)_x$ defined by the unique flat section s for D , defined around x , with $s(x) = v$. This isomorphism $E|_{X \setminus S} \longrightarrow \text{kernel}(D')|_{X \setminus S}$ clearly extends to a homomorphism

$$D_K : E \otimes \mathcal{O}_X(-S) \longrightarrow \text{kernel}(D') \subset J^1(E) \quad (4.1)$$

over X . Let

$$Q_D := \text{kernel}(D') / (E \otimes \mathcal{O}_X(-S))$$

be the quotient, which is a torsion sheaf; its support is contained in S , because on $X \setminus S$ both $\text{kernel}(D')$ and $E \otimes \mathcal{O}_X(-S)$ are identified with $E|_{X \setminus S}$. Note that we have

$$\text{degree}(\text{kernel}(D')) = \text{degree}(E \otimes \mathcal{O}_X(-S)) + \text{degree}(Q_D). \quad (4.2)$$

The homomorphism D' is surjective, because the residue of D at each point $x_i \in S$ is an isomorphism. From this surjectivity it follows that

$$\text{degree}(\text{kernel}(D')) = \text{degree}(J^1(E)) - \text{degree}(E \otimes K_X \otimes \mathcal{O}_X(S)) = -3m.$$

Since $\text{degree}(E \otimes \mathcal{O}_X(-S)) = -3m$, from (4.2) it follows that $\text{degree}(Q_D) = 0$. Since Q_D is a torsion sheaf of degree zero, we conclude that $Q_D = 0$. This implies that the homomorphism D_K in (4.1) is an isomorphism.

Let $p_0^{(1)} : J^1(E) \longrightarrow J^1(K_X^{-1/2} \otimes \mathbb{L})$ be the homomorphism constructed as in (2.4) for the projection p_0 in (3.4). The composition

$$p_0^{(1)} \circ D_K : E \otimes \mathcal{O}_X(-S) \longrightarrow J^1(K_X^{-1/2} \otimes \mathbb{L}),$$

where D_K is the isomorphism in (4.1), clearly vanishes over S . So $p_0^{(1)} \circ D_K$ produces a homomorphism

$$\widetilde{p_0^{(1)} \circ D_K} : E \longrightarrow J^1(K_X^{-1/2} \otimes \mathbb{L}). \quad (4.3)$$

From Corollary 3.7 we know that the section $\widetilde{S_D(K_X^{1/2} \otimes \mathbb{L})} \in H^0(X, \mathcal{O}_X(S))$ in (3.23) does not vanish on $X \setminus S$. Using this it follows that $\widetilde{p_0^{(1)} \circ D_K}$ in (4.3) is an isomorphism over $X \setminus S$. Therefore, $J^1(K_X^{-1/2} \otimes \mathbb{L})/E$ is a torsion sheaf of degree zero, because $\text{degree}(E) = -m = \text{degree}(\widetilde{p_0^{(1)} \circ D_K})$. Hence we have $J^1(K_X^{-1/2} \otimes \mathbb{L})/E = 0$, implying that $\widetilde{p_0^{(1)} \circ D_K}$ in (4.3) is an isomorphism over entire X .

Over $X \setminus S$, the isomorphism $\widetilde{p_0^{(1)} \circ D_K}$ evidently takes the subbundle

$$(K_X^{1/2} \otimes \mathbb{L})|_{X \setminus S} \subset E|_{X \setminus S}$$

in (3.4) to the subbundle

$$((K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X)|_{X \setminus S} = (K_X^{1/2} \otimes \mathbb{L})|_{X \setminus S} \subset J^1(K_X^{-1/2} \otimes \mathbb{L})|_{X \setminus S}$$

in (2.3). Consequently, the isomorphism $\widetilde{p_0^{(1)} \circ D_K}$ over X takes the subbundle $K_X^{1/2} \otimes \mathbb{L} \subset E$ to the subbundle $K_X^{1/2} \otimes \mathbb{L} \subset J^1(K_X^{1/2} \otimes \mathbb{L})$. \square

Remark 4.5. It should be clarified that the isomorphism $\widetilde{p_0^{(1)} \circ D_K}$ in (4.3) depends on the logarithmic connection D .

Theorem 4.6. *The space of parabolic SL_2 -opers on X is an affine space for the vector space $H^0(X, K_X^2 \otimes \mathcal{O}_X(S))$.*

Proof. Let D be a connection on E_* defining a parabolic SL_2 -oper on X . Consider the isomorphism $\widetilde{p_0^{(1)} \circ D_K}$ constructed in (4.3). Let

$$D_1 = (\widetilde{p_0^{(1)} \circ D_K})_* D$$

be the logarithmic connection on $J^1(K_X^{-1/2} \otimes \mathbb{L})$ given by the logarithmic connection D on E using this isomorphism. The composition homomorphism

$$\begin{aligned} J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) &\xrightarrow{D_1} J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S) \longrightarrow J^0(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S) \quad (4.4) \\ &= (K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S) = K_X^{1/2} \otimes \mathbb{L} \otimes \mathcal{O}_X(S) = K_X^{1/2} \otimes \mathbb{L}^* \end{aligned}$$

will be denoted by D'_1 (see (3.8)); the projection

$$J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S) \longrightarrow J^0(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S)$$

in (4.4) is the homomorphism $J^1(K_X^{-1/2} \otimes \mathbb{L}) \longrightarrow J^0(K_X^{-1/2} \otimes \mathbb{L})$ in (2.3) tensored with the identity map of $K_X \otimes \mathcal{O}_X(S)$. Consider the homomorphism \mathbf{t} in (2.5). Let D''_1 denote the composition

$$J^2(K_X^{-1/2} \otimes \mathbb{L}) \xrightarrow{\mathbf{t}} J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) \xrightarrow{D'_1} K_X^{1/2} \otimes \mathbb{L} \otimes \mathcal{O}_X(S) = K_X^{1/2} \otimes \mathbb{L}^*.$$

From the construction of the isomorphism $\widetilde{p_0^{(1)} \circ D_K}$ in (4.3) it follows that this homomorphism D''_1 vanishes identically. Therefore, the image of the composition $D_1 \circ \mathbf{t}$ lies in the subbundle

$$(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes (K_X \otimes \mathcal{O}_X(S)) \subset J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes (K_X \otimes \mathcal{O}_X(S));$$

see (2.3) for the above inclusion $(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \hookrightarrow J^1(K_X^{-1/2} \otimes \mathbb{L})$. In other words, we have the homomorphism

$$\begin{aligned} J^2(K_X^{-1/2} \otimes \mathbb{L}) &\xrightarrow{D_1 \circ \mathbf{t}} (K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes K_X \otimes \mathcal{O}_X(S) \\ &= K_X^{3/2} \otimes \mathbb{L} \otimes \mathcal{O}_X(S) = K_X^{3/2} \otimes \mathbb{L}^* \subset J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S). \end{aligned}$$

Consequently, we have a holomorphic differential operator

$$D_1 \circ \mathbf{t} \in H^0(X, \mathrm{Diff}^2(K_X^{-1/2} \otimes \mathbb{L}, K_X^{3/2} \otimes \mathbb{L} \otimes \mathcal{O}_X(S))) = H^0(X, \mathrm{Diff}^2(K_X^{-1/2} \otimes \mathbb{L}, K_X^{3/2} \otimes \mathbb{L}^*)) \quad (4.5)$$

of order two (see (2.6)).

While $D_1 \circ \mathbf{t}$ is constructed above from the homomorphism D_1 in (4.4), we shall now show that the homomorphism D_1 can also be recovered back from the differential operator $D_1 \circ \mathbf{t}$. For this, consider the two subsheaves

$$\mathbf{t}(\ker(D_1 \circ \mathbf{t})) \quad \text{and} \quad J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X$$

of $J^1(J^1(K_X^{-1/2} \otimes \mathbb{L}))$ (see (2.3) for the second subsheaf). The composition

$$J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \hookrightarrow J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) \longrightarrow J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) / (\mathbf{t}(\ker(D_1 \circ \mathbf{t})))$$

vanishes on S . Therefore, this composition gives a homomorphism

$$D_2 : J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S) \longrightarrow J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) / (\mathbf{t}(\ker(D_1 \circ \mathbf{t}))) \quad (4.6)$$

which is an isomorphism over $X \setminus S$. On the other hand, we have

$$\deg(J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S)) = \deg(J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) / (\mathbf{t}(\ker(D_1 \circ \mathbf{t})))) .$$

Since any homomorphism between two holomorphic vector bundles of same degree is an isomorphism if it is generically an isomorphism, we now conclude that the homomorphism D_2 in (4.6) is an isomorphism. Finally, the composition

$$\begin{aligned} J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) &\longrightarrow J^1(J^1(K_X^{-1/2} \otimes \mathbb{L})) / (\mathbf{t}(\ker(D_1 \circ \mathbf{t}))) \\ &\xrightarrow{D_2^{-1}} J^1(K_X^{-1/2} \otimes \mathbb{L}) \otimes K_X \otimes \mathcal{O}_X(S) \end{aligned}$$

coincides with D_1 in (4.4). Therefore, D_1 is uniquely determined by $D_1 \circ \mathbf{t}$.

It should be clarified that this does not imply that the map $D \mapsto D_1$ is injective. In fact, this map is *not* injective.

The symbol of the differential operator $D_1 \circ \mathbf{t}$ in (4.5) coincides with the section of $\mathcal{O}_X(S)$ given by the constant function 1. The condition in Definition 4.1 that the connection on $\bigwedge^2 E = \mathcal{O}_X(-S)$ induced by D coincides with the tautological logarithmic connection on $\mathcal{O}_X(-S)$ given by the de Rham differential, implies that the connection on $\bigwedge^2 J^1(K_X^{-1/2} \otimes \mathbb{L}) = \mathcal{O}_X(-S)$ induced by D_1 coincides with the tautological logarithmic connection on $\mathcal{O}_X(-S)$ given by the de Rham differential. This is equivalent to the condition that the first order part of the differential operator $D_1 \circ \mathbf{t}$ in (4.5) vanishes. The space of second order holomorphic differential operators satisfying the above conditions is an affine space for the vector space

$$H^0(X, \operatorname{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{3/2} \otimes \mathbb{L} \otimes \mathcal{O}_X(S))) = H^0(X, K_X^2 \otimes \mathcal{O}_X(S)) .$$

Using this it follows that the space of parabolic SL_2 -oper on X is an affine space for the vector space $H^0(X, K_X^2 \otimes \mathcal{O}_X(S))$. \square

5. PARABOLIC SL_r -OPERS

5.1. Parabolic tensor product and parabolic dual. In [MY] an equivalent formulation of the definition of parabolic bundles was given. We shall recall it now.

Let $V \longrightarrow X$ be a holomorphic vector bundle. Let

$$V_{x_i} =: F_{i,1} \supsetneq \cdots \supsetneq F_{i,j} \supsetneq \cdots \supsetneq F_{i,a_i} \supsetneq F_{i,a_i+1} = 0$$

be a quasiparabolic filtration over each point x_i of S . Fix parabolic weights

$$0 \leq \alpha_{i,1} < \cdots < \alpha_{i,j} < \cdots < \alpha_{i,a_i} < 1$$

associated to these quasiparabolic flags ([MS], [Se, p. 67]).

Now, for a point $x_i \in S$ and $t \in [0, 1]$, let

$$V^{i,t} \subset V$$

be the coherent subsheaf defined as follows: if $t \leq \alpha_{i,1}$, then

$$V^{i,t} = V,$$

if $t > \alpha_{i,1}$, then $V^{i,t}$ is defined by the short exact sequence of sheaves

$$0 \longrightarrow V^{i,t} \longrightarrow V \longrightarrow V/F_{i,j+1} \longrightarrow 0,$$

where $j \in [1, a_i]$ is the largest number such that $\alpha_{i,j} < t$. For $t \in [0, 1]$, define

$$V^{(t)} = \bigcap_{i=1}^m V^{i,t} \subset V.$$

Now we have a filtration of coherent sheaves $\{V_t\}_{t \in \mathbb{R}}$ defined by

$$V_t := V^{(t-[t])} \otimes \mathcal{O}_X(-[t]S),$$

where $[t]$ is the integral part of t , meaning $0 \leq t - [t] < 1$. From the construction of the filtration $\{V_t\}_{t \in \mathbb{R}}$ it is evident that the parabolic vector bundle $(V, \{F_{i,j}\}, \{\alpha_{i,j}\})$ can be recovered from it. This description of parabolic bundles was introduced in [MY].

Let

$$\iota : X \setminus S \hookrightarrow X$$

be the inclusion of the complement. Let $\{V_t\}_{t \in \mathbb{R}}$ and $\{W_t\}_{t \in \mathbb{R}}$ be the filtrations corresponding to two parabolic vector bundles V_* and W_* respectively. Consider the torsionfree quasi-coherent sheaf $\iota_*((V_0 \otimes W_0)|_{X \setminus S})$ on X . Note that $V_s \otimes W_t$ is a coherent subsheaf of it for all s and t . For any $t \in \mathbb{R}$, let

$$\mathcal{E}_t \subset \iota_*((V_0 \otimes W_0)|_{X \setminus S})$$

be the quasi-coherent subsheaf generated by all $V_\alpha \otimes W_{t-\alpha}$, $\alpha \in \mathbb{R}$. It is easy to see that \mathcal{E}_t is a coherent sheaf, and the collection $\{\mathcal{E}_t\}_{t \in \mathbb{R}}$ satisfies all the three conditions needed to define a parabolic vector bundle on X with parabolic structure over S . The parabolic vector bundle defined by $\{\mathcal{E}_t\}_{t \in \mathbb{R}}$ is denoted by $V_* \otimes W_*$, and it is called the *tensor product* of V_* and W_* .

Now consider the torsionfree quasi-coherent sheaf $\iota_*((V_0^* \otimes W_0)|_{X \setminus S})$ on X . For any $t \in \mathbb{R}$, let

$$\mathcal{F}_t \subset \iota_*((V_0^* \otimes W_0)|_{X \setminus S})$$

be the quasi-coherent subsheaf generated by all $V_\alpha^* \otimes W_{\alpha+t}$, $\alpha \in \mathbb{R}$. This \mathcal{F}_t is a coherent sheaf, and the collection $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ satisfies the three conditions needed to define a parabolic vector bundle with parabolic structure over S . The parabolic vector bundle defined by $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is denoted by $\text{Hom}(V_*, W_*)$. When W_* is the trivial parabolic line bundle with trivial parabolic structure, then $\text{Hom}(V_*, W_*)$ is the parabolic dual V_*^* .

For any parabolic structure on V and any integer $d \geq 2$, the tensor product $V^{\otimes d}$ is a subsheaf of the vector bundle underlying the parabolic tensor product $(V_*)^{\otimes d}$. Let \mathcal{S}_d denote the group of permutations of the set $\{1, \dots, d\}$. The natural action of \mathcal{S}_d on $V^{\otimes d}$ that permutes the factors extends to an action of \mathcal{S}_d on the vector bundle underlying the parabolic bundle $(V_*)^{\otimes d}$. The parabolic symmetric product $\text{Sym}^d(V_*)$ is the parabolic bundle given by the fixed point locus of this action of \mathcal{S}_d on $(V_*)^{\otimes d}$; note that this fixed point locus is equipped with the induced parabolic structure.

The exterior products of a parabolic vector bundle are defined in a similar way. The parabolic determinant bundle of a parabolic vector bundle V_* is the m -th degree parabolic exterior product of V_* , where m is the rank of V_* .

5.2. Connections on a parabolic symmetric product. Consider the rank two parabolic bundle E_* in Section 3. For any integer $r \geq 2$, the parabolic symmetric product $\text{Sym}^{r-1}(E_*)$ will be denoted by \mathbb{E}_r . So \mathbb{E}_r is a parabolic vector bundle of rank r .

Corollary 5.1. *The parabolic bundle \mathbb{E}_r admits a connection.*

Proof. A connection on E_* induces a connection on the parabolic tensor product $(E_*)^{\otimes(r-1)}$. This connection on $(E_*)^{\otimes(r-1)}$ preserves the parabolic subbundle $\mathbb{E}_r = \text{Sym}^{r-1}(E_*) \subset (E_*)^{\otimes(r-1)}$. Therefore, from Proposition 3.2(1) it follows that \mathbb{E}_r admits a connection. \square

Note that the parabolic determinant bundle $\det E_*$ for E_* , which is the same as the second parabolic exterior product of E_* , is the trivial line bundle on X with the trivial parabolic structure. We observe that the determinant of the underlying bundle E equals $\mathcal{O}_X(-S)$ with parabolic weights 1 on S , which corresponds to the trivial parabolic line bundle \mathcal{O}_X with parabolic weights 0 on S . Using this it is straight-forward to deduce that parabolic determinant bundle $\det \mathbb{E}_r$, which is the same as the r -th parabolic exterior product of \mathbb{E}_r , is also the trivial line bundle on X with the trivial parabolic structure.

A connection on the parabolic vector bundle \mathbb{E}_r induces a connection on the parabolic line bundle $\det \mathbb{E}_r$. Indeed, a connection on \mathbb{E}_r induces a connection on the parabolic tensor product $(\mathbb{E}_r)^{\otimes r}$. This induced connection on $(\mathbb{E}_r)^{\otimes r}$ preserves the parabolic subbundle $\det \mathbb{E}_r \subset (\mathbb{E}_r)^{\otimes r}$.

In the proof of Corollary 5.1 we saw that a connection D on the parabolic vector bundle E_* produces a connection on the parabolic vector bundle \mathbb{E}_r . Now if D is a SL_2 -oper, meaning the logarithmic connection on $\bigwedge^2 E = \mathcal{O}_X(-S)$ induced by D coincides with the one given by the de Rham differential, then the connection on the parabolic determinant line bundle $\bigwedge^2 E_*$ coincides with the trivial connection on the trivial line bundle given by the de Rham differential. Using this it follows that the connection on the parabolic bundle \mathbb{E}_r given by D has the property that the connection on the parabolic line bundle $\det \mathbb{E}_r$ induced by it also coincides with the trivial connection on the trivial line bundle given by the de Rham differential.

Two connections on the parabolic bundle \mathbb{E}_r are called *equivalent* if they differ by a holomorphic automorphism of the parabolic vector bundle \mathbb{E}_r .

Definition 5.2. A *parabolic SL_r -oper* is an equivalence class of connections D on the parabolic bundle \mathbb{E}_r such that the connection on $\det \mathbb{E}_r$ induced by D coincides with the trivial connection on $\det \mathbb{E}_r = \mathcal{O}_X$ given by the de Rham differential.

6. SPECIAL RATIONAL PARABOLIC WEIGHTS

Henceforth, we assume that

$$c_i := \mathbf{c}(x_i) \in \mathbb{N}$$

for all $x_i \in S$, where \mathbf{c} is the function in (3.2). We also assume that the integer r is odd.

There is a ramified Galois covering

$$\rho : Y \longrightarrow X \tag{6.1}$$

satisfying the following conditions:

- ρ is unramified over the complement $X \setminus S$;
- for every $x_i \in S$ and each point $y \in \rho^{-1}(x_i)$, the order of ramification of ρ at y is c_i .

Such a covering ρ exists; see [Na, p. 26, Proposition 1.2.12].

The Galois group $\mathrm{Gal}(\rho)$ for ρ will be denoted by Γ . The holomorphic cotangent bundle of Y will be denoted by K_Y . The action of Γ on Y produces an action of Γ on K_Y .

A Γ -equivariant vector bundle on Y is a holomorphic vector bundle V on Y equipped with a lift of the action of Γ such that the action of any $\gamma \in \Gamma$ on V maps V_y to $V_{\gamma(y)}$ linearly for every $y \in Y$.

Since the parabolic weights of E_* at $x_i \in S$ are integral multiples of $\frac{1}{2c_i}$, the parabolic weights, at x_i , of the parabolic symmetric product $\mathrm{Sym}^j(E_*)$ are integral multiples of $\frac{j}{2c_i}$. In particular, the parabolic weights of $\mathrm{Sym}^{2j}(E_*)$ at x_i are integral multiples of $\frac{1}{c_i}$. Recall that for each point $y \in \rho^{-1}(x_i)$, the order of ramification of ρ at y is c_i . Consequently, there is a unique Γ -equivariant vector bundle \mathcal{E}_r on Y of rank r that corresponds to the parabolic vector bundle $\mathrm{Sym}^{r-1}(E_*) = \mathbb{E}_r$ [Bi1].

This action of Γ on \mathcal{E}_r induces an action of Γ on the holomorphic line bundle $\bigwedge^r \mathcal{E}_r = \det \mathcal{E}_r$. Since the Γ -equivariant bundle \mathcal{E}_r corresponds to the parabolic bundle \mathbb{E}_r , the Γ -equivariant line bundle $\det \mathcal{E}_r$ corresponds to the parabolic line bundle $\det \mathbb{E}_r$. As noted in Section 5.2, the parabolic line bundle $\det \mathbb{E}_r$ is the trivial line bundle with the trivial parabolic structure. Therefore, the Γ -equivariant line bundle $\det \mathcal{E}_r$ corresponding to \mathbb{E}_r is the trivial line bundle on Y equipped with the trivial action of Γ ; this means that Γ act diagonally on the trivial line bundle $Y \times \mathbb{C}$ using the trivial action of Γ on \mathbb{C} and the Galois action of Γ on Y .

Lemma 6.1. *The holomorphic vector bundle \mathcal{E}_r admits a Γ -invariant holomorphic connection D_Γ such that the holomorphic connection on $\det \mathcal{E}_r = \mathcal{O}_Y$ induced by D_Γ coincides with the de Rham differential d .*

Proof. Since the parabolic bundle \mathbb{E}_r corresponds to the Γ -equivariant bundle \mathcal{E}_r , there is a natural bijection between the connections on \mathbb{E}_r and the Γ -invariant holomorphic connections on \mathcal{E}_r (see [Bi2]). Now, Corollary 5.1 says that \mathbb{E}_r admits a connection. Hence \mathcal{E}_r admits a Γ -invariant holomorphic connection. Let D' be a Γ -invariant holomorphic connection on \mathcal{E}_r . Note that any holomorphic connection on the trivial bundle \mathcal{O}_Y is of the form $d + \omega$, where ω is a holomorphic 1-form on Y . Let $d + \omega$ be the holomorphic connection on $\det \mathcal{E}_r = \mathcal{O}_Y$ induced by D' . Now it is straight-forward to check that

$$D_\Gamma := D' - \frac{1}{r}\omega$$

is a Γ -invariant holomorphic connection on \mathcal{E}_r satisfying the condition that the holomorphic connection on $\det \mathcal{E}_r = \mathcal{O}_Y$ induced by D_Γ coincides with the de Rham differential d . \square

Proposition 6.2. *Let D be a holomorphic connection on \mathcal{E}_r satisfying the condition that the holomorphic connection on $\det \mathcal{E}_r$ induced by D coincides with the de Rham differential d . Then D is a SL_r -oper on Y .*

Proof. The filtration of E_* in (3.4) produces a filtration of subbundles of the parabolic symmetric product

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = \mathbb{E}_{r,0}, \quad (6.2)$$

where $\mathbb{E}_{r,0}$ is the holomorphic vector bundle underlying the parabolic bundle \mathbb{E}_r . Each F_j is a holomorphic subbundle of $\mathbb{E}_{r,0}$ of rank j . We shall describe the quotient line bundles F_j/F_{j-1} equipped with the parabolic structure induced by the parabolic structure of \mathbb{E}_r .

Let \mathbb{K} be the parabolic line bundle whose underlying holomorphic line bundle is the canonical bundle K_X , and the parabolic weight of \mathbb{K} at any $x_i \in S$ is $\frac{c_i-1}{c_i}$. If K' is the line subbundle $K_X^{1/2} \otimes \mathbb{L}$ in (3.4) equipped with the parabolic structure induced by E_* (this simply means that the parabolic weight of K' at any $x_i \in S$ is $\frac{2c_i-1}{2c_i}$), then \mathbb{K} is identified with the parabolic tensor product $K' \otimes K'$.

For any $1 \leq j \leq r$, the quotient F_j/F_{j-1} in (6.2) equipped with the parabolic structure induced by the parabolic structure of \mathbb{E}_r is identified with the parabolic tensor product (and dual in case the exponent is negative) $\mathbb{K}^{(r+1)/2-j}$ (recall that r is an odd integer).

Since the parabolic weight of \mathbb{K} at any $x_i \in S$ is an integral multiple of $\frac{1}{c_i}$, there is a unique Γ -equivariant line bundle on Y that corresponds to the parabolic line bundle \mathbb{K} . It is straight-forward to check that this Γ -equivariant line bundle on Y is the canonical bundle K_Y equipped with the action of Γ induced by the action of Γ on Y .

It was noted above that the quotient F_j/F_{j-1} in (6.2) equipped with the induced parabolic structure is $\mathbb{K}^{(r+1)/2-j}$. Since the correspondence between parabolic bundles and equivariant bundles is compatible with the tensor product and dualization operations, the parabolic line bundle F_j/F_{j-1} corresponds to the Γ -equivariant line bundle $K_Y^{(r+1)/2-j}$, because \mathbb{K} corresponds to K_Y .

Consequently, from (6.2) we conclude that that Γ -equivariant bundle \mathcal{E}_r has a filtration of holomorphic subbundles

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V_r = \mathcal{E}_r \quad (6.3)$$

such that for all $1 \leq j \leq r$,

- $\mathrm{rank}(V_j) = j$,
- $V_j/V_{j-1} = K_Y^{(r+1)/2-j}$, and
- the action of Γ on \mathcal{E}_r preserves the subbundle V_j .

Let D be a holomorphic connection on \mathcal{E}_r . From (6.3) it follows that $D(V_1) \subset V_2 \otimes K_Y$, and more generally, using induction,

$$D(V_j) \subset V_{j+1} \otimes K_Y$$

for all $1 \leq j \leq r-1$. Therefore, D produces an \mathcal{O}_Y -linear homomorphism

$$\phi_j : V_j/V_{j-1} \longrightarrow (V_{j+1}/V_j) \otimes K_Y \quad (6.4)$$

for all $1 \leq j \leq r-1$. Note that both V_j/V_{j-1} and $(V_{j+1}/V_j) \otimes K_Y$ are identified with $K_Y^{(r+1)/2-j}$, because $V_i/V_{i-1} = K_Y^{(r+1)/2-i}$ for all $1 \leq i \leq r$. This implies that ϕ_j in (6.4) is either an isomorphism, or identically zero.

Assume that $\phi_j = 0$ for some $1 \leq j \leq r-1$. Then the connection D on \mathcal{E}_r preserves the subbundle V_j . This implies the

$$\mathrm{degree}(V_j) = 0. \quad (6.5)$$

Since $V_i/V_{i-1} = K_Y^{(r+1)/2-i}$ for all $1 \leq i \leq r$, we have

$$\mathrm{degree}(V_j) = j \frac{r-j}{2} \mathrm{degree}(K_Y) = j(r-j)(g_Y - 1), \quad (6.6)$$

where g_Y is the genus of Y . Since $\mathrm{genus}(X) \geq 1$, and $S \neq \emptyset$, it follows that $g_Y > 1$. Hence (6.6) contradicts (6.5).

In view of the above contradiction we conclude that $\phi_j \neq 0$ for all $1 \leq j \leq r-1$. As noted before, this implies that ϕ_j is an isomorphism for all $1 \leq j \leq r-1$.

Therefore, the holomorphic connection D on \mathcal{E}_r defines a SL_r -oper if the holomorphic connection on $\det \mathcal{E}_r$ induced by D coincides with the de Rham differential d on \mathcal{O}_Y . This completes the proof of the proposition. \square

The action of Γ on \mathcal{E}_r produces an action of Γ on the space of all holomorphic connections on \mathcal{E}_r . This action of Γ on the space of all holomorphic connections on \mathcal{E}_r evidently preserves the space $\mathcal{D}(\mathcal{E})$ consisting of all holomorphic connections on \mathcal{E}_r that induce the trivial connection on $\det \mathcal{E}_r = \mathcal{O}_Y$.

Since the SL_r -opers on Y are the equivalence classes of holomorphic connections on \mathcal{E} lying in $\mathcal{D}(\mathcal{E})$, we obtain an action of Γ on the space of all SL_r -opers on Y . A SL_r -oper on Y is called Γ -invariant if it is fixed by this action of Γ on \mathcal{E} .

Theorem 6.3. *Parabolic SL_r -opers on X are in a natural bijection with the Γ -invariant SL_r -opers on Y .*

Proof. As noted before, since the Γ -equivariant bundle \mathcal{E}_r corresponds to the parabolic bundle $\mathrm{Sym}^{r-1}(E_*) = \mathbb{E}_r$, the parabolic connections on \mathbb{E}_r are in a natural bijective correspondence with the Γ -invariant holomorphic connections on \mathcal{E}_r . We recall that the SL_r -opers on

Y are the equivalence classes of holomorphic connections D' on \mathcal{E}_r such that the connection on $\bigwedge^r \mathcal{E}_r = \mathcal{O}_Y$ induced by D' coincides with the one given by the de Rham differential d . For such a Γ -invariant holomorphic connection D' on \mathcal{E}_r , the connection D on the parabolic bundle \mathbb{E}_r has the property that the connection of the parabolic line bundle $\bigwedge^r \mathbb{E}_r$ induced by D is the trivial connection on \mathcal{O}_X (the connection on \mathcal{O}_X given by the de Rham differential d). The theorem follows from these. \square

A projective structure P_Y on Y is called Γ -invariant if the automorphisms of Y given by the action of Γ on Y preserve P_Y . Let $\mathcal{P}(Y)^\Gamma$ denote the space of all projective structures on Y that are Γ -invariant. Any element of $\mathcal{P}(Y)^\Gamma$ defines a logarithmic projective structure on X (see [BDM]). This $\mathcal{P}(Y)^\Gamma$ is an affine space for

$$H^0(Y, K_Y^{\otimes 2})^\Gamma = H^0(X, K_X^{\otimes 2} \otimes \mathcal{O}_X(S)).$$

The space of all SL_r -opers on Y is in bijection with

$$\mathcal{P}(Y) \times \left(\bigoplus_{i=3}^r H^0(Y, K_Y^{\otimes i}) \right)$$

(see [Bi3, Theorem 4.9] and [Bi3, Eq. (5.4)]). Consequently, the space of all Γ -invariant SL_r -opers on Y is in bijection with

$$\mathcal{P}(Y)^\Gamma \times \left(\bigoplus_{i=3}^r H^0(Y, K_Y^{\otimes i})^\Gamma \right) = \mathcal{P}(Y)^\Gamma \times \left(\bigoplus_{i=3}^r H^0(X, K_X^{\otimes i} \otimes \mathcal{O}_X((i-1)S)) \right).$$

Now from Theorem 6.3 we conclude that the space of all parabolic SL_r -opers on X is in bijection with

$$\mathcal{P}(Y)^\Gamma \times \left(\bigoplus_{i=3}^r H^0(X, K_X^{\otimes i} \otimes \mathcal{O}_X((i-1)S)) \right).$$

Setting $r = 2$ we get that the parabolic SL_2 -opers on X are identified with $\mathcal{P}(Y)^\Gamma$. Consequently, we have the following:

Theorem 6.4. *The space of all parabolic SL_r -opers on X is in a natural bijection with*

$$\mathcal{P}_2 \times \left(\bigoplus_{i=3}^r H^0(X, K_X^{\otimes i} \otimes \mathcal{O}_X((i-1)S)) \right),$$

where \mathcal{P}_2 denotes the space of all parabolic SL_2 -opers on X .

7. FURTHER COMMENTS

There are other natural generalizations of SL_r -opers to the parabolic set-up (see also [ABF]). Following the original definition of [BD] one can also define a parabolic SL_r -oper for a given parabolic divisor $S = \sum_i x_i$, with $x_i \in X$, and with real parabolic weights $\alpha_{i,j}$ such that

$$0 \leq \alpha_{i,1} < \cdots < \alpha_{i,j} < \cdots < \alpha_{i,r} < 1 \text{ for each } x_i \in S,$$

and $m_i = \sum_j \alpha_{i,j} \in \mathbb{N}^*$, as a triple (E, E_\bullet, D) , where

- E is a rank r vector bundle,

- $E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ is a filtration by subbundles such that $\mathrm{rk}(E_i) = i$,
- $D : E \rightarrow E \otimes K_X \otimes \mathcal{O}_X(S)$ is a logarithmic connection on E such that $D(E_i) \subset E_{i+1} \otimes K_X \otimes \mathcal{O}_X(S)$ and the induced \mathcal{O}_X -linear map $\overline{D} : E_i/E_{i-1} \rightarrow E_{i+1}/E_i \otimes K_X \otimes \mathcal{O}_X(S)$ factorizes through an isomorphism $E_i/E_{i-1} \cong E_{i+1}/E_i \otimes K_X$ followed by the natural sheaf inclusion $E_{i+1}/E_i \otimes K_X \subset E_{i+1}/E_i \otimes K_X \otimes \mathcal{O}_X(S)$.
- For each $x_i \in S$ the residue map $\mathrm{Res}(D, x_i)$ preserves the full flag $(E_\bullet)_{x_i}$ and acts as multiplication by $\alpha_{i,r+1-j}$ on $(E_j/E_{j-1})_{x_i}$
- $\det E = \mathcal{O}_X(-\sum_i m_i x_i)$ and $\det D$ is the logarithmic de Rham differential $f \mapsto df$ restricted to $\mathcal{O}_X(-\sum_i m_i x_i)$.

It can be easily shown that for $r = 2$ and rational parabolic weights the two definitions coincide (up to a choice of a theta-characteristic and a square-root of $\mathcal{O}_X(-S)$). For $r > 2$ we note that the above definition is more general than Definition 5.2. It would be interesting to know whether they coincide for the special rational parabolic weights of the SL_r -opers in Section 6 and whether Theorem 6.4 holds for general real weights. We will address these questions in a future paper.

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